INVARIANTS OF SINGULARITIES VIA INTEGRATION

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This is a note written for the AIM workshop on "Numerical invariants of singularities and of higher-dimensional algebraic varieties". Its goal is to give an informal introduction to some of the results relating invariants of singularities (most notably, the log canonical threshold) with various integration theories.

The general (though rather imprecise) framework is the following. We will deal with a field K having an absolute value $|\cdot|$ and a measure. On K^n we have the corresponding product measure. The goal is to relate the singularities of a polynomial f in $K[x_1, \ldots, x_n]$ with the asymptotic behavior of $\mu\left(\{x \in K^n \mid |f(x)| < \epsilon\}\right)$, when ϵ goes to zero. This can be done by studying the behavior of certain integrals. The key is to use a log resolution of singularities for f and some version of the Change of Variable Formula. The main examples we will consider are $K = \mathbb{C}$, $K = \mathbb{Q}_p$ (or more general p-adic fields) and $K = \mathbb{C}((t))$.

In the first section we deal with the Archimedean case: $K = \mathbb{C}$. We start with a question of Gelfand about the meromorphic extension of complex powers, and describe its solutions. The first solution, due independently to Bernstein and Gelfand and to Atiyah, is a first instance of combining integration with resolution of singularities. The second solution, due to Bernstein, uses the existence of the Bernstein-Sato polynomial and integration by parts. In the second section we give a brief overview of the p-adic side of the story. We review p-adic integration, define the Igusa zeta function and state the main result of Igusa about the rationality of the zeta function. In particular, we see the connection between the largest pole of this function and the log canonical threshold.

In the third section we cover some basic facts about spaces of arcs and motivic integration. The main goal is to underline the similarities and the differences with the p-adic case. The fourth section presents the geometric results relating the approaches to singularities via spaces of arcs and via divisorial valuations. In particular, we give the description of the log canonical threshold in terms of the codimensions of certain subsets of the space of arcs. In the last section we deviate slightly from the main topic of these notes to present one of the basic applications of the theory of motivic integration: the definition of stringy invariants of varieties with mild singularities.

1. Complex powers and singularities

A good introduction to the results in this section is Igusa's book [Ig]. We consider the case of a polynomial with complex coefficients, but similar results hold over $\mathbb{R}$.

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a nonconstant polynmial. One can show that if $s \in \mathbb{C}$ is such that $\mathrm{Re}(s) > 0$ and if $\Phi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ is a $\mathcal{C}^\infty$-function with compact support on $\mathbb{C}^n$, then

$$Z_{f,\Phi}(s) := \int_{\mathbb{C}^n} |f(z)|^{2s}\Phi(z)dz\, d\overline{z} \tag{1}$$

is well-defined, and by taking $s \to Z_{f,-}(s)$ we get a holomorphic map on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ with values in the space of distributions on $\mathbb{C}^n$. This is the *complex power* associated to f. Gelfand asked whether every complex power can be extended meromorphically to $\mathbb{C}$.

The first solution was due independently to Bernstein and Gelfand and to Atiyah (see [Ig] for details and for precise references). The idea is to use a log resolution of singularities for f. This is a morphism $\pi\colon Y \to \mathbb{C}^n$ that is proper and birational, with Y nonsingular, and such that in local coordinates $y_1, \ldots, y_n$ on Y we can write

$$f(\pi(y)) = u(y) \cdot y_1^{a_n} \cdots y_n^{a_n},\ \det(\mathrm{Jac}(\pi))(y) = v(y) \cdot y_1^{k_1} \cdots y_n^{k_n},$$

where u and v do not vanish anywhere.

The idea is to use the Change of Variable Formula to reduce the computation of the integral in (1) to integrals of the form

$$\int_{\mathbb{C}^n} \prod_{i=1}^{n} |z_i|^{2sa_i+k_i} \cdot \psi(z)dzd\overline{z}, \tag{2}$$

for a suitable $\psi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$. These integrals are easy to analyze directly and one gets

Theorem 1.1 (Bernstein-Gelfand, Atiyah)**.** *With the above notation, there is a meromorphic function on $\mathbb{C}$ with values in the space of distributions on $\mathbb{C}^n$ such that if* $\mathrm{Re}(s) > 0$, *its value is given by $Z_{f,-}(s)$. Moreover, every pole is of the form $-\frac{k_i+m}{a_i}$ for some positive integer m, some local chart on Y and some i as above.*

Recall that the log canonical threshold of f (more precisely, of the pair $(X, V(f))$ is defined in terms of a resolution of singularities by

$$\mathrm{lc}(f) := \min_i \frac{k_i + 1}{a_i}, \tag{3}$$

where the minimum is taken over all i and over all local charts on Y as above. It follows from Theorem 1.1 that the complex power associated to f is holomorphic in the region $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > -\mathrm{lc}(f)\}$.

One can compare the above result with the following "local" version that is often used to describe the log canonical threshold. The proof proceeds along the same lines, using the resolution of singularities and the Change of Variable Formula.

Theorem 1.2. *If f is a polynomial in $\mathbb{C}[x_1, \ldots, x_n]$, then*

$$\mathrm{lc}(f) = \sup\left\{ s > 0 \mid \frac{1}{f(z)^s} \in L^2_{\mathrm{loc}}(\mathbb{C}^n) \right\}.$$

We now describe the second solution to Gelfand's question, due to Bernstein. In fact, motivated by this problem, Bernstein introduced what is now called the Bernstein-Sato polynomial of f. In order to prove its existence he developed the basics of the theory of modules over the Weyl algebra.

