SINGULARITIES IN BIRATIONAL GEOMETRY: MINIMAL LOG DISCREPANCIES AND THE LOG CANONICAL THRESHOLD

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These are introductory notes written for the AIM workshop on "Numerical invariants of singularities and of higher-dimensional algebraic varieties". They are by no means exhaustive. The main goal is to present the definition of two fundamental invariants of singularities that appear in birational geometry, state some important open questions about these invariants, and provide some references to the relevant literature. We do not aim for the utmost generality in the definitions and conjectures: for these the reader should go to the original papers.

1. Definitions

Let X be a complex n-dimensional algebraic variety. We assume that X is normal and $\mathbb{Q}$-Gorenstein. The first condition implies that there is a unique (up to linear equivalence) Weil divisor K_X on X whose restriction to the smooth locus $X^{\rm reg}$ of X satisfies $\mathcal{O}(K_X|_{X^{\rm reg}}) \simeq \Omega^n_{X^{\rm reg}}$. The second condition says that K_X is $\mathbb{Q}$-Cartier, i.e. that there is a positive integer r such that rK_X is a Cartier divisor. Note that since K_X is $\mathbb{Q}$-Cartier, for every dominant morphism $\pi\colon X' \to X$ we can define the pull-back $\pi^*(K_X)$: this is $\frac{1}{r}\pi^*(rK_X)$ where we take r as above.

If $\pi\colon X' \to X$ is a birational morphism and X' is normal, there is a unique $\mathbb{Q}$-divisor $K_{X'/X}$ on X' with the following properties:

1) $K_{X'/X}$ is $\mathbb{Q}$-linearly equivalent with $K_{X'} - \pi^*(K_X)$.
2) $K_{X'/X}$ is supported on the exceptional locus of π (that is, for every prime divisor E that appears in $K_{X'/X}$ we have $\dim(\pi(E)) < n-1$).

For example, if both X and X' are smooth, then $K_{X'/X}$ is the effective divisor locally defined by the Jacobian of π.

In order to define invariants, we will consider various *divisors over* X: these are prime divisors $E \subset X'$, where $\pi\colon X' \to X$ is a birational morphism and X' is normal. Every such divisor E gives a discrete valuation ${\rm ord}_E$ of the function field $K(X') = K(X)$, corresponding to the DVR $\mathcal{O}_{X',E}$. We will identify two divisors over X if they give the same valuation of $K(X)$. In particular, we may always assume that X' and E are both smooth. The *center* of E is the closure of $\pi(E)$ in X and denoted by $c_X(E)$.

We will consider pairs (X, Y), where Y stands for a formal sum $\sum_{i=1}^r q_iY_i$, with q_i real numbers and Y_i closed subschemes of X. We will say that Y is effective if all q_i are nonnegative. An important special case is that of an $\mathbb{R}$-Cartier divisor, i.e. when all Y_i are effective Cartier divisors on X.

Let E be a divisor over X. If Z is a closed subscheme of X, then we define $\mathrm{ord}_E(Z)$ as follows: we may assume that E is a divisor on X' and that the scheme-theoretic inverse image $\pi^{-1}(Z)$ is a divisor. Then $\mathrm{ord}_E(Z)$ is the coefficient of E in $\pi^{-1}(Z)$. If (X, Y) is a pair as above, then we put $\mathrm{ord}_E(Y) := \sum_i q_i \,\mathrm{ord}_E(Y_i)$. We also define $\mathrm{ord}_E(K_{-/X})$ as the coefficient of E in $K_{X'/X}$. Note that both $\mathrm{ord}_E(Y)$ and $\mathrm{ord}_E(K_{-/X})$ do not depend on the particular X' we have chosen.

Suppose now that (X, Y) is a pair as above and that E is a divisor over X. The *log discrepancy* of (X, Y) with respect to E is

$$a(E; X, Y) := \mathrm{ord}_E(K_{-/X}) - \mathrm{ord}_E(Y) + 1.$$

In order to define invariants that do not depend on the choice of a divisor, one takes minima over all divisors E with restriction on their centers. For example, if W is a closed subset of X, then the *minimal log discrepancy* of (X, Y) along W is

$$\mathrm{mld}(W; X, Y) := \min\{a(E; X, Y) \mid c_X(E) \subseteq W\}.$$

There are also other versions of minimal log discrepancies, but the study and properties of all these variants are more or less equivalent. For example, if W is irreducible and η_W denotes the generic point of W, then

$$\mathrm{mld}(\eta_W; X, Y) := \min\{a(E; X, Y) \mid c_X(E) = W\}.$$

The pair (X, Y) is called *log canonical* if $\mathrm{mld}(X; X, Y) \geq 0$ and *log terminal* if $\mathrm{mld}(X, X, Y) > 0$. One can show that if (X, Y) is not log canonical and $\dim(X) \geq 2$, then $\mathrm{mld}(X; X, Y) = -\infty$. Other classes of singularities of pairs, such as *canonical* or *terminal* can also be defined in terms of minimal log discrepancies.

