AIM Workshop on Representations of Surface Groups (March 19-23, 2007):

Questions and open problems

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1 Representations into $\mathrm{Sp}(4,\mathbb{R})$

Question 1.1 (Bill Goldman)**.** Can every maximal representation in $\mathrm{Sp}(4,\mathbb{R})$ be deformed to a proper Zariski closed subgroup?

Comment 1.2 (Bill Goldman). Every maximal representation of the unitary group can be deformed in this way.

Comment 1.3 (Anna Wienhard). The problem should be easier for $\mathrm{Sp}(2n,\mathbb{R})$ with $n > 2$ (it is still not known though). For $n = 2$ progress was made at the conference on many components, however there exist $2(g-2)$ ″exotic″ components for which nothing is currently known.

2 Geometry of the Hitchin component

Question 2.1 (John Loftin)**.** Does there exist a mapping class group-invariant Kähler structure on the Hitchin component for $\mathrm{SL}(3,\mathbb{R})$?

Comment 2.2 (Richard Wentworth). Theorem 1.0.2 of [Lab06] shows that the Hitchin component for $\mathrm{SL}(3,\mathbb{R})$ is given by the bundle of cubic differentials on Teichmüller space. There is a mapping class group-invariant *complex* structure on this space.

Comment 2.3 (John Loftin). There is evidence for a Kähler structure, since transverse to the fibres there is a Kähler metric (Weil-Petersson), and on the fibres there is a Kähler metric.

Question 2.4 (Richard Wentworth, Francois Labourie)**.** Let S be a closed surface, $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathbb{R})$ a representation in the Hitchin component. Given a complex structure J on S there exists a unique ρ-equivariant harmonic map $u : (\tilde{S}, J) \to X = \mathrm{SL}(n, \mathbb{R})/\,\mathrm{SO}(n)$ (Corollary 3.5 of [Cor88]). Is there a unique complex structure J such that u is also conformal?

Comment 2.5 (Richard Wentworth). This is true for $n = 2, 3$ (see Theorem 9.3.2 of [Lab06]).

Comment 2.6 (Anna Wienhard). One can ask the same question for maximal representations.

3 Surface Bundles

Question 3.1 (Dieter Kotschick)**.** Fix a closed Riemann surface B of genus $g \geq 3$, and fix $h \geq 2$. There exist at most finitely many non-isotrivial holomorphic genus h fibrations over B, $F_h \to X \to B$.

This gives a conjugacy class of representations $\rho : \pi_1(B) \to \mathrm{MCG}(F_h)$. How to characterise the representations which are the holonomy of a holomorphic fibration?

Comment 3.2 (Dieter Kotschick). When the fibration is holomorphic there is a Kähler structure on the surface bundle X, and so the cohomology of X satisfies certain constraints from Hodge theory (e.g. $h^1(X)$ is even). These constraints give some restrictions on the representations, but they are not enough to give an "if and only if" statement.

Question 3.3 (Dieter Kotschick)**.** Fix g, h with $h \geq 2$. Does there exist a locally trivial fibration of surfaces $F_h \to X \to B_g$ such that X admits a metric of strictly negative curvature?

Comment 3.4 (Dieter Kotschick). A necessary condition is that the monodromy is pseudo-Anosov. If the curvature is also constant then the signature must be zero and so the Toledo invariant vanishes.

4 Invariants of representations

Question 4.1 (Marc Burger)**.** Let M be a compact manifold foliated by surfaces, with transverse measure μ. Denote the foliation by $\mathcal{F}$. Let X be a Hermitian symmetric space, $G = \mathrm{Isom}(X)^0$ and $\rho : \pi_1(M) \to G$. Given a ρ-equivariant map $f : \tilde{M} \to X$ we have $f^*\omega \in H^2(\mathcal{F})$, and let C be a Ruelle-Sullivan cycle. Find bounds on $\langle f^*\omega, C\rangle \in \mathbb{R}$ and study the maximal representations.

Question 4.2 (Oscar Garcia-Prada)**.** Consider a real group G with a symmetric space $X = G/K$ of quaternion-Kähler type. Let Ω be a 4-form on X, M a 4-manifold and $\rho : \pi_1(M) \to G$. Study the Toledo invariant $\langle f^*\Omega, [M]\rangle$. An interesting special case is when M is a Kähler surface.

Comment 4.3 (Dieter Kotschick). It might be interesting to study the case where M is aspherical.

