Thompson's Group at 40 Years

Preliminary Problem List

AIM Research Conference Center, January 2004

The names associated with some of the questions are potential contact people for the problem– they may have suggested the problem or expressed interest in it.

1 Sunday Problem Session: Amenability

Moderator: Martin Bridson

Q 1.1 (Matt Brin) Consider conditions related to amenability, either conditions which imply amenability, are equivalent to amenability, or are implied by amenability.

Theorem 1.1 *If a group G is amenable, then all group rings kG where K is any field, have common right multipliers.*

Having common right multipliers means that $\forall x \forall y \exists a, b$ so that $xa = yb$ where $x, y, a, b \in kG - \{0\}$.
It is unknown whether F satisfies this condition. It is open whether amenability is equivalent to this condition of having common right multipliers.
Question: How important is the field k? Might this work with one field but not another?
Can you find a field k so that kF has common right multiples?
An intermediate question: Right Ore Condition: Does $\mathbb{Z}F$ have a ring of right fractions? Orderability implies that $\mathbb{Z}F$ embeds in a division ring.
Are there other conditions related to amenability which are not known for F?

Q 1.2 In general, if G satisfies the right Ore condition for any field k, is G amenable? (In general, not just for F?)
This is related to the amenability of algebras (see a recent Journal of Algebra paper by Elek.) Is the algebra $\mathbb{Z}F$ amenable?
Remark: For F work with a monoid instead of the group. Equivalently, consider kM where M is a positive monoid with respect to some natural generating set, since it is crucial here to talk about right multiples.

Q 1.3 (Victor Guba) We can take linear combinations of elements to some degree. All elements are polynomials, without loss of generality homogeneous polynomials. Let $M_0 \subset M$ be the elements of fixed degree.
Partial results (due to Guba) If you take two linear polynomials

$$x = ax_0 + bx_1 + \cdots$$

$$y = a'x_0 + b'x_1 + \cdots$$

then common right multiples exist. But maybe this is not true if you take nonlinear polynomials. Perhaps this is a way so show nonamenability.
(Slava Grigorchuk) Show that the amenability of F is equivalent to left amenability of the submonoid M.
For purely quadratic terms in kM, can one find common right multiples?
(Matt Brin) Bounded cohomology is trivial for an amenable group. It is known that F satisfies this condition. Are there other related conditions for amenability which are not known for F?
(Slave Grigorchuk) Consider the bounded cohomology condition. It is enough to find a uniformly bounded cation on a Banach space with a homology condition.

Q 1.4 With respect to bounded cohomology, is F of cohomological dimension one or two?
We can even try unitary representations of F since nothing is known about this.
What are suitable coefficient modules?
(Tatiana Smirnova-Nagnibeda) Consider the statistics of F.
There is a percolation criterion for any graph. Let G be a group with S a symmetric set of generators. Let $p \in [0, 1]$ denote the probability of there being an open edge of the Cayley graph between two elements. Do this independently for every edge. Now study the connected subgraphs of this graph. Let $f(p)$ be the probability that there exists a connected infinite subgraph containing the identity.
We can show that $f(p)$ is nondecreasing and look at the phase transitions. There is a critical p_c after which $f(p) > 0$. p_c is an invariant of the graph. If p is close to 1, then you get a unique cluster. If p is close to p_c then you get infinitely many clusters. There exists a critical value p_u for uniqueness, i.e.

$$\exists p_u > p_c \text{ so that } \forall p > p_u \; Prob(\exists! \text{ connected infinite subgraph }) > 0.$$

Theorem 1.2 *If G is amenable, then there is always one infinte cluster; $p_u = p_c$ for any generating set.*

Conjecture 1.3 (Benjamini and Tram) *The reverse implication is true.*

This is open and very hard. The forward implication is proven by Igor Pak and Tatiana.