In general, the singularities of (X, Y) are "good" if the minimal log discrepancies are large. For example, if X is smooth and if W is a smooth subvariety of codimension r, then $\mathrm{mld}(W; X, \emptyset) = r$.

Another invariant of singularities that can be defined in terms of log discrepancies is the *log canonical threshold*. Suppose that (X, Y) is a pair as above, with Y effective and non-empty, and such that X is log terminal (i.e. the pair $(X, \emptyset)$ is log terminal). The log canonical threshold of (X, Y) is

$$\mathrm{lc}(X, Y) := \sup\{t > 0 \mid (X, tY)\,\text{is log canonical}\} = \inf_E \frac{1 + \mathrm{ord}_E(K_{-/X})}{\mathrm{ord}_E(Y)}.$$

For example, if X is nonsingular and Y is a nonsingular subvariety of codimension r, then $\mathrm{lc}(X, Y) = r$. For a more interesting example, take H the hypersurface in $\mathbb{C}^n$ defined by $x_1^{a_1} + \ldots + x_n^{a_n} = 0$, for which $\mathrm{lc}(\mathbb{C}^n, H) = \min\{1, \sum_{i=1}^n \frac{1}{a_i}\}$. In general, larger log canonical thresholds correspond to "better" singularities.

We mention that one can define minimal log discrepancies and the log canonical threshold in a slightly different set-up: X is not necessarily $\mathbb{Q}$-Gorenstein, but Y is an $\mathbb{R}$-divisor such that $K_X + Y$ is $\mathbb{R}$-Cartier.

A basic fact about the log canonical threshold and minimal log discrepancies is that they can be computed using a log resolution of singularities. Indeed, suppose that $\pi\colon X' \to X$ is a proper, birational morphism such that X' is smooth, the union between the exceptional locus of π and the $\pi^{-1}(Y_i)$ (and $\pi^{-1}(W)$ if we are considering $\mathrm{mld}(W; X, Y)$) is a divisor with simple normal crossings on X'. Write

$$K_{X'/X} = \sum_i k_i E_i, \pi^{-1}(Y) = \sum_i a_i E_i.$$

With this notation we have

$$\mathrm{lc}(X, Y) := \min \frac{k_i + 1}{a_i}$$

(note that $k_i + 1 > 0$ for every i since X is log terminal). Similarly, we have

$$\mathrm{mld}(W; X, Y) := \min\{1 + k_i - a_i \mid c_X(E_i) \subseteq W\}$$

if $1 + k_i - a_i \geq 0$ for all i such that $c_X(E_i) \cap W \neq \emptyset$ (otherwise, if $\dim(X) \geq 2$ we have $\mathrm{mld}(W; X, Y) = -\infty$). In particular, these formulas show that if Y has rational coefficients, then the above invariants are rational.

Despite the similar definitions, the minimal log discrepancy is a much more subtle invariant than the log canonical threshold. At least in the case when X is smooth, the log canonical threshold has been related to various other points of view on singularities: to integrals on vanishing cycles, roots of the Bernstein-Sato polynomial, integrability conditions of various kinds (Lebesgue, p-adic or motivic), invariants in positive characteristic and so on. On the other hand, not much is known about minimal log discrepancies. However, they seem more important from the point of view of birational geometry: certain conjectures on their behavior (see the next section) are related to the Termination Conjecture in the Minimal Model Program.

2. Conjectures

We start with a conjecture describing a semicontinuity property of minimal log discrepancies along various points in X.

Conjecture 2.1 (Ambro)**.** Given a pair (X, Y), with Y effective, the function $x \to \mathrm{mld}(x; X, Y)$ is lower semicontinuous.

If x is a smooth point of X, then $\mathrm{mld}(x; X, \emptyset) = \dim(X)$, hence a positive answer to the above conjecture would imply the following boundedness conjecture.