Bernstein proved that there is a nonzero polynomial in one variable $b(s)$ such that we have a relation of the form

$$b(s)f^s = P(s, x, \partial_x) \bullet f^{s+1}, \tag{4}$$

for some polynomial differential operator $P \in \mathbb{C}[s, x, \partial_x]$, where $\bullet$ stands for the action of this operator. This relation has to be interpreted formally, but it has the obvious meaning whenever we can make sense of f^s. The polynomials $b(s)$ for which there is P as above form an ideal, and the monic generator of this ideal is called the Bernstein-Sato polynomial of f, and denoted by $b_f(s)$.

Theorem 1.3 (Bernstein)**.** *If $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a nonconstant polynomial, then there is a meromorphic function on $\mathbb{C}$ with values in the space of distributions on $\mathbb{C}^n$ such that if $\mathrm{Re}(s) > 0$, its value is given by $Z_{f,-}(s)$. Moreover, if s is a pole of this function, then there is a root λ of b_f and a nonnegative integer r such that $s = \lambda - r$.*

The idea of the proof is to use the equation (4) and integration by parts to write

$$|b(s)|^2 \cdot \int_{\mathbb{C}^n} |f(z)|^{2s}\Phi(z)dz\, d\overline{z} = \int_{\mathbb{C}^n} |f(z)|^{2(s+1)} \cdot \Psi(z)dz\, d\overline{z}, \tag{5}$$

for a suitable $\Psi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$. As we have seen, the right-hand side is defined and holomorphic in s when $\mathrm{Re}(s) > -1$. We now multiply by $|b(s+1)|^2$, and continuing this way, we get the assertion in the theorem.

In general, the Bernstein-Sato polynomial is a very subtle invariant of the singularities of f. Comparing the assertions about the poles in Theorems 1.1 and 1.3 one can speculate on connections between the roots of b_f and the invariants coming from the resolution of singularities. In fact, there is such a precise connection:

Theorem 1.4 (Kashiwara, Lichtin)**.** *If $f \in \mathbb{C}[x_1, \ldots, x_n]$, then all roots of b_f are negative rational numbers. Moreover, with the above notation for a resolution of f, every root λ of b_f is of the form $\lambda = -\frac{k_i+m}{a_i}$ for some i and some positive integer m.*

Theorem 1.5 (Kollár, Lichtin)**.** *If $f \in \mathbb{C}[x_1, \ldots, x_n]$, then the largest root of $b_f(s)$ is $-\mathrm{lc}(f)$.*

Theorem 1.5 was proved in [Kol]. Note first that Theorem 1.4 implies that for every root λ of b_f we have $\lambda \leq -\mathrm{lc}(f)$. On the other hand, one shows that there is a root of b_f in $[-\mathrm{lc}(f), 0)$ using the description of $\mathrm{lc}(f)$ in Theorem 1.2 and using the idea in the proof of Theorem 1.3 of combining equation (4) and integration by parts.

The rationality statement in Theorem 1.4 was proved by Kashiwara in [Ka] using deep results in the theory of D-modules. Buliding on Kashiwara's work, Lichtin described the relation between the roots of b_f and the resolution of singularities in [Li].

2. Igusa zeta functions

In this section we give an overview of the p-adic point of view. Again, our main reference for p-adic integration and for Igusa zeta functions is Igusa's book [Ig]. The idea is that if f is a polynomial with integer coefficients, then the asymptotic behavior of the number of solutions of f in $\mathbb{Z}/p^m\mathbb{Z}$, when m goes to infinity, is closely related to the singularities of f.

We will work in the following more general setup. Let p be a prime and let K be a p-adic field, i.e. a finite extension of the field of p-adic rational numbers $\mathbb{Q}_p$. The integral closure of the ring $\mathbb{Z}_p$ of p-adic integers is denoted by O_K. It is a discrete valuation ring with fraction field K, and we denote a generator of the maximal ideal $\mathfrak{m}$ by π. The residue field of O_K is a finite extension of F_p, so it is equal to F_q for some $q = p^e$.

On K we have the $\mathfrak{m}$-adic topology, induced by the metric

$$d(u,v) = |u-v|_K := \left(\frac{1}{q}\right)^{\mathrm{ord}_\pi(u-v)},$$

where ord_π is the valuation on K corresponding to O_K. In other words, an element u in O_K is close to zero if it is divisible by a large power of π. A basis of open neighborhoods of the origin is given by the powers of the maximal ideal in O_K. Note that O_K is compact, and therefore K is locally compact.

We have on K the Haar measure μ: it is the unique measure that is invariant under translations and such that $\mu(O_K) = 1$. These requirements imply that $\mu(\mathfrak{m}^\ell) = \frac{1}{q^\ell}$ for $\ell \geq 0$. We have the product measure on K^n such that $\mu(\prod_i \mathfrak{m}^{\ell_i}) = \left(\frac{1}{q}\right)^{\sum_i \ell_i}$. Since K^n is locally compact, the standard result of measure theory hold in this setting.

Suppose now that $f \in O_K[x_1, \ldots, x_n]$ is a nonconstant polynomial. One can show that if we put

$$Z_f(s) := \int_{O_K^n} |f(x)|_K^s dx, \tag{6}$$

this defines a holomorphic function Z_f on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$, the *Igusa zeta function* of f. In fact there are several variations, involving an auxiliary function Φ as in the previous section, integrating on a different subset of K^n, or involving also a character of the group of units in K. We refer to [Ig] for this more general definition.

