Brief Guide to Some of the Literature on F-singularities

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Many of the tools of higher dimensional complex birational algebraic geometry—including singularities of pairs, multiplier ideals, and log canonical thresholds— have "characteristic p" analogs arising from ideas in tight closure theory. Tight closure, introduced by Hochster and Huneke in [17], is a closure operation performed on ideals in commutative rings of prime characteristic, and has an independent trajectory as an active branch of commutative algebra. Huneke's book [19] gives a nice introduction from the algebraic perspective. Here we review some of the connections with birational geometry that have developed since the survey [30] appeared. Basic references for the birational geometry terms used here are [24], and [21] or [20].

Let X be an reduced, irreducible scheme of finite type over a perfect field k of positive characteristic p. Even if we are mostly interested in complex varieties, such schemes arise in practice by reduction to characteristic p.

Our goal is to understand the singularities of X (or of subschemes or divisors of X) in terms of the Frobenius morphism $F : X \to X$. On the underlying topological spaces, the Frobenius map is simply the identity map, but the corresponding map of rings of functions $\mathcal{O}_X \to F_*\mathcal{O}_X$ is the p-th power map. The Frobenius is a finite map of schemes of degree p, though it is not a map of k-varieties unless $k = \mathbb{F}_p$.

1 F-singularities.

The following simple fact is fundamental: the scheme X is smooth over k if and only if the Frobenius is a vector bundle—that is, if and only if the $\mathcal{O}_X$-module $F_*\mathcal{O}_X$ is locally free [22]. By weakening the property that $F_*\mathcal{O}_X$ is locally free in various ways, different classes of "F-singularities" arise.

1.1 F-purity and log canonical singularities.

Suppose that the Frobenius map $\mathcal{O}_X \to F_*\mathcal{O}_X$ is locally split as a map of $\mathcal{O}_X$-modules. This property of "local Frobenius splitting," called *F-purity* in the commutative algebra literature, was introduced by Hochster and Roberts in their proof of the Cohen-Macaulayness of rings of invariants [18].

Local F-splitting should not be confused with global F-splitting, in which there is a global splitting for the map $\mathcal{O}_X \to F_*\mathcal{O}_X$, though of course there is no difference for affine X. Global F-splitting (usually called "Frobenius splitting") was introduced by Mehta and Ramanathan to study cohomology of Schubert varieties [25]. The new book [8] gives a comprehensive overview of this rich subject. On the other hand, local and global F-splitting are related as follows: Fix a projective variety X together with a choice of ample line bundle $\mathcal{L}$ on it. Then X is globally F-split if and only if the affine cone

$$\text{Spec } \oplus_{n\geq 0} H^0(X, \mathcal{L}^n)$$

over X with respect to $\mathcal{L}$ is locally F-split [31].

Local F-splitting is expected to be the "characteristic p" analog of *log canonical singularities.* Fix a complex variety $X_{\mathbb{C}}$. Because $X_{\mathbb{C}}$ is defined by finitely many equations in finitely many unknowns, we may consider a ring A finitely generated over the integers over which "all the defining equations of $X_{\mathbb{C}}$ are defined." We can then construct a scheme X_A of finite type over A, and thus recover $X_{\mathbb{C}}$ as $X_A \times_A \text{Spec}\mathbb{C}$. Furthermore, we may assume that $X_A \to \text{Spec } A$ is faithfully flat (by inverting an element of A if necessary). The closed fibers of the map $X_A \to \text{SpecA}$ are considered "prime characteristic models"; these fibers are schemes of finite type over finite fields of different (positive) characteristics. The reduction to prime characteristic process is described carefully in many places, for example, in [30].

The following theorem was first proved by Watanabe in the mid-nineties although the published version appeared (in greater generality) several years later in a joint paper with Hara [15].

Theorem 1 *Let $X_{\mathbb{C}}$ be a normal complex variety for which the canonical class is $\mathbb{Q}$-Cartier. If, for some choice of A as above, there is a dense set of closed points of Spec A such that the corresponding prime characteristic models are F-pure, then $X_{\mathbb{C}}$ has log canonical singularities.*

The converse statement is an important **open problem**.

1.2 F-regularity and log terminal singularities.

The notion of *F-regularity*, which is a slightly stronger property than F-purity, is the prime characteristic analog of log terminal singularities. For each natural number e, consider the iterated Frobenius map $F^e : X \to X$, which on the level of sections is the p^e-th power map. When X is affine, it is said to be *strongly F-regular* if, for every non-zero function $c \in \mathcal{O}_X$, the map

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X \tag{1}$$

$$s \mapsto cs^{p^e}$$

splits for all sufficiently large e. For non-affine X, we require this condition locally.[1] In fact, it turns out that one need not consider all c, but may instead chose any c such that the complement of the divisor defined by c is smooth.