Conjecture 2.2 (Shokurov)**.** If (X, Y) is a pair with Y effective, then for every x in X we have $\mathrm{mld}(x; X, Y) \leq \dim(X)$.

The following conjecture is known as Inversion of Adjunction. It relates minimal log discrepancies on a variety with those on a hypersurface, and therefore provides a useful tool for induction on dimension.

Conjecture 2.3 (Shokurov and Kollár)**.** Consider a pair (X,Y) with Y effective and let $H\subset X$ be a normal effective Cartier divisor on X that is not contained in the support of Y. If W is a proper closed subset of H, then

$$\mathrm{mld}(W;H,Y|_H)=\mathrm{mld}(W;X,Y+H).$$

Arguably the hardest conjectures on these invariants involve the so-called ACC property. One says that a set or real numbers has the *ascending chains property* (ACC, for short) if it contains no infinite strictly increasing sequence. One similarly defines the DCC property (the *descending chains property*).

Conjecture 2.4 (Shokurov)**.** Let us fix a positive integer n and a set $\Gamma\subset\mathbb{R}_+$ having the DCC property. Consider the set of minimal log discrepancies $\mathrm{mld}(W;X,Y)$, where $\dim(X)=n$ and Y is an $\mathbb{R}$-Cartier divisor such that when we write $Y=\sum_i b_iD_i$ with D_i prime divisors, all b_i are in Γ. This set has ACC.

There is a similar conjecture for log canonical thresholds.

Conjecture 2.5 (Shokurov)**.** For every positive integer n, the set

$$\mathcal{T}_n:=\{\mathrm{lc}(X,Y)\,|\,\dim(X)=n,\ X\,\text{log terminal},\ Y\,\text{effective Cartier divisor}\}$$

has ACC.

3. Some references

We review here a few basic references for the basics on minimal log discrepancies and log canonical thresholds. We also mention what is known about the conjectures listed in the previous section.

A basic reference for singularities of pairs and their applications in birational geometry is [Kol]. In particular, §8 contains the definition and some fundamental properties of the log canonical threshold. In §9 one relates the log canonical threshold with the the complex singularity exponent, an invariant introduced by Arnold in terms of the asymptotic behavior of certain integrals over vanishing cycles. In §10 one proves that if f is a polynomial, then $-\mathrm{lc}(\mathbb{C}^n,V(f))$ is the largest root of the Bernstein-Sato polynomial of f, an invariant that comes out of D-module theory.

An introduction to minimal log discrepancies and their basic properties can be found in [Am].

A few words about the conjectures stated in the previous section. Conjecture 2.1 on the semicontinuity of minimal log discrepancies was stated in [Am]. It is proved there that the conjecture holds in dimension at most three, or in the toric setting.

The Inversion of Adjunction Conjecture is discussed in [Kol], §7. A more thorough discussion of the conjecture and of its connections with the Minimal Model Program appears in §17 of [K+]. We remark that the inequality $\mathrm{mld}(W;X,Y+H)\leq \mathrm{mld}(W;H,Y|_H)$ in Conjecture 2.3 is elementary and can be found in [Kol]. Moreover, it is shown there using vanishing theorems that one of the two minimal log discrepancies in the statement is positive if and only if the other one is (this is the log terminal case of the conjecture). A more recent result in [Kaw] says that one of the minimal log discrepancies is nonnegative if and only if the other one is (this is the log canonical case of the conjecture). We mention that these results hold in a more general framework than ours, in particular H is not assumed to be Cartier.

A description of minimal log discrepancies in terms of spaces of arcs was given in [EMY]. Using this description, Conjectures 2.1 and 2.3 were proved when the ambient variety is locally a complete intersection in [EMY] and [EM].

The main motivation for Conjectures 2.4 and 2.5 is that they are connected with the Termination Conjecture for log flips in the Minimal Model Program. For the precise connection between the Semicontinuity Conjecture 2.1 and the ACC Conjecture 2.4 with the Termination Conjecture see [Sho]. More recently, it was shown in [Bir] that termination in the case of nonnegative Kodaira dimension follows from the ACC conjecture for log canonical thresholds and a conjecture of Alexeev and Borisov on boundedness of log Fano varieties.

Not much is known about the ACC conjectures. The fact that they hold in dimension two follows from [Al]. A different proof of Conjecture 2.5 in dimension two was given in [FJ] using the valuation tree.

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