Theorem 1.4 (Igor Pak and Tatiana Smirnova-Nagnibeda) *If G is nonamenable, and S is any generating set, then $\Gamma(G, S^k)$ has nonunicity of percolation, i.e. p_u is strictly greater than p_c. (Must check exact definition of S^k).*

This is a characterization of nonamenability.
If you could produce two statistically independent clusters then that would imply nonamenability. It is hard to find this critical value for F. Estimate it in terms of the spectral radius, and the isoperimetric constant.

Q 1.5 Can one prove that for some Cayley graph of F (with respect to some finite generating set) there are two infinite clusters, so $p_u > p_c$? This would prove nonamenability. (There are some techniques for this from percolation theory.)

Q 1.6 (Ken Brown) Is the isoperimetric constant for F equal to $\frac{1}{2}$ for all finite generating sets? It is true for $S = \{x_0, x_1\}$. Is $i(F, S) \geq \frac{1}{2}$ for all finite generating sets S for F? The sets used in examples are very natural. Can one do better with different sets?
(Jose Burillo) The isoperimetric constant depends on the generating set. One can find groups with positive isoperimetric constant which approaches 0 as you change (finite) generating set. For example, $BS(2,3)$ has nonuniform amenability, meaning that the infimum over all generating sets S of the isoperimetric constant if 0.
6. Is F non-uniformly amenable?
(Claas Rover) Is it known that there are no paradoxical decompositions of F?
(Kai-Uwe Bux) Have any paradoxical decompositions been tried?
(Jose Burillo) Ask the same question for Folner sets: what types of Folner sets have been tried?
(Slava Grigorchuk) What is the Tarski number for F, i.e. the number of sets in a paradoxical decomposition for F? For the free group F_2 it is 4.
Can one estimate the Tarski number from below? If a group is amenable, then the number is infinite. So show that for F, the Tarski number is not 5, must be at least 6. We know it's not 4 because that is equivalent to the existence of a free subgroup.

Q 1.7 (Victor Guba) Let G be a finitely generated group, with Cayley graph Γ with respect to some finite generating set. Consider a flow on Γ, as follows. For each edge, assign a real number, with the condition that if $w(e) = x$ where e is an edge, then $w(\overline{e}) = x$ as well, where $\overline{e}$) is the same edge with the opposite orientation. The w is a map from the set of edges to $\mathbb{R}$.
Let v be a vertex in Γ. Compute the sum of the weights of the incoming edges, call this $w(v)$. A group G is nonamenable iff there exists a flow and constants $\epsilon, c > 0$ so that $|w(e)| \leq c$ for any edge e, and for any vertex v, we have $|w(v)| \geq \epsilon$.
If you can do this on a large family of finite sets, chosen so that up to left translation, every set can be put inside one of these sets, then the group will be nonamenable.
For F, we can assume these sets live inside of M^{-1}. If you fail to construct these sets in a natural way, then you may get a good family of Folner sets.
There is an algorithm for constructing the maximal flow on finite sets, for integer weights.
By considering triples of trees, one might have a computerized approach towards constructing a maximal flow. (?)
Try to construct a flow on $\Gamma(F, S)$ to either suggest Folner sets or prove nonamenability.

Q 1.8 (Ross Geoghegan) Ballman and Thomas showed

Theorem 1.5 *If G acts properly discountinously on X, a proper $CAT(0)$ spaces by isometries so that the induced topological action on the boundary $\partial_\infty X$ has no fixed point then G is non-amenable.*

and we have

Theorem 1.6 *If G is amenable then every topological action of G on a compact metric space Y has an invariant probability measure.*

Can these be used to approach the amenability of F?

Q 1.9 What is the spectral radius of F?
(Martin Bridson) Can we use the action in the infinite dimensional cube complex? Often this helps to understand amenable subgroups.
(Ross Geoghegan) Consider looking at the compact boundary of the cube complex. There is an action of F on the boundary with no fixed point. Does this imply nonamenability? What about the faithful action on the dyadic solenoid? Nicola Monod produced an action but with a fixed point.