Theorem 2 *Let $X_{\mathbb{C}}$ be a normal complex variety for which the canonical class is $\mathbb{Q}$-Cartier. Then $X_{\mathbb{C}}$ has log terminal singularities if and only if for some choice of A as above, there is a dense set of closed points of Spec A such that the corresponding prime characteristic models are F-regular.*

The "if" direction of Theorem 2 is proved in [15], and the "only if" direction in [12]. In some ways, F-regularity is easier to work with than F-purity, which is more complicated from an arithmetical point of view. For example, the cone over an elliptic curve of prime characteristic is F-pure if and only if the elliptic curve is ordinary (that is, not supersingular) [30, 4.3]. This difference may account for the openness of the converse to Theorem 1.

1.3 F-rationality and Rational Singularities.

The prime characteristic analog of rational singularities is F-rationality, a fact that historically preceded and motivated Theorems 1 and 2. By definition, a local ring of prime characteristic is *F-rational* if every ideal generated by a system of parameters is tightly closed; an alternative characterization more in our spirit is that a d-dimensional point x on X is an F-rational point if and only if the local cohomology module $H^d_{\{x\}}(\mathcal{O}_X)$ has no non-trivial proper submodules stable under the action of the Frobenius module [29].

Replacing the words "F-regular" and "log terminal" with "F-rational" and "rational" in Theorem 2, we arrive at the theorem relating F-rationality with rational singularities. The proof of the "if" direction uses the characterization of F-rationality in terms of local cohomology [29], while the converse statement, due to Nobuo Hara, also invokes a variant of Kodaira vanishing [12]. In fact, the equivalence of F-rational and rational singularities implies Theorem 2, by virtue of the "canonical cover trick"; see [30, 4.12].

The Frobenius action on local cohomology is a crucial ingrediant also in the proofs of Theorems 1 and 2, and indeed, both F-regularity and F-purity admit characterizations in terms of the Frobenius action on certain local cohomology modules, which come down to a criterion for purity of a map due to Hochster; see [15] for precise statements in the language here.

[1] We caution the reader of a technical point: there are three different notions of F-regularity in the literature, all of which are expected but not known to be equivalent. A ring R of prime characteristic is weakly F-regular if all its all ideals are tightly closed, and F-regular if the same is true in any of its localizations. On the other hand, we are mainly interested here in the case of normal varieties over a perfect field for which the canonical class is $\mathbb{Q}$-Cartier, and in this case, the three notions are all known to be equivalent [1]. This technical issue arises partially from the vexing **open question** as to whether the operation of tight closure commutes with localization.

1.4 F-injectivity and DuBois singularities.

Another closely related concept is F-injectivity. The scheme X is defined to be *F-injective* at a d-dimensional point x if the map of local cohomology $F : H^d_{\{x\}}(\mathcal{O}_X) \to H^d_{\{x\}}(\mathcal{O}_X)$ is injective. Karl Schwede has proposed that F-injectivity should be the prime characteristic analog of DuBois singularities. In his 2006 PhD thesis, he gives much evidence, including a proof of an analog of Theorem 1 establishing that "F-injective implies DuBois" [27].

2 Singularities of Pairs.

Log terminal and log canonical singularities are most important in the context of *pairs;* therefore it is natural to develop a theory of F-regularity, F-purity, and indeed tight closure of a pair (X, Δ), where Δ is an effective $\mathbb{Q}$-divisor on normal X. The first to undertake this project were Hara and Watanabe in [15].

For any effective $\mathbb{Q}$-divisor D, we have an inclusion $\mathcal{O}_X \to \mathcal{O}_X(\lfloor D \rfloor)$ given by a section of $\lfloor D \rfloor$, and hence composition maps induced by Frobenius

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X \to F^e_* \mathcal{O}_X(\lfloor D \rfloor),$$

where $\lfloor D \rfloor$ denotes the "round-down" of D. Hara and Watanabe then defined:

Definition 1 *(i.) The pair (X, Δ) is F-pure if and only if the map $\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor)$ splits for every e.*

(ii.) For X affine, the pair (X, Δ) is (strongly) F-regular if and only if for every non-zero $c \in \mathcal{O}_X(X)$, the composition map

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor) \xrightarrow{\cdot c} F^e_* \mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor)$$

splits for all sufficiently large e.

When $\Delta = 0$, this recovers the classical definitions of F-purity and (strong) F-regularity.

