Topological phases in condensed matter physics

The American Institute of Mathematics

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A.1 Bais, Ferdinand

Topological interactions in gauge theories
Some problems + an Introduction

Defects: magnetic versus electric degrees of freedom. Some poorly understood features of topological defects in gauge theories have to do with the interactions they have with fundamental degrees of freedom as well as with each other. In particular, the interplay of topological and non-topological quantum numbers on fluxes and monopoles in the presence of nion abelian unbroken symmetries is worth studying, also in view of the search for truly non-abelian dualities.

Quite a lot has been learned from the study of non-abelian discrete gauge theories [Bais:1980vd]. These theories feature topological (magnetic flux) sectors which are characterized by non-abelian quantum numbers (conjugacy classes to be specific), whereas the allowed dyonic sectors carry electric charges falling into representations of the stabilizer group of the conjugacy class. This implies an intricate interdependence of admissible electric and magnetic quantum numbers in the semi-classical excitation spectrum of the theory. Important was the discovery that (in d=2) this interplay is a manifestation of an underlying quantum group structure, a Hopf algebra, whose representation theory exactly accounts for the structure just mentioned [Bais:1991pe], [DeWildPropitius:1998ck]. These theories are now of interest as models for physical systems which are ideal for topological quantum computing (Kitaev, Presskill a.o.).

For monopoles (in d=3) in phases with residual non-abelian continuous symmetries interesting results have been obtained [Bais:1997qy], [Schroers:1998pg] which show that a similar picture emerges, but the precise mathematical structure has not yet been established. In other words the algebraic and physical structure reflecting the rather well studied geometry of some moduli spaces is still an important topic.

Topological Interactions. Defects in phases with non-abelian residual symmetries, often exhibit nontrivial topological interactions with the other (fundamental and/or topological) degrees of freedom in the theory. These interactions are a consequence of the nontrivial connectivity properties of the solution space of a single defect.

From a physical point of view these topological interactions manifest themselves through the possibility of nontrivial entanglements and consequently of various scattering phenomena which can be described as non-abelian generalizations of the Ahoronov-Bohm effect [Overbosch:2001xb], [DeWildPropitius:1998ck]. Another consequence is the possibility of exotic, non-abelian quantum statistics for collective excitations in certain phases of the theory, as occurring for example in the fractional quantum Hall effect [Slingerland:2001ea]. Also the peculiar situation of general relativity in (2+1) dimensions - which as was shown by Witten, is a topological field theory (A Chern-Simons theory with Poincare or Euclidean group as gauge group) - was studied from this perspective and the quantum symmetry underlying its structure was constructed [Bais:1998yn], [Koornwinder:1998xg]. The Hopf algebra in question turned out to be the quantum double of the group SU(2). Recently we have shown how gravitational scattering of particles, with and without spin, can be treated consistently within this framework [Bais:2002yn].

Broken Hopf-symmetry and confinement. We have emphasized the importance of quantum symmetries, generally speaking (quasi triangular) Hopf-algebra's, which have the important feature that they allow one to treat topological (say magnetic) and ordinary (say color-electric) quantum-numbers on equal footing. These algebra's therefore provide the natural language for d=2 quantum mechanical systems where entanglements etc play a role. Having states labeled in this universal way it is interesting to study the question of duality and conceivable condensation phenomena.

We have recently given a phase classification of certain non-abelian (topological) field theories and the many distinct but allowed types of confinement one may have. These phases are precisely characterized by the the breaking of the Hopf symmetry through a condensate of certain well defined (bosonic) order-parameter fields (electric, magnetic or both) in the theory [Bais:2002zb],[Bais:2002yb]. These findings may be linked to certain duality properties of the Hopf symmetries we have studied [Koornwinder:1999bg]. This phenomenon of the breaking of quantum-symmetries appears to be important and generic in 2-dimensional physics and we should look for physical realisations.

Core instabilities of monopoles. Interesting in the above context are also phases of theories where different types of topological excitations can coexist, in which case the topological interactions between them may lead to rather exotic physical properties. As we showed long ago, topological excitations (monopoles an instantons) in these theories may exhibit core deformations [Bais:1991zp].