Note that the Igusa zeta function gives a convenient way to encode the numbers

$$c_r = \#\{u \in (O_K/\mathfrak{m}^r)^n \mid f(u) = 0\} \tag{7}$$

(with the convention $c_0 = 1$). Indeed, we have

$$Z_f(s) = \sum_{m \in \mathbb{N}} \mu\left(\{u \in O_K^n \mid \mathrm{ord}_\pi(f(u) = m\}\right) \cdot \frac{1}{q^{ms}}$$

and the measures in this formula can be computed as

$$\mu\left(\{u \in O_K^n \mid \mathrm{ord}_\pi(f(u) = m\}\right) = \mu\left(\{u \in O_K^n \mid \mathrm{ord}_\pi(f(u) \geq m\}\right)$$

$$-\mu\left(\{u \in O_K^n \mid \mathrm{ord}_\pi(f(u) \geq m+1\}\right) = c_m \cdot \frac{1}{q^{mn}} - c_{m+1} \cdot \frac{1}{q^{(m+1)n}}.$$

The advantage of the integral formula in the definition of the Igusa zeta function is that p-adic integrals, too, satisfy a Change of Variable Formula. As in the case of Theorem 1.1, one can use a log resolution of f (defined over $\mathbb{Q}_p$) to reduce the computation of Z_f to the computation of integrals involving only monomial expressions. Arguing in this way one gets the following

Theorem 2.1 (Igusa)**.** *The function Z_f admits a meromorphic extension to $\mathbb{C}$, that is in fact a rational function of $\left(\frac{1}{q}\right)^s$. Moreover, with the notation for a log resolution from §1, if λ is a pole of Z_f, then there is i such that* $\mathrm{Re}(s) = -\frac{k_i+1}{a_i}$.

Using the previously described connection between the Igusa zeta function and the numbers c_m, one can deduce from the above theorem the rationality of the generating series associated to f, a statement that had been conjectured by Borevich.

Corollary 2.2. *If f is in $O_K[x_1, \ldots, x_n]$, then the power series*

$$P_f := \sum_{m \in \mathbb{N}} \frac{c_m}{q^{mn}} t^m$$

is a rational function.

Note that Theorem 2.1 implies that Z_f is holomorphic in the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > -\mathrm{lc}(f)\}$. On can reformulate this by saying that the radius of convergence of the series P_f is at least $q^{\mathrm{lc}(f)}$, or equivalently

$$\limsup_{m \to \infty} c_m^{1/m} \leq q^{n-\mathrm{lc}(f)}.$$

One can not expect to get also a lower bound for the numbers c_m in general. However, this holds after possibly enlarging the field K.

Theorem 2.3 (Igusa)**.** *After possibly passing to a finite extension of K, we can find a pole λ of Z_f with* $\mathrm{Re}(\lambda) = -\mathrm{lc}(f)$. *Therefore*

$$\limsup_{m \to \infty} c_m^{1/m} = q^{n-\mathrm{lc}(f)}.$$

A proof of this theorem can be found in [VZG]. Note that in *loc. cit.* one develops the whole story not just for a polynomial, but for an arbitrary ideal. We will switch to this setup in the following section, when discussing spaces of arcs.

We end this section with what is arguably the most interesting open problem concerning Igusa zeta functions.

Conjecture 2.4 (Igusa)**.** Let f be a polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$. For every prime p, we denote by $Z_{f,p}$ the Igusa zeta function constructed over $\mathbb{Q}_p$. If p is large enough, then for every pole λ of $Z_{f,p}$, its real part is a root of the Bernstein-Sato polynomial $b_f(s)$.

A slightly weaker version of the conjecture, in terms of eigenvalues of the monodromy instead of roots of the Bernstein-Sato polynomial, is known as the Monodromy Conjecture. We refer to [Den] for a discussion of this conjecture, and to [Ve3] for some recent progress.

3. Spaces of arcs and motivic integration

We replace now the field $\mathbb{Q}_p$ (or one of its extensions) by the field $\mathbb{C}((t))$ of Laurent power series. In this case the set of solutions in $\mathbb{Z}/p^m\mathbb{Z}$ is replaced by the set of solutions in $\mathbb{C}[t]/(t^m)$. This is the set of closed points of a scheme, the $(m-1)$st jet scheme H_{m-1} of the hypersurface H defined by f. Instead of counting the number of elements, we compute the dimensions of these schemes, and we have the following analogue of Theorem 2.3.

Theorem 3.1 (Mustaţă). *If $f \in \mathbb{C}[x_1, \ldots, x_n]$ defines a hypersurface H in $\mathbb{C}^n$, then*

$$\lim_{m\to\infty} \frac{\dim\, H_{m-1}}{m} = n - \mathrm{lc}(f).$$

As in the p-adic setting, one can define integrals on the space $(\mathbb{C}[[t]])^n$. This is an infinite-dimensional space and this time the integrals will take value in a more complicated ring. We sketch in this section the basics of motivic integration. There are by now several introductions to this topic that we recommend to the reader: [Bli], [Cr] and [Ve2], as well as some survey papers [DL1] and [Lo]. Due to Kontsevich [Kon] (see [Ba1]), this theory was generalized by Denef and Loeser in [DL3] to the case of an ambient singular variety. There are constructions in a much more general set-up (see [CL]), but these go beyond the scope of these notes. For applications to singularities, it turns out that it is more effective to use the geometry behind motivic integration, and we will explain this in the next section. In particular, Theorem 3.1 above will be a consequence of more general geometric results.