2 Monday Problem Session: Subgroup Structure

Moderator: Bill Harvey

Q 2.1 (Matt Brin) Conjecture: subgroups of F are elementary amenable or contain a copy of F.
Note: It is easy to contain a copy of F, so not containing one is a strong condition. If not, there should be interesesting things to say– any counter-example would be interesting.

Q 2.2 What if we consider $PLF(I)$ instead of F and remove the condition that slopes are powers of 2 but still have finitely many breakpoints– is it true that subgroups are elementary amenable or contain a copy of F?
Start by thinking about this case or even $PLF(\mathbb{R})$- are these questions equivalent for $\mathbb{R}$ and $[0,1]$?
Why would one think this:

Theorem 2.1 (Ubiquity Theorem, Brin) *If $H < PLF(I)$, with $U = (a,b)$ a component of $I - fix(H)$. If there is $f \in H$ moving points in U near a but not near b or vice versa then H contains a subgroup isomorphic to F.*

(That is, if there there is a trivial germ at one end but not at the other)

Q 2.3 What subgroups of F are amenable? Examples- finitely generated wreath products, ...

Q 2.4 What can we say about 2-generator subgroups of F?

Q 2.5 (Mark Sapir) If $H < F$, take all elements represented as spherical(X,X) diagrams. Make a 2-dimensional automaton which represents H- that is, it reads diagrams in H. Consider all possible diagrams read from this automata and consider that the closure $\overline{H}$. In a free group, subgroups are closed and we have $\overline{H} = H$, but in F we may get larger subgroups. Such closed subgroups have no distortion. What are their other properties? Is the membership problem solvable? Example: the derived subgroup is closed, all centralizers of elements are closed. Note that closure does not preserve elementary amenability. Sapir has shown that closed subgroups are diagram groups. What is a measure of a non-closed subgroup's distance from being closed?

In F, there is the non-associative operation of addition from $F \times F$ to F. Does this induce an operation on homology?
Look at substructures– F is the local group of a groupoid and closure can be described in terms of this structure.

Q 2.6 (Slava Grigorchuck) What are the maximal subgroups of F of infinite index?

Q 2.7 Does an elementary amenable but not solvable subgroup of F contain a copy of a Brin-like subgroup ($\mathbb{Z}$ wr $\mathbb{Z}$ wr $\mathbb{Z}$ wr $\ldots$)

Q 2.8 (Ross Geoghegan) Does F contain finitely presented subgroups not of type FP_3?
Remark: Bieri-Neumann-Strebel implies there are normal subgroups which are finitely generated but not finitely presented. Can this be extended with some kind of fibre product to get subgroups which have appropriate finiteness properties?

Q 2.9 (Matt Brin) $V_{n,r}$ is the group of homeomorphisms of r copies of the Cantor set with slopes which are powers of n. Higman partially classified the isomorphism type- what is the full classification?

Q 2.10 (Jim Cannon) What are the finite generating sets of F? Is there an algorithm for deciding if a given set of elements generates F? Are there better finite/infinite generating sets than the ones commonly in use? What about two generator sets? Is there a classification of two-element generating sets of F?

Q 2.11 (Mark Sapir) Solve the membership problem for F.
Related questions: Are there subgroups of F with exponential distortion?
All known subgroups have at most polynomial distortion. What distortions can occur?

3 Tuesday Problem Session: Connections with other areas

Moderator: Ross Geoghegan
Areas with ties to Thompson's groups:

1. Infinite dimensional Lie Algebras
2. Jonsson-Tarski
3. Kac-Moody algebras
4. p-adic
5. quasi-symmetric
6. dynamics

7. mapping class groups
8. ergodic theory
9. computation
10. Four color theorem
11. topology
12. boundaries
13. string theory
14. mirror symmetry

Q 3.1 (Matt Brin) Can F be used to shorten the proof of the four color theorem? Reference: paper on the four color theorem in the Notices by Robin, which refers to a paper by Louis Kaufman.
The four color theorem starts with a trivalent graph which can be rearranged to look like an element of F. The four colorability of the regions is equivalent to the three colorability of the graph. We can show that the three colorability of any representative of $w \in F$ is equivalent to the three colorability of any other representative of w. So the four color theorem is true if every $w \in F$ can be three colored. Is the set of three colorable elements of F closed under multiplication?