A simple but interesting theory of this type is Alice electrodynamics (proposed by A.Schwarz (1982)), a theory whose gauge group is the usual U(1) enhanced with a local Z_2 realization of charge conjugation symmetry. One of the exotic properties of the theory is the emergence of the topological concept of "cheshire charge', a nonlocalizable manifestation of electric and/or magnetic charge. We have shown that this elusive concept for certain parameter ranges in the theory manifests itself through a core instability of magnetic monopoles, where the point defect decays into a ring-shaped object carrying non-localisable magnetic charge [Striet:2002zx].

Charge instabilities in d=2. Alice electrodynamics (AED) is a theory of electrodynamics in which charge conjugation is a local gauge symmetry. We have described a charge instability in (2+1)-dimensions that is due to this gauging of charge conjugation[Striet:2003zy]. The metastability manifests itself through the creation of a pair of alice fluxes. The final state is one in which the charge is completely delocalized, i.e., it is carried as Cheshire charge by a flux pair that gets infinitely separated. We determine the decay rate in terms of the parameters of the model. The relation of this phenomenon with other salient features of 2-dimensional compact QED, could be studied.

Open problems: In the context sketched above, the following problems stand out as rather important/interesting:

A. Monopoles in theories with nonabelian unbroken symmetries have an intricate interplay between magnetic (topological) quantum numbers and electrical (ordinary) quantumnumbers as may be concluded from the structure of their modulispaces in a few known examples. What is the underlying algebraic structure characterizing their quantumnumbers and allowing for example to determine their "fusion rules"?

B. We have proposed a generic mechanism of spontaneous breaking of Hopf (quantum) symmetries and related it to certain confinement phenomena in 2 dimensional physics. What are explicit examples and realisations in condensed matter systems (for example certain hierarchies in quantum Hall physics)?

C. "Charge" instability due to the Cheshire phenomenon in 2 dimensional physics may have some condensed matter analogues. It is an interesting mechanism that gets rid of charge by a completely new type of screening. Are their realistic systems which feature this phenomenon?

D. Are there global (i.e. non-gauged) analogues of the topological effects mentioned above like for example the Ahoronov-Bohm effect?

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A.2 Belinschi, Serban

My main interest is in operator algebras. Derived from here, I have recently been focusing on two different directions: free probability theory and modular tensor categories.

Free probability theory originates in Voiculescu's work on free products of C^*-algebras and type II_1 factors from the late '80s and early '90. It is a noncommutative probability theory in which the concept of independence is replaced by freeness: in a noncommutative probability space (i.e. a unital algebra $\mathcal{A}/\mathbb{C}$ with a liniar functional φ such that $\varphi(1) = 1$) two unital subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are free if $\varphi(a_{i_1}a_{i_2}\cdots a_{i_n}) = 0$ whenever $a_{i_j} \in \mathcal{A}_{i_j}$ such that $\varphi(a_{i_j}) = 0$, $i_j = 1, 2$, and $i_j \neq i_{j+1}$. A number of notions from the classical probability theory have free analogues. My main focus until now was on regularity questions for free convolutions and on free entropy.

My other (more recent) interest is in the classification of modular tensor categories (MTCs). Classification of MTCs of a given rank would give a classification of topological quantum field theories (TQFTs) of the same rank. (the connection of TQFTs with operator algebras can be found, for example, in the work of Ocneanu) I hope that attending this conference will give me the opportunity to better understand both the mathematics involved in the theory of TQFTs and especially their applications.

A.3 Fendley, Paul

I'm interested in 2+1 dimensional theories which are built by defining a quantum Hamiltonian acting on a two-dimensional classical theory. The 2d theory can be either a lattice model or a field theory. The most famous example of such is the Rokhsar-Kivelson model of quantum dimers, where the underlying two-dimensional model consists of classical dimers with equal weights. This model has the remarkable property that the ground-state wavefunction is the equal-amplitude sum over all configurations. Thus computations of correlators in the ground states are exactly computations in the 2d classical theory

One promising direction is to consider field theories directly instead of lattice models. The Rokhsar-Kivelson model is believed to be in the same universality class as the bosonic field theory with Hamiltonian density

$$H = \left(\frac{\partial\phi}{\partial t}\right)^2 + (\nabla^2\phi)^2$$

One finds that the correlators in the ground state here are precisely correlators in theclassical theory of a free boson. This opens up the questions of whether any 2d theory can be "elevated" into such a 2+1d theory. In particular, is there a non-abelian version? There are some partial results on this, but there are many open questions.