Let X be an arbitrary scheme of finite type over $\mathbb{C}$. Its mth *jet scheme* X_m is characterized by its functor of points as follows:

$$\mathrm{Hom}(\mathrm{Spec}\, A, X_m) \simeq \mathrm{Hom}(\mathrm{Spec}\, A[t]/(t^{m+1}), X) \tag{8}$$

for every $\mathbb{C}$-algebra A. In particular, the set of closed points of X_m is equal to the set of m-jets $\mathrm{Hom}(\mathrm{Spec}\ \mathbb{C}[t]/(t^{m+1}), X)$ of X. For example, $X_0 = X$ and $X_1 = TX$, the total tangent space of X.

Existence is proved locally, first for affine schemes and then suitably gluing the jet schemes of affine charts. Note that if $X = \mathbb{C}^n$, then $X_m = (\mathbb{C}[t]/t^{m+1})^n \simeq \mathbb{C}^{(m+1)n}$. If X is a closed subscheme of $\mathbb{C}^n$, then we get a corresponding closed embedding $X_m \hookrightarrow (\mathbb{C}^n)_m$. Moreover, the equations of X_m can be obtained by "formally differentiating" m times the equations defining X.

Example 3.2. If $X \hookrightarrow \mathbb{C}^2 = \mathrm{Spec}\, \mathbb{C}[u,v]$ is defined by $f = u^2 - v^3$, then $X_2 \subseteq (\mathbb{C}^2)_2 = \mathrm{Spec}\, \mathbb{C}[u,v,u',v',u'',v'']$ is defined by (f, f', f''), where

$$f' = 2uu' - 3v^2v',\ \ f'' = 2uu'' + 2(u')^2 - 6v(v')^2 - 3v^2v''.$$

The truncation maps $\mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}[t]/(t^m)$ induce morphisms $X_m \to X_{m-1}$. The projective limit of the X_m is the *space of arcs* X_∞ of X, whose $\mathbb{C}$-valued points correspond to elements of $\mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[[t]], X)$. The above constructions are functorial: for a morphism $f\colon X \to Y$, we get morphisms $f_m\colon X_m \to Y_m$ and $f_\infty\colon X_\infty \to Y_\infty$.

As a side remark, we mention that the space of arcs of a variety attracted some attention before the theory of motivic integration. For example, Kolchin proved the following theorem (see [IK] for a modern proof).

Theorem 3.3 (Kolchin)**.** *If X is an irreducible variety, then X_∞ is also irreducible.*

Suppose now that $\pi\colon Y \to X$ is a resolution of singularities of X, i.e. π is proper, birational and Y is smooth. While X_∞ is irreducible by the above theorem, the irreducible components of $(\pi_\infty)^{-1}(X_{\mathrm{sing}})$ contain a lot of information about the singularities of X. Nash conjectured that they are in bijection with the divisors that appear on every resolution of singularities of X (see [IK] for the precise statement). While the conjecture was disproved in general in *loc. cit.*, it still attracts a great deal of interest.

If X is nonsingular of dimension n, then every projection $X_m \to X_{m-1}$ is locally trivial (in the Zariski topology), with fiber $\mathbb{C}^n$. In particular, we have $\dim X_m = (m+1)n$. In what follows we will assume that we work in an ambient nonsingular variety of dimension n, and we denote by $\phi_m\colon X_\infty \to X_m$ the canonical projection.

The space of arcs X_∞ is infinite-dimensional, but in what follows we will deal with some subsets of finite codimension. A *cylinder* in X_∞ is a subset of the form $C = \psi_m^{-1}(S)$, where $S \subseteq X_m$ is a constructible subset. It is clear that the cylinders form an algebra of subsets of X_∞. Since each projection $X_m \to X_{m-1}$ is locally trivial, we may define

$$\mathrm{codim}(C) := \mathrm{codim}(S, X_m) = (m+1)n - \dim(S).$$

Similarly, one says that C is irreducible (closed, locally closed) if S is.

Cylinders are the most important "measurable sets" for motivic integration. The other subsets that appear are of the form Y_∞ for some proper closed subset of X. These sets, however, have "measure zero" or "infinite codimension", hence they can be ignored in this theory. The key property is that a cylinder can never be contained in such a set.

The interesting functions on X_∞ are given by the order of vanishing along closed subschemes of X. If $Y \hookrightarrow X$ is such a subscheme defined by the ideal I_Y, then one defines $\mathrm{ord}_Y\colon X_\infty \to \mathbb{N} \cup \{\infty\}$ by

$$\mathrm{ord}_Y(\gamma) = \mathrm{ord}(\gamma^{-1} I_Y) \tag{9}$$

(note that $\gamma^{-1} I_Y$ is an ideal in $\mathbb{C}[[t]]$, and by convention it is zero if and only if its order is ∞). The finite level sets are locally closed cylinders:

$$\mathrm{ord}_Y^{-1}(m+1) = \phi_m^{-1}(Y_m) \smallsetminus \phi_{m+1}^{-1}(Y_{m+1}),$$

while the set $\mathrm{ord}_Y^{-1}(\infty) = Y_\infty$ can be ignored for the purpose of motivic integration.