Q 3.2 (Vlad Sergiescu) Give a satisfying explanation for the fact that

$$H^*(T, \mathbb{R}) \cong H^*_{cont}(\mathrm{Diff}^+(S^1), \mathbb{R})$$

There is no known map, although they are known to be isomorphic as rings. The Gobillion-Vey invariant does not pull back.

Q 3.3 (Fred Gardiner) Can you make a connection between the "universal idempotent up to conjugacy" and the renormalization equation equation for the Feigenbaum fixed point (of the renormalization operator).
Namely, is this a renormalization equation: $egg = ge$ where the operation is composition, and $e : \mathbb{R} \to \mathbb{R}$ is given by $e(x) = \lambda z$ and $g : [0, 1] \to [0, 1]$ is given by $g(x) = \lambda x(1 - x)$.
(Kai-Uwe Bux) What is the semigroup generated by e and g?

Q 3.4 (Patrick deHornoy) Consider F in terms of generators which allow a rotation at every point on the tree. Can you compute geodesics with respect to this generating set? The Cayley graph with respect to this generating set is not locally finite because of the infinite generating set.
Computing rotation distance between elements of F should be equivalent to finding the distance between two triangulations with the same number of triangles. Distance between triangulations is computed by counting the number of elementary transformations needed, i.e. just changing the diagonal in a square.
The theorem of Sleator-Tarjan-Thurston is an asymptotic result, and finds $2n - 6$ as a sharp upper bound, for large enough n, on the rotation distance between two trees with n carets each. Is there an elementary combinatorial proof?

Q 3.5 (Susan Hermiller) Is there a finite complete rewriting system for F? Could one prove that one does not exist using the generators x_0 and x_1? Perhaps there might be one for a larger but finite generating set.
(Patrick deHornoy) Try using the above family of generators, which allows rotations at all nodes of the tree, combined with an analogue of Guba's "square" for rewriting. Might need to do it twice for rewriting system.

Q 3.6 (Christophe Kapoudjian) What p-adic analogues of Thompson's groups could we construct?
Let N be the Neretin spheromorphism group which is analogous to $\mathrm{Diff}^+(S^1)$. Is N a possible analogue of Thompson's groups?
What should we expect to be a satisfactory p-adic analogue of Thompson's groups?
N is satisfactory from a cohomological point of view.
(Slava Grigorchuk) Is there a finitely generated and finitely presented pro-p group which imitates Thompson's groups, with no free subgroups, non-elementary, and with a balanced presenatation?
N is generated by two subgroups: V and the full automorphism group of the tree. What replaces finiteness properties in a p-adic world?

Q 3.7 (Linda Keen) Study Fibonacci dynamical systems. What does the group generated have in common with F? Is F a subgroup of it? Is it amenable?
In general, Cantor repellers give rise to mapping class spaces. What is the Teichmuller group, and how does it relate to F? Does it contain F? Does it contain the other Thompson groups? Is this group related to Sean Cleary's groups with irrational slopes?

Q 3.8 (Tatiana Smirnova-Nagnibeda) Describe the Poisson boundary of the symmetric random walks on F. A recent proof shows this is nontrivial. (if trivial, then the group is amenable.)
(Slava Grigorchuk) Describe the Martin boundary of this space. A nontrivial Poisson boundary implies there exists nonconstant bounded harmonic functions on the group.

Q 3.9 (Slava Grigorchuk) Prove a von Neumann statistical ergodic theorem for F. Also, prove a Birkhoff individual ergodic theorem for actions of F, when the action is by measure preserving transformations on measure spaces.

Q 3.10 (Slava Grigorchuk) If $Homeo(S^1)$ is nonamenable, then F is nonamenable.
(Fred Gardiner) If the group of quasi-symmetric maps of S^1 is nonamenable, does that imply that F is nonamenable?