For example, a Chern-Simons theory can be related to a 2d conformal field theory in a similar way, a la Witten. However, the Chern-Simons theory breaks time-reversal symmetry, and the corresponding 2d theory is chiral. The theories here are T-symmetric, and the 2d theories are not chiral.

Some open questions are:

1) How generic are these (lattice or field theory) constructions?

2) How well can these models be understood? With a few exceptions, it is even difficult to construct the models, much less understand them in detail.

3) Is there a lattice model of this type with excitations with non-abelian statistics?

A.4 Franko, Jennifer

I am a graduate student of Professor Zhenghan Wang. While I have for the past year been studying the fields of topology and quantum computation, I am quite new to the area of condensed matter physics. Thus, I do not come to this conference with many questions of my own, but rather with a desire to broaden the context in which my interests lie.

A.5 Haghverdi, Esfan

I have been working in the field of categorical logic, models of linear logic (A new logic invented by J.-Y. Girard) and models for Geometry of Interaction (A new semantics invented by J.-Y. Girard). During my research I have come across many interesting categorical models. Currently I am doing research in quantum and topological quantum computing. I am very much interested in fusion and modular categories that seem to play an increasingly important role in topological quantum field theory and consequently in topological quantum computing.

The main problems that I have been working on are the classification problems for fusion and modular tensor categories and the importance of such categorical models in theoretical foundation for topological quantum computing. There are also some important indications that some examples of modular categories can lead to the invention of new quantum algorithms. Moreover going back to my work in logic, I am very keen on finding a correspondence between the physical/computational models and the logical formalisms that I have studied, this is motivated by the similarity of the respective mathematical models.

A.6 Jetchev, Dimitar

I am interested in studying mathematical algorithms related to topological quantum computing models, constructed from the abstract study of anyonic systems.

More precisely, I am studying sequences of certain moves on a triangular lattice (which represents the boundary of a 2-dimensional manifold). By definition, a "configuration" on the lattice is a pair of special dimers (perfect matchings or 1-factors) - a red perfect matching which is always the fixed sublattice of the triangular lattice shown on the picture and a blue perfect matching which differs in finitely many edges from the fixed red perfect matching (i.e. all but finitely many edges are both red and blue). Alternatively, the two perfect matchings can be thought of as forming the boundary of a 2-dimensional manifold.

Given an initial configuration, consider a sequence of three types of moves (together with their inverse moves) on that configuration. Type 1 move is a choice of a rhombus with two opposite blue edges, one of which is also red and a flipping of the two blue edges to the other two edges of the rhombus (consider its inverse move as well). Type 2 move is a choice of a rhombus with two opposite blue edges and a diagonal red edge and a flipping of the blue edges to the other two edges of the rhombus (its inverse move is in fact again a type 2 move). Type 3 move is a choice of a bowtie (an alternating cycle of four red and four blue segments) and a replacement of the bowtie with four simple circles (red-blue edges).

The main hypothesis is that one can go from every initial configuration to any other configuration by the above three types of moves together with their inverses. Another important question is the one about minimizing the number of moves necessary to convert a configuration to the trivial one. Very little progress has been done so far towards these questions which are very important for the topological quantum computing models.

I am also interested in studying simple deformations on the lattice (e.g. a rotated bowtie) and how they affect the ergodicity of the three types of moves. The hypothesis is that the three types of moves, together with a special additional move are still ergodic.

Finally, I am trying to do some computer simulations based on a version of Monte Carlo algorithm to see whether it is possible to reduce arbitrary initial configuration to the trivial one.