The above setup is very similar to that in the p-adic theory. The key new idea in motivic integration concerns the ring where the measure takes its values. This is a localization of the *Grothendieck ring of varieties* over $\mathbb{C}$. This Grothendieck ring, denoted by $K_0(\mathrm{Var}/\mathbb{C})$, is defined as the quotient of the free abelian group on the set of isomorphism classes $[V]$ of complex varieties, modulo the relations

$$[V] = [W] + [V \smallsetminus W],$$

where V is a complex variety and W is a closed subvariety of V. $K_0(\mathrm{Var}/\mathbb{C})$ becomes a ring with the product $[V] \cdot [W] = [V \times W]$, the unit being the class of a point. Note that if $f\colon V \to W$ is locally trivial in the Zariski topology, with fiber F, then $[V] = [W] \cdot [F]$ in the Grothendieck ring.

One denotes by $\mathbb{L}$ the class of the affine line $\mathbb{A}^1$ in the Grothendieck ring. The *motivic measure* takes values in the localization $K_0(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}]$. If $C = \phi_m^{-1}(S)$ is a cylinder, then $\mu(C) = [S] \cdot \mathbb{L}^{-mn}$.

In fact, in order to compute integrals one has to work in a suitable completion $\widehat{K_0}$ of the above localization (a more recent construction of Cluckers and Loeser [CL] shows that, in fact, it is enough to just invert some elements in the Grothendieck ring). If $f\colon X_\infty \to \mathbb{Z} \cup \{\infty\}$ is such that

(1) For every $m \in \mathbb{Z}$, the set $f^{-1}(m)$ is a cylinder.
(2) The set $f^{-1}(\infty)$ is contained in the space of arcs of a proper closed subset of X,

then we may consider the sum

$$\int_{X_\infty} \mathbb{L}^{-f} := \sum_{m \in \mathbb{Z}} \mu(f^{-1}(m)) \cdot \mathbb{L}^{-m}.$$

If this sum is convergent in the above mentioned completion of the localized Grothendieck ring, one says that f is integrable, and the above sum is called *the motivic integral of f*. One can define in a similar way the integral of $\mathbb{L}^{-f}$ over a cylinder in X_∞

One can do this slightly more generally by allowing also rational powers of $\mathbb{L}$ (by adjoining $\mathbb{L}^{1/m}$ for a suitable m). In this case one can integrate also functions that take values in $\frac{1}{m}\mathbb{Z} \cup \{\infty\}$, for some m.

A basic fact about motivic integrals is that they are easy to compute for divisors with simple normal crossings. More precisely, one has the following

Theorem 3.4 (Kontsevich)**.** *Suppose that $D = a_1 D_1 + \cdots + a_r D_r$ is a $\mathbb{Q}$-divisor with simple normal crossings on X and let $\mathrm{ord}_D := \sum_{i=1}^r a_i \,\mathrm{ord}_{D_i}$. Then ord_D is integrable if and only if all $a_i > -1$, and in this case its motivic integral admits the following "rational expression"*

$$\int_{X_\infty} \mathbb{L}^{-\mathrm{ord}_D} = \sum_{J \subseteq \{1,\ldots,r\}} [D_J^\circ] \cdot \prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1},$$

where for every J, one puts $D_J = \bigcap_{j \in J} D_j \smallsetminus \bigcup_{i \notin J} D_i$.

The key result of the theory is the following Change of Variable Theorem. Together with resolution of singularities and the previous result, this can be used to compute motivic integrals of functions of the form ord_Y.

Let $\pi\colon X'\to X$ be a proper, birational morphism of nonsingular complex varieties. Denote by $K_{X'/X}$ the discrepancy divisor. This is an effective divisor, locally defined by the determinant of the Jacobian matrix of π.

Theorem 3.5 (Kontsevich)**.** *With the above notation, $f\colon X_\infty\to\frac{1}{m}\mathbb{Z}\cup\{\infty\}$ is integrable if and only if so is $f\circ\pi_\infty+\mathrm{ord}_{K_{X'/X}}$. In this case, for every cylinder C in X_∞, we have*

$$\int_C \mathbb{L}^{-f}=\int_{\pi_\infty^{-1}(C)}\mathbb{L}^{-f\circ\pi_\infty-\mathrm{ord}_{K_{X'/X}}}.$$

The proof of this theorem follows formally from a more geometric result that we will state in the next section (see Theorem 4.2 below). To continue the parallel with the p-adic setting, we mention that Denef and Loeser [DL4] have defined a motivic Igusa zeta function by putting for a polynomial $f\in\mathbb{C}[x_1,\ldots,x_n]$

$$Z_f:=\int_{X_\infty}\mathbb{L}^{-s\cdot\mathrm{ord}_f},$$

where one has to interpret this time the symbol $\mathbb{L}^{-s}$ as a new variable. Theorems 3.4 and 3.5 have analogues in this setting, and one gets in this way a rationality result as in the case of the p-adic Igusa zeta function. There is also a version of Conjecture 2.4 above for motivic Igusa zeta functions, for which we refer to [Ve2].

4. Singularities via spaces of arcs

We have seen that motivic integration is an analogue of p-adic integration that satisfies a Change of Variable Theorem. In particular, we can use it to interpret invariants defined via resolution of singularities in terms of codimensions of certain cylinders.