Q 3.11 (Claas Rover) What is the language complexity (context sensitive) of the word problem of all Thompson's groups? Can you find a nicer class under context sensitive?

Q 3.12 (Ross Geoghegan) Is there an interplay known between Thomspon's groups and Kac-Moody?

Q 3.13 Thompson's groups are automorphism groups of the Jonsson-Tarski algebra, i.e. the algebra formed from the entire binary tree. V is the full automorphism group, T is the order preserving automorphism group, and F is the strict order preserving automorphism group. This setup should be analogous to automorphisms of vector spaces, i.e. linear groups. Are there analogies between Thompson's groups and arithmetic groups, or virtual duality groups? Are Thompson's groups like "infinite duality groups"? What is the significance of the Jonsson-Tarski algebra?
Reference suggested by R. Thompson: *Set theory without variables* by Tarski and Givant.

Q 3.14 (Slava Grigorchuk) Describe all reasonable topologies on Thompson's groups.

Q 3.15 (Martin Bridson) Is there some deep duality behind Cannon and Woodruff's construction of the number of tree pair diagrams, or are they numerical coincidences?

4 Tuesday Problem Session: Everything else

Moderator: José Burillo

Q 4.1 (Matt Brin) What are the automorphisms of Thompson's groups? This is known for F and T. What about for $F(n)$, and other generalizations of F? Not even all outer automorphisms are known for these groups.

Q 4.2 Is F automatic? Synchronously or asynchronously combable? What languages are associated with F? Guba and Sapir give an example of a nongeodesic language.
(Victor Guba) Prove that an automatic group cannot contain $\mathbb{Z}^\infty$.
(Martin Bridson) There are no known examples of torsion free infinite dimensional automatic groups, nor for combable groups.

Q 4.3 (Martin Bridson) What about higher dimensional isoperimetric inequalities for Thompson's groups? Automatic groups satisfy Euclidean isoperimetric inequalities in every dimension. Try to prove that F has higher dimensional isoperimetric inequalities which are not Euclidean.
(Victor Guba) What is the annular isoperimetric function for F? Is there any relation of this to automaticity?
(José Burillo) Can we compute other filling invariants for F?

Q 4.4 (Ross Geoghegan) Are all finitely generated diagram groups torsion free of type FP_∞? Are all diagram groups of type FP_∞ Thompson-like? (philosophically) For example, consider groups of homeomorphisms of the line - are there genuinely different examples?
(Mark Sapir) Starting with F you can manufacture different examples - very different but based on F.
(Victor Guba) All known FP_∞ groups are diagram groups. Are there others?

Q 4.5 (Victor Guba) Is it true that for any n, F has subgroups of index n isomorphic to F? (probably yes)
(Ken Brown) Are there finite index subgroups of F which are not isomorphic to F? Cannon has an example which is a candidate, but no proof.
What finitely presented groups (maybe finitely generated) have this property?

Q 4.6 Is the Cayley complex of F semi-stable?

Q 4.7 (Victor Guba) What are the simple subgroups of F? For example, F', and $F(p)'$ for all embeddings of $F(p) \hookrightarrow F$. Are there any others? The motivation for this is a result of Guba and Sapir which says that every simple diagram group is inside F.

Q 4.8 (Jen Taback) What is the quasi-isometry group of F? Are F(2) and F(3) quasi-isometric?

Q 4.9 (Martin Bridson) Are F and $F \times \mathbb{Z}$ quasi-isometric?

Q 4.10 (Ross Geoghegan) One possible solution might be to look at invariants at infinity. Build a $K(\pi, 1)$ and look at the homology, or the fundamental group at infinity. Perhaps one of these could distinguish between F and $F \times Z$. Could K-theory distinguish these groups? Taylor's example worked for something similar in the 1970's.
Related question: Is F quasi-isometric to $F \times F$?