The hard part is to realize what these definitions should be. Having done so, Hara and Watanabe then show that many of the basic properties of F-pure and F-regular varieties generalize to pairs. Indeed, they show that the proofs of Theorems 1 and 2 generalize easily to this setting, interpreting log-terminal to be "Kawamata log terminal" [15].

A notion of *divisorially F-regular* is also introduced in [15], which Takagi has recently shown to correspond to plt singularities [33].

3 Multiplier Ideals and Test Ideals.

The prime characteristic analogs of multiplier ideals are test ideals. The test ideal of a commutative ring of prime characteristic is an important aspect of the theory of tight closure, defined originally by Hochster and Huneke without any regard for "pairs". By definition, the test ideal of R is the set of all elements c such that $cI^* \subset I$ for all ideals I in R. Here I^* denotes the tight closure of I, but its definition is not important for the discussion. The non-obvious fact the that test ideal of virtually any ring is non-zero is proved by Hochster and Huneke in [17].

Obviously the test ideal is the unit ideal if and only if all ideals of R are tightly closed, that is, if and only if the ring is (weakly) F-regular. Indeed, the test ideal defines the non-F-regular locus of Spec R, endowing it with a natural scheme structure[2]. Similarly, the multiplier ideal of a normal $\mathbb{Q}$-Gorenstein scheme defines the non-log-terminal locus, where here we mean multiplier ideal of the pair $(X, 0)$. Thus Theorem 2 can be interpreted loosely as saying that, after reducing to prime characteristic, the multiplier ideal and the test ideal have the same radical. In fact, a much stronger statement holds:

Theorem 3 *Let $X_{\mathbb{C}}$ be a normal complex (affine) variety for which the canonical class is $\mathbb{Q}$-Cartier. Let $J_{\mathbb{C}}$ be its multiplier ideal, and J_A be the corresponding ideal on X_A, for any choice of A as above. Then, for all the closed points in some dense open subset of Spec A, the pullback of J_A to each of the corresponding prime characteristic models is the test ideal of that model.*

This theorem was proved independently in [13] and [32].

The theory of tight closure, and hence of test ideals, has been generalized to pairs by Hara and Yoshida [16]. Fix a domain of prime characteristic p, and an ideal $\mathfrak{a}$. For each ideal $J \subset R$, they define the tight closure of J with respect to the pair $(R, \mathfrak{a})$ as the ideal

$$J^{*\mathfrak{a}} = \{z \mid \textit{there exists } c \neq 0 \textit{ s.t. } c\mathfrak{a}^{p^e} z^{p^e} \in J^{[p^e]} \textit{ for all } \ e \gg 0\},$$

where $J^{[p^e]} = F^e_* F^{e*}(J)$ is the ideal of R generated by the p^e-th powers of the generators of J. Taking $\mathfrak{a}$ to the unit ideal of R, we arrive at Hochster and Huneke's original definition of the tight closure of J.[3] They also define tight closures for pairs $(R, \mathfrak{a}^t)$, where t is a positive rational number by replacing $\mathfrak{a}^{p^e}$ by $\mathfrak{a}^{\lceil tp^e \rceil}$ in the definition above, the importance of allowing fractional coefficients for multiplier ideals being well-known.

Having defined tight closure, Hara and Yoshida then define the test ideal of the pair $(R, \mathfrak{a})$ to be the ideal $\tau(\mathfrak{a}) \subset R$ of all elements c such that $cJ^{\mathfrak{a}*} \subset J$ for all ideals J, and show that many of the basic properties of test ideals hold in this setting, including the obvious generalization of Theorem 3. The theory is further developed and refined in the papers papers [36] and [14]. Many properties of multiplier ideals,[4] such as the subadditivity property, the restriction theorem, the Briancon-Skoda theorem, the description in the toric case are proved for test ideals using simple "frobenius"

[2] Here, we brush aside the difficulties regarding the behavior of tight closure under localization, and assume that weak and strong F-regularity are equivalent.

[3] We caution the reader that the terminology is misleading in this more general setting; this is not actually a closure operation on J if $\mathfrak{a} \neq (1)$. That is, $J^{\mathfrak{a}*} \neq (J^{\mathfrak{a}*})^{\mathfrak{a}*}$ in general.

[4] See the book [24].

arguments. Thus issues of resolution of singularities and vanishing theorems are avoided. So many of the techniques of multiplier ideals that have proven so powerful in complex geometry can be now carried out in prime characteristic as well.