A.7 Kitaev, Alexei

I am interested in identifying a mathematical structure that would faithfully describe robust properties of two-dimensional "quantum media". A quantum medium is informally defined as the ground state Ψ of a spin (or similar kind) Hamiltonian on a lattice. We assume that there is a finite energy gap, which conjecturedly implies the existence of a "content Hamiltonian" (the name was suggested by M.Freedman), i.e., Ψ can be charecterized as the joint null state of a set of quasi-local operators. The states Ψ_1 and Ψ_2 are said to be equivalent if they can be transformed one to another by a quasi-local automorphism of the operator algebra; stable equivalence is defined by extending the Hilbert space of each spin. We are interested in characterizing all possible equivalence classes.

Although there is little hope of making this into a rigorous theory, one can use physical arguments or study examples. It appears to be established that a two-dimensional quantum medium is partially characterized by a unitary modular category, which describes the properties of anyons. However, a fuller description is given by a TQFT, which assignes to any four-manifold M the rational number $F(M) = (c/8)\sigma(M)$, where c is a constant, called the "central charge" (it actually corresponds to the difference $c_L - c_R$ for an associated conformal theory). The modular category only defines c modullo 8.

One may wonder if further refinement is possible. For example, one can construct two quantum media with $c = 16$ such that the corresponding *boundary excitation* are described by different conformal theories (defined by non-isomorphic even self-dual lattices in R^{16}). The question is whether these two media are equivalent *in the bulk*.

A.8 Nayak, Chetan

I would like the workshop to feature discussions (a) the various topological phases which are realistic possibilities in correlated electron systems; (b) microscopic models in which might occur; (c) materials in which they might be found; (d) prospects for their detection or construction.

A.9 Sachdev, Subir

I am interested in the classification of ground states of lattice quantum spin system, with an eye to fairly realistic applications to the cuprate superconductors and allied strongly correlated systems. Such systems also exhibit interesting quantum phase transitions, which are so far only poorly understood. Questions related to these issues are:

- What are the different conventional order parameters (those breaking spin-rotation or lattice symmetries of the Hamiltonian) which can characterize ground states in two spatial dimensions, and what microscopic couplings are needed to obtain them ?
- Many states with topological order are known, characterized by discrete gauge theories: is there a general relation between such states and non-collinear spin correlations ?
- Are fractionalized states with excitations carrying $U(1)$ gauge charges possible in two dimensions, apart from isolated critical points ?
- Are there general rules which govern which conventional/topological phases can be generically separated by second order quantum phase transitions ?
- What role do Berry phases play at quantum critical points ?

For more information, please see my web site http://pantheon.yale.edu/∼subir

A.10 Wang, Zhenghan

The abstract properties of anyons are described by topological quantum field theories (TQFTs). In a collaboration with Belinschi, Haghverdi and Stong, I am working on the classification of TQFTs. A unitary TQFT assigns a Hilbert space to each oriented closed surface (with some additional decorations, e.g. a Lagrange subspace in the first Homology). The Hilbert space associated with the torus has an algebra structure, and is called the Verlinder algebra of the TQFT.

The main conjecture is a rigidity property of TQFTs:

If the dimenson of the Verlinder algebara is fixed, there are only finitely many TQFTs.

The classification can be achieved if the dimensions of the Verlinder algebra is 1,2,3,4.

Other questions include the search of topological phases in the extended Hubbard models.

A.11 Wen, Xiao-Gang

How to classify topological/quantom orders

The traditional many-body theory is based on two cornerstones, Landau's Fermi liquid theory and Landau's symmetry breaking theory.[L3726,GL5064] Landau's symmetry breaking theory points out that the reason that different phases are different is because they have different symmetries. A phase transition is simply a transition that changes the symmetry. Landau's symmetry breaking theory describes almost all the known phases, such as solid phases, ferromagnetic and anti-ferromagnetic phases, superfluid phases, etc, and all the phase transitions between them.

Using Landau's symmetry breaking theory, we can understand the origin of the gapless phonon. In Landau's symmetry breaking theory, a phase can have gapless excitations if the ground state of the system has a special property called spontaneous breaking of the continuous symmetry.[N6080,G6154] Gapless phonons exist in a solid because the solid breaks the continuous translation symmetries. Thus we can say that the origin of gapless phonons is the translation symmetry breaking in solids.