Let X be a nonsingular variety and $Y\hookrightarrow X$ a closed subscheme. As in §1, we consider a log resolution of singularities $\pi\colon X'\to X$ for the pair (X,Y). We recall the definition: π is proper and birational, X' is nonsingular, and the union of $\pi^{-1}(Y)$ with the exceptional locus of π is a divisor with simple normal crossings. We write

$$\pi^{-1}(Y)=\sum_{i=1}^r a_iD_i,\, K_{X'/X}=\sum_{i=1}^r k_iD_i. \tag{10}$$

In addition, we may assume that π is an isomorphism over $X\smallsetminus Y$, hence $k_i=0$ if $a_i=0$.

Define the contact locus of order m of Y by $\mathrm{Cont}^m(Y):=\mathrm{ord}_Y^{-1}(m)$, and similarly $\mathrm{Cont}^{\geq m}(Y):=\mathrm{ord}_Y^{-1}(\geq m)$. If we apply Theorem 3.5 to the function $f=0$ and integrate over the cylinder $\mathrm{Cont}^m(Y)$, then we get

$$\mu(\mathrm{Cont}^m(Y))=\sum_{\nu=(\nu_i)_i\in\mathbb{N}^r}\mu\left(\bigcap_{i=1}^r\mathrm{Cont}^{\nu_i}(D_i)\right)\mathbb{L}^{-\sum_i k_i\nu_i}, \tag{11}$$

where the sum is over those ν such that $\sum_{i=1} a_i\nu_i = m$ (hence this is a finite sum).

By considering the codimensions of the corresponding cylinders, one deduces the following

Theorem 4.1 ([ELM]). *With the above notation, for every m we have*

$$\mathrm{codim}\ \mathrm{Cont}^m(Y) = \min_{\nu} \sum_{i=1}^{r} \nu_i(k_i+1),$$

where the sum is over all $\nu = (\nu_i)_i \in \mathbb{N}^r$ such that $\cap_{\nu_i\geq 1} D_i \neq \emptyset$ and $\sum_{i=1}^{r} \nu_i a_i = m$.

The formula for the log canonical threshold in Theorem 3.1 is then an easy consequence, using the fact that $\mathrm{codim}\,\mathrm{Cont}^{\geq m}(Y) = \mathrm{codim}(Y_{m-1}, X_{m-1})$. Similar descriptions can be given for other invariants of singularities, the minimal log discrepancies. Moreover, the generalization of motivic integration from [DL3], one can give a description of minimal log discrepancies in terms of the codimensions of certain sets of arcs even when the ambient variety is singular (see [EMY]). We note that for these invariants it is not clear how to give an interpretation in terms of Lebesgue or p-adic integrals.

In fact, in order to relate invariants of singularities with the codimension of the contact loci one does not need to use the formalism of motivic integration. One can use instead the following geometric result, that is the key ingredient in the Change of Variable Theorem. The advantage of this approach is that, as we will see, it makes more transparent the connection between the usual approach to singularities (via divisorial valuations) and that via cylinders in the space of arcs.

Theorem 4.2 (Kontsevich). *Let $\pi\colon X' \to X$ be a proper, birational morphism between nonsingular complex varieties. Given an integer $e \geq 0$, consider the contact locus*

$$\mathrm{Cont}^e(K_{X'/X})_m := \{\gamma \in X'_m \mid \mathrm{ord}_{K_{X'/X}}(\gamma) = m\}.$$

If $m \geq 2e$, then the locus $\mathrm{Cont}^e(K_{X'/X})_m$ is a union of fibres of $\pi_m\colon X'_m \to X_m$, each of which is isomorphic to $\mathbb{A}^e$. Moreover, if γ, $\gamma' \in \mathrm{Cont}^e(K_{X'/X})_m$ lie in the same fiber of π_m, then they lie over the same jet in X'_{m-e}.

We want to stress one point: the morphism π_∞ is "almost everywhere" a bijection. More precisely, if π is an isomorphism over $X \smallsetminus Z$, then the induced map $X'_\infty \smallsetminus \pi^{-1}(Z)_\infty \to X_\infty \smallsetminus Z_\infty$ is bijective by the Valuative Criterion for properness. On the other hand, this map is very far from being an isomorphism. In fact, the theorem implies that if we consider the decomposition

$$X'_\infty \smallsetminus \pi^{-1}(Z)_\infty = \coprod_{e\in\mathbb{N}} \mathrm{Cont}^e(K_{X'/X}),$$

then on the eth piece π_∞ behaves like a fibration in the sense that it increases the codimension of the cylinders by e: if $C \subseteq \mathrm{Cont}^e(K_{X'/X})$ is a cylinder, then $\pi_\infty(C)$ is also a cylinder and $\mathrm{codim}\ \pi_\infty(C) = \mathrm{codim}\ C + e$.

For a proof of the theorem we refer to Looijenga's Bourbaki talk in [Lo]. We now explain the applications to the study of singularities. Invariants of singularities like the log canonical threshold or minimal log discrepancies are defined by considering all divisorial valuations of the function field of our variety X (one then proves that it is enough to consider those valuations corresponding to divisors on a suitable resolution of singularities). We now show how to recover all divisorial valuations using cylinders in the space of arcs.

Let X be a nonsingular variety. A *divisor over* X is a prime divisor E on a normal variety X' such that there is a birational morphism $X' \to X$. Every such divisor defines a valuation ord_E of the function field $K(X') = K(X)$, and we identify two such divisors if they give the same valuation. A *divisorial valuation* of $K(X)$ is a valuation of the form $q \cdot \mathrm{ord}_E$ for some divisor E over X. A fundamental invariant of a divisorial valuation $v = q \cdot \mathrm{ord}_E$ is its *log discrepancy*: this is $q(k_E + 1)$, where k_E is the coefficient of E in the divisor $K_{X'/X}$.