Furthermore, several new results for complex varieties have been discovered via tight closure for pairs after reducing to prime characteristic. Notable are two results of Takagi. In [35], he proves an inversion of adjunction result for pairs (X, Z) where Z is an arbitrary subscheme of a smooth complex variety X. In [34], he shows the "subadditivity" property for multiplier ideals can be adapted to the singular case by multiplying by the Jacobian ideal. No direct proofs of these results are known in the complex case, an **interesting challenge** for complex geometers.

4 F-thresholds.

We now briefly discuss prime characteristic analogs of log-canonical thresholds and their refinements called "jumping coefficients" as defined in [10].

Let Z be a subscheme of a smooth complex variety $X_{\mathbb{C}}$. The log canonical threshold of a pair $(X_{\mathbb{C}}, Z)$ is the supremum of all rational numbers t, such that the pair $(X_{\mathbb{C}}, tZ)$ is log canonical. Equivalently, it is the infimum over all t such that the multiplier ideal of the pair $(X_{\mathbb{C}}, tZ)$ is non-trivial. Naturally, if X has prime characteristic, one could consider an analogous number using F-pure pairs. This is done by Takagi and Watanabe, who named this number the *F-pure threshold* of a pair and proved some basic properties analogous to those of log canonical thresholds [37].

The jumping coefficients of the pair $(X_{\mathbb{C}}, Z)$ are the values of t where the multiplier ideal of $(X_{\mathbb{C}}, tZ)$ makes a discrete jump—that is, the values of t such that $\mathcal{J}(X_{\mathbb{C}}, tZ) \neq \mathcal{J}(X_{\mathbb{C}}, (t-\epsilon)Z)$ for every $\epsilon > 0$. In prime characteristic, Mustata, Takagi and Watanabe introduced a notion of *F-thresholds* for pairs which turn out to be jumping coefficients for test ideals [26]. Fix a pair $(R, \mathfrak{a})$, where R is a regular local ring of prime characteristic p. For every ideal $J \subset m$ containing $\mathfrak{a}$ in its radical, they define the F-threshold of the pair $(R, \mathfrak{a})$ with respect to J as

$$\lim_{e \to \infty} \frac{max\{r | \mathfrak{a}^r \nsubseteq J^{[p^e]}\}}{p^e},$$

where again $J^{[p^e]}$ is the ideal generated by the p^e-th powers of the generators of J. In particular, taking $J = m$, they recover the F-pure threshhold. Like jumping coeffients, F-thresholds turn out to be discrete and rational [7]; this is not obvious from the definition, but is based on a different way to think about F-thresholds inspired by the work on D-module generators of Alvarez-Montaner, Blickle and Lyubeznik [2].

There are many subtleties involving F-thresholds, which depend on the characteristic. For example, fix a polynomial $f \in \mathbb{Z}[X_1, \ldots, x_n]$, and consider the F-thresholds of the pairs $(\mathbb{F}_p[[X_1, \ldots, x_n]], f \bmod p)$, as we vary the prime number p. It is believed but **not known** that there are infinitely many p for which the F-pure threshold of this pair agrees with the log canonical threshold of the corresponding complex hypersurface. This problem harks back to the subtle arithmetic nature of F-purity.

For a fixed chacteristic p, another interesting **open problem** is whether the F-pure thresholds of a family of divisors on a fixed variety can have accumulation points. The corresponding problem for log canonical thresholds is well known in birational geometry.

5 Connections with D-modules.

The connection between the log canonical threshold c of a complex polynomial f and its Bernstein-Sato polynomial b_f is well-known: $-c$ is always a root of b_f, in fact the smallest root greater than -1. Also the jumping coefficients of f are roots of b_f, as is shown in [10]. Mustata, Takagi and Watanabe study F-pure thresholds for varying p, connecting them to the roots of the Bernstein-Sato polynomial of (the complex polynomial) f as well [26]. In this way, they produce new roots of the Bernstein-Sato polynomial. In the case where f is monomial, this process recovers all the roots [9]. An interesting **open problem** would be to understand which roots one recovers in general.

The theory of D-modules in prime characteristic appears to have deep connections with the theory of F-regularity, F-purity and tight closure. This possibility is first considered in [28], where an F-pure ring is shown to be F-regular if and only if it is simple as a D-module. Deeper connections are reported in the beautiful work of Manuel Blickle [5], [6], including an interesting connection with the intersection homology D-module, which has had nice applications to equivariant D-modules in prime characteristic [23]. Other interesting papers on D-modules and their interaction with Frobenius are Lyubeznik's paper on "F-modules", Emerton and Kisin's paper on the prime characteristic Riemann-Hilbert correspondence [11], and Blickle's Cartier isomorphism for toric varieties [4]. Presumably, there are as yet unrevealed connections with F-crystals and crystalline coholomogy—some connections are discussed in Blickle's thesis [3].

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