It is quite interesting to see that our understanding of a gapless excitation — phonon — is rooted in our understanding of the phases of matter. Knowing light as a massless excitation, one may wonder maybe if light, just like a phonon, is also a Nambu-Goldstone mode from a broken symmetry. However, experiments tell us that a gauge boson, such as light, is really different from a Nambu-Goldstone mode in 3+1 dimensions.

In the late 1970's, we felt that we understood, at least in principle, all the physics about phases and phase transitions based on Landau's symmetry breaking theory. In such a theory, the only way to get gapless excitations is via spontaneous breaking of a continuous symmetry, which will lead to scalar bosonic gapless collective excitations. It seems that there is no way to obtain gapless gauge bosons and gapless fermions from symmetry breaking. This may be the reason why people think our vacuum (with massless gauge bosons and nearly-gapless fermions) is very different from condensed matter systems (which contain only gapless scalar bosonic collective excitations, such as phonons). It seems there does not exist any order that give rise to massless light and massless fermions. Because of this, we put light and fermions into a different category than phonons. We regard them as elementary and introduce them by hand into our theory of nature.

However, if we really believe that light and fermions, just like phonons, exist for a reason, then such a reason must be a certain order in our vacuum that protects the their masslessness. Now the question is what kind of order can give rise to light and fermion, and protect their masslessness. From this point of view, the very existence of light and fermions indicates that our understanding of the states of matter is incomplete. We should deepen and expand our understanding of the states of matter. There should be new states of matter that contain new kinds of orders. The new orders will produce light and fermions, and protect their masslessness.

Our understanding of this new kind of order starts at an unexpected place — fractional quantum Hall (FQH) systems.[TSG8259,L8395] What is really new in FQH states is that we lost the two cornerstones of the traditional many-body theory. FQH systems contain many different phases at zero temperature which have the same symmetry. Thus those phases cannot be distinguished by symmetries and cannot be described by Landau's symmetry breaking theory.

Since FQH states cannot be described by Landau's symmetry breaking theory, it was proposed that FQH states contain a new kind of order — topological order.[Wrig,Wtoprev] Topological order is new because it cannot be described by symmetry breaking, long range correlation, or local order parameters. None of the usual tools that we used to characterize a phase applies to topological order. Despite this, topological order is not an empty concept since it can be characterized by a new set of tools, such as the number of degenerate ground states,[HR8529] quasiparticle statistics,[ASW8422] and edge states.[H8285,Wedgerev]

Question 1: Topological order is closely related to topological field theory. Topological field theory is related to knot theory. To classify all possible topological orders, we ned to classify all topological field theories. But how?

The concept of topological order was recently generalized to quantum order[Wqoslpub] to describe new kinds of orders in gapless quantum states. Topological field theory always has a finit energy gap. So the quantum order is beyond the topological order. It appears that one class of quantum orders arises from a condensation of nets of strings (or simply string-net condensation).[Walight,LWsta,Wqoem] This class of quantum orders shares some similarities with the symmetry breaking orders of "particle" condensation.

We know that different symmetry breaking orders can be classified by symmetry groups. Using group theory, we can classify all the 230 crystal orders in three dimensions. The symmetry also produces and protects gapless Nambu-Goldstone bosons. Similarly, different string-net condensations (and the corresponding quantum orders) can be classified by a mathematical object called projective symmetry group (PSG).[Wqoslpub] Using PSG, we can classify over 100 different 2D spin liquids that all have the same symmetry. Just like the symmetry group, PSG can also produce and protect gapless excitations. However, unlike the particle condensation, string-net condensation produces and protects gapless gauge bosons and fermions.[Wqoslpub,Wlight,WZqoind] Because of this, we can say that light and massless fermions can have a unified origin. They can emerge from string-net condensations. Light is the fluctuation of condensed nets of strings. Fermions are ends of condensed strings.

Question 2: PSG is a very rough characterization of string-net condensations. What is the mathematical structure behind the string-net condensations? (The group theory is the mathematical structure behind the particle condensations and symmetry breaking.) How to classify different string-net condensations?

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