We can get valuations starting also from irreducible cylinders in the space of arcs of X. Indeed, we may assume that $X = \mathrm{Spec}(A)$, and if C is such a cylinder, then we define

$$\mathrm{ord}_C(f) := \min_{\gamma \in C} \mathrm{ord}_{V(f)}(\gamma)$$

for every $f \in A$. This extends uniquely to a valuation of $K(X)$, which is nontrivial if and only if C does not dominate X.

Example 4.3. Suppose that E is a prime divisor on X' such that we have a birational morphism $\pi\colon X' \to X$. We may assume that X' is smooth, and let $C^q(E)$ be the closure of $\pi_\infty(\mathrm{Cont}^q(E))$. This is a closed, irreducible cylinder that depends only on the valuation corresponding to E and

$$\mathrm{ord}_{C^q(E)} = q \cdot \mathrm{ord}_E.$$

The following results relate divisorial valuations with spaces of arcs. For proofs, see [ELM].

Theorem 4.4. *If C is an irreducible cylinder in X_∞ that does not dominate X, then there are a unique positive integer q and a unique divisor E over X such that $C \subseteq \mathrm{Cont}^q(E)$ and $\mathrm{ord}_C = \mathrm{ord}_{C^q(E)}$. In particular, ord_C is a divisorial valuation.*

The applicability of this result in studying singularities comes from the fact that the log discrepancy of E translates as the codimension of the corresponding cylinder in X_∞, as follows.

Theorem 4.5. *For every q and E, the codimension of $C^q(E)$ is equal to the log discrepancy of the valuation $q \cdot \mathrm{ord}_E$.*

5. Stringy invariants

One of the first applications of motivic integration was towards defining stringy invariants of singular algebraic varieties. These are invariants that take into account the singularities of the variety, and behave well with respect to birational transformations. We review here the definition of stringy Hodge and Betti numbers. The starting point is the following result.

Theorem 5.1 (Kontsevich)**.** *Two K-equivalent nonsingular projective varieties have the same Hodge numbers.*

Recall that two nonsingular projective varieties X_1 and X_2 are called K-equivalent if there are projective birational morphisms $Y \to X_1$ and $Y \to X_2$ such that K_{Y/X_1} and K_{Y/X_2} are numerically equivalent (in fact, in this case one can show that these two divisors are equal). For example, this is the case with two birational Calabi-Yau varieties.

An earlier version of the above theorem, proved by Batyrev in [Ba2], stated that birational Calabi-Yau varieties have the same Betti numbers. Its proof used p-adic integration to show that the two varieties have the same number of points over finite fields, and then the Weil conjectures to deduce that the two varieties have the same Betti numbers.

Before explaining the proof of Theorem 5.1, let us review the definition of the Hodge-Deligne polynomial. Recall that if X is a complex nonsingular projective variety, with $\dim(X) = n$, its Hodge polynomial is

$$E(X; u, v) = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q} u^p v^q,$$

where $h^{p,q}$ is the Hodge number $h^q(X, \Omega_X^p)$. The Poincaré polynomial of X can be obtained as $E(X; t, t)$, hence the Euler-Poincaré characteristic of X is equal to $E(X; 1, 1)$.

It is a theorem of Deligne [Del] that the Hodge polynomial can be extended additively to arbitrary complex varieties, i.e. one can define a polynomial with integer coefficients $E(Y; u, v)$ for every complex variety Y, such that if Z is a closed subvariety of Y, then $E(Y; u, v) = E(Z; u, v) + E(Y \smallsetminus Z; u, v)$. In other words, E induces a group homomorphism

$$K_0(\mathrm{Var}/\mathbb{C}) \to \mathbb{Z}[u, v]. \tag{12}$$

In fact, this is a ring homomorphism: since the left-hand side is generated by classes $[X]$, with X nonsingular and projective, it is enough to show that for such X_1 and X_2 we have $E(X_1 \times X_2; u, v) = E(X_1; u, v) \cdot E(X_2; u, v)$. This follows from the Küneth formula. $E(Y; u, v)$ is called the Hodge-Deligne polynomial of Y, and $E(Y; t, t)$ the virtual Poincaré polynomial of Y. It is known that the Euler-Poincaré charcateristic $\chi(Y)$ of every Y is equal to the Euler-Poincaré characteristic $\chi_c(Y)$ for the cohomology with compact support, and that it gives an additive function. Equivalently, for every Y we have

$$E(Y; 1, 1) = \chi(Y) = \chi_c(Y).$$

The definition of $E(Y; u, v)$ is in terms of the mixed Hodge structure on the cohomology with compact support of Y. However, in order to compute it one does not need to know the definition: one can do the computation by induction on dimension, compactifying and resolving singularities to reduce to the case of a nonsingular, projective variety. For example, we have

$$E(\mathbb{A}^1; u, v) = E(\mathbb{P}^1; u, v) - E(\mathrm{pt}; u, v) = (1 + uv) - 1 = uv.$$

We deduce that $E(\mathbb{A}^n; u, v) = (uv)^n$.