Q 4.11 (Mark Sapir) Show that the asymptotic cone of F is not a direct product. It is known that the asymptotic cone is infinite dimensional, and simply connected.

Q 4.12 (Vlad Sergiescu) Are other versions of the Novikov Conjecture interesting for F? People think that F should satisfy the integral Novikov conjecture, but a proof is not known.

Q 4.13 (Dan Farley) Does F have property A? Property A is a weak form of amenability which implies uniform embeddability in a Hilbert space. (Guoliang Yu defined property A.)

Q 4.14 (Matt Brin) Is $PIP(\Delta_2)$ finitely generated? finitely presented?

Q 4.15 (Claas Rover) Does F arise in the abstract commensurator group of other groups (besides branch groups), as it does for Grigorchuk's group? Claas has some examples.

Q 4.16 (Mark Sapir) Develop algorithms to solve equations over F, especially quadratic equations. Even elementary theories are not known. Is the elementary theory decidable or not?
(Slava Grigorchuk) Describe the set of solutions to an equation over F.
Miklos Abert has a criteria to show that F does not have identities. (This will be true for any branch group.) If the point stabilizers are all different, then there will be no identities. This was proved by Brin and Squier as well.

Q 4.17 (José Burillo) What is the growth function for F? What about co-growth? That would decide amenability as well.

Q 4.18 (Ross Geoghegan) Related to yesterday's problem by Slave Grigorchuk to describe all topologies on F. Classify all (locally compact) groups in which Thompson's groups embed as dense subgroups.

Q 4.19 (Victor Guba) What is the representation variety of F as a group of homeomorphisms of $\mathbb{R}$?

Q 4.20 (Victor Guba) How do you decide algorithmically if $w \in F$ is a commutator? Every element of F is a product of two commutators. What is an example of an element which is not a commutator, so $x, y \in F'$ but $xy \neq F'$?

Q 4.21 (Jen Taback) Are there analogues of Fordham's methods for V and T? Of Guba and Sapir's diagram groups methods for determining word length in F?

Q 4.22 (Ross Geoghegan) If F acts freely and properly discontinuously on a proper CAT(0) space, is there a fixed point in the boundary? By boundary, we mean the cone topology boundary. A theorem of Ballman and Adams says that a negative answer implies that F is not amenable.

Q 4.23 What is the cogrowth of F?

Q 4.24 (Vlad Sergiescu) Is there a generalization of Thompson's groups acting on spaces with boundary the Serpinski or Menger curves? So can the Cantor set be replaced by these spaces to produce Thompson-like groups? Gilbert Levitt studies homeomorphisms of Menger curves, and may have a result in this direction.

Q 4.25 (Bill Harvey) Is there an analog of Moonshine for F? F seems to have a strange connection with modular forms (look at dimensions of representations). Could one find an interesting set of numbers related to F which yielded a power series and a modular form?

5 Supplementary Problems

Q 5.1 (Matt Brin) What is the relation between F or $Aut(F)$ and arbitrary finite-state automata regarded as transducers?

Q 5.2 (Matt Brin) To what extent is the F construction universal in embedding infinitary algebraic structures in finitary structures? Or: classify the techniques to embed infinitary algebraic structures in finitary structures.

Q 5.3 (Matt Brin) Explore the group ring structure of the Thompson groups and their various submonoids

Q 5.4 (Matt Brin) What relation exists between Thompson groups and constructions in K-theory, particularly those of Waldhausen. Two aspects: details of Waldhausen's constructions and viewing K-theory as studying structures arising from two levels of equivalence.

Q 5.5 (Matt Brin) What relation is there between Thompson's groups and aspects of self-similarity? F acts more naturally on the self-similar "ruler space" than it does on $\mathbb{R}$.

Q 5.6 (Matt Brin) What relations are there between Thompson's groups and wavelets, which use invariants under shift and dilation?

Q 5.7 (Matt Brin) What are the proper homotopy properties of the ends of Thompson groups? For example, what about semi-stability at infinity?