The Riemann Hypothesis

The American Institute of Mathematics

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This document is a preliminary version of a planned comprehensive resource on the Riemann Hypothesis.

Suggestions and contributions are welcome and can be sent to rh(at)aimath.org

Table of Contents

The Riemann Hypothesis is an assertion about the zeros of the Riemann ζ-function. Generalizations of the ζ-function have been discovered, for which the analogue of the Riemann Hypothesis is also conjectured.

These generalizations of the ζ-function are known as "zeta-functions" or "L-functions." In this section we describe attempts to determine the collection of functions that deserve to be called L-functions.

A.1 Terminology and basic properties

For a more detailed discussion, see the articles on the Selberg class[11] and on automorphic L-functions[12].

A.1.a Functional equation of an L-function. The Riemann ζ-function[4] has functional equation

$$\begin{aligned} \xi(s) &= \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \\ &= \xi(1-s). \end{aligned} \tag{1}$$

Dirichlet L-functions[9] satisfy the functional equation

$$\begin{aligned} \xi(s,\chi) &= \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}+a)L(s,\chi) \\ &= **\,