There is also an alternative way of proving the existence of the morphism (12). Bittner showed in [Bit] that the Grothendieck group $K_0(\mathrm{Var}/\mathbb{C})$ admits a presentation as the free abelian group on isomorphism classes of nonsingular complex projective varieties, with relations

$$[\mathrm{Bl}_Z X] - [E] = [X] - [Z],$$

where $\mathrm{Bl}_Z X$ is the blowing-up of the nonsingular projective variety X along the nonsigular closed subvariety Z, and E is the exceptional divisor. The key ingredient in this result is the Weak Factorization Theorem of [AKMW]. Therefore in order to construct (12) it is enough to show that the Hodge polynomial of nonsingular projective varieties satisfies the above "blowing-up relation", which can be done by a direct computation.

Suppose now that X is a nonsingular variety. Using the morphism (12) one can specialize the motivic measure from the localized Grothendieck ring $K_0(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}]$ to $\mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$. We consider the completion of this ring given by Laurent power series in u^{-1} and v^{-1}, and we compute integrals with respect to the corresponding topology on this ring. These are the *Hodge realizations* of the motivic integrals we have defined in the previous section. We denote the integral corresponding to the function f by $\int_{X_\infty}(uv)^{-f}$. Of course, theorems 3.4 and 3.5 have analogues in this setting.

Let us give now the proof of Theorem 5.1. Suppose that $Y \to X_1$ and $Y \to X_2$ are projective, birational morphisms of nonsingular projective varieties such that $K_{Y/X_1} = K_{Y/X_2}$. Note that by definition we have

$$\int_{(X_1)_\infty} (uv)^0 = E(X_1; u, v),$$

and similarly for X_2. On the other hand, the Change of Variable Formula gives

$$\int_{(X_1)_\infty} (uv)^0 = \int_{Y_\infty} (uv)^{-\mathrm{ord}_{K_{Y/X_1}}} = \int_{Y_\infty} (uv)^{-\mathrm{ord}_{K_{Y/X_2}}} = \int_{(X_2)_\infty} (uv)^0,$$

which completes the proof.

One can use similar ideas to define the stringy E-polynomial for varieties with mild singularities. Suppose that X is a variety with Gorenstein canonical singularities. If $Y \to X$ is a resolution of singularities of X, then we have a discrepancy divisor $K_{Y/X}$ and by our assumption on the singularities of X this is an integral, effective divisor. Therefore we may define

$$E_{\mathrm{st}}(X; u, v) := \int_{Y_\infty} (uv)^{-\mathrm{ord}_{K_{Y/X}}}.$$

It can be easily deduced from the Change of Variable Formula that this is well-defined, i.e. it does not depend on the resolution we have chosen.

Using Theorem 3.4 one can write down an explicit formula in terms of the resolution: if $K_{Y/X} = \sum_{i=1}^r a_i D_i$, then

$$E_{\mathrm{st}}(X; u, v) = \sum_{J \subseteq \{1,\ldots,r\}} E(D_J^\circ; u, v) \cdot \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1}. \tag{13}$$

Note that the stringy E-function is in general not a polynomial, but a rational function. Of course, this expression makes sense without any mention of motivic integration. The merit of this theory is in proving the independence of resolution. Nowadays, however, this can be achieved also using the Weak Factorization Theorem of [AKMW].

One can define $E_{\mathrm{st}}(X;u,v)$ more generally when X has log terminal singularities. In this case, the discrepancy divisor $K_{Y/X}$ is a $\mathbb{Q}$-divisor, and its coefficients a_i are > -1. In particular, the above integral makes sense, though we need to allow fractional powers of u and v in the expression of $E_{\mathrm{st}}(X;u,v)$. One can specialize this invariant to the stringy Poincaré function by putting $P_{\mathrm{st}}(Y;t) = E_{\mathrm{st}}(X;t,t)$ and further to the stringy Euler-Poincaré characteristic by putting

$$\chi_{\mathrm{st}}(X) := \lim_{u,v\to 1} E_{\mathrm{st}}(X;u,v).$$

Note that (13) becomes

$$\chi_{\mathrm{st}}(X) = \sum_{J\subseteq\{1,\ldots,r\}} \chi(D_J^\circ)\cdot\prod_{j\in J}\frac{1}{a_j+1}.$$

Another remarkable application of motivic integration is to the proof of the so-called McKay correspondence. If G is a finite group actiong on a smooth variety M preserving its canonical class, then the quotient $X = M/G$ has Gorenstein canonical singularities, so the above $E_{\mathrm{st}}(X;u,v)$ is defined. On the other hand, one can define *orbifold Hodge numbers* and a polynomial $E_{\mathrm{orb}}(X;u,v)$ in terms of the action of G on M. The McKay correspondence at the level of Hodge numbers asserts that

$$E_{\mathrm{st}}(X;u,v) = E_{\mathrm{orb}}(X;u,v).$$

This was proved by Batyrev [Ba3] and Denef and Loeser [DL2] in the case when $M=\mathbb{C}^n$ and $G\subset SL(n)$ and by Lupercio and Poddar [LP] and Yasuda [Yas] in general.

We recall that in the definition of the stringy E-function we had to assume that our variety has log terminal singularities. Veys [Ve1] extended this definition to more general singularities, under the assumption that the Minimal Model Program holds.

By now there are stringy version of other invariants, too. For example, in [dFLNU] one constructs a stringy analogue of the total Chern class of a smooth projective variety. The main ingredient is a realization of motivic integration with values in a ring of constructible functions. A stringy version of the elliptic genus is constructed in [BL]. However, so far there is no interpretation of the elliptic genus in terms of motivic integration. Therefore the independence of the resolution is proved in thais case using the Weak Factorization Theorem of [AKMW].

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