Comment 4.4 (Domingo Toledo). Other interesting special cases are: When M is a complex hyperbolic surface, when M has constant curvature and when M is the product of two surfaces.

Question 4.5 (Marc Burger)**.** Let k be a field and V a symplectic vector space over k. There exists a central extension

$$1 \to W(k) \to \widetilde{\mathrm{Sp}(V)} \to \mathrm{Sp}(V) \to 1$$

where $W(k)$ is the Witt group. Given a representation $\rho : \pi_1(S) \to \mathrm{Sp}(V)$, the Toledo invariant $\tau(\rho)$ is an element of $W(k)$. Let A be an algebraic set and consider a map $u : A \to \mathrm{Rep}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. To this map associate the tautological representation $\rho : \pi_1(S) \to \mathrm{Sp}(2n, k(A))$, and define the invariant $\tau(\rho) \in W(k(A))$. Study this.

Comment 4.6 (Marc Burger). When A is a point this is the classical Toledo invariant.

Question 4.7 (Olivier Guichard)**.** Following on from the previous question, there is also a central extension for $\mathrm{SL}(n)$

$$1 \to K_2(A) \to E(n, A) \to \mathrm{SL}(n, A) \to 1$$

Let H_n be the Hitchin component, and let $A = \mathbb{Q}(H_n)$. Given a representation $\rho : \pi_1(S) \to \mathrm{SL}(n, A)$ we obtain $q \in K_2(A)$. There is a map $d\log : K_2(A) \to \Omega^2(H_n)$. Is $d\log(q)$ the Weil-Petersson form?

5 Other questions

Question 5.1 (Oscar Garcia-Prada)**.** The cyclic group of order n acts on the Hitchin component of $\mathrm{Rep}(\pi_1(S), \mathrm{SL}(n, \mathbb{R}))$ (multiply the Higgs field by roots of unity). Call the subspace of $\mathbb{Z}/n\mathbb{Z}$-invariant Higgs fields "cyclotomic Higgs fields", which are in one-to-one correspondence with $H^0(S, K^n)$. Is this space independent of the complex structure on S?

Question 5.2 (Anna Wienhard)**.** Do there exist embeddings $\mathrm{SL}(2, \mathbb{R}) \hookrightarrow G$ such that the component in $\mathrm{Rep}(\pi_1(S), G)$ containing the representation

$$\begin{tikzcd}
\pi_1(S) \arrow[r] \arrow[dr, "Fuchsian"'] & G \\
& \mathrm{SL}(2, \mathbb{R}) \arrow[u, "\varphi"]
\end{tikzcd}$$

consists entirely of discrete and faithful representations?

Comment 5.3 (Anna Wienhard). For G being a split real form and $\mathrm{SL}(2, \mathbb{R}) \to G$ an irreducible representation this gives the Hitchin component.

For G being of Hermitian type and $\mathrm{SL}(2, \mathbb{R}) \to G$ a tight representation, then this gives one connected component in the space of maximal representations.

Question 5.4 (Marc Burger)**.** Count rational representations whose image lies in the set of integer points of the symplectic group.

Question 5.5 (Bill Goldman)**.** Let S be a closed orientable surface with $\chi(S) < 0$, and G a semisimple Lie group. Define $E_\rho : \mathcal{T}_S \to \mathbb{R}$ to be the energy function associated to a representation $\rho : \pi_1(S) \to G$ (see [GW05] for more details). What are the most general conditions on ρ for which E_ρ is proper?

Comment 5.6 (Bill Goldman). If G is split and ρ is a Hitchin representation or maximal symplectic representation then E_ρ is proper (see [Lab05]).

If ρ is convex co-compact then E_ρ is proper (see [GW05]).

References

[Cor88] Kevin Corlette, *Flat G-bundles with canonical metrics*, J. Diff. Geom 28 (1988), 361–382.

[FG06] Vladimir Fock and Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. (2006), 1–211.

[GW05] William M. Goldman and Richard A. Wentworth, *Energy of twisted harmonic maps of Riemann surfaces*, math.DG/0506212.

[Lab05] Francois Labourie, *Cross ratios, Anosov representations and the energy functional on Teichmüller space*, math.DG/0512070.

[Lab06] Francois Labourie, *Flat Projective Structures on Surfaces and Cubic Holomorphic Differentials*, math.DG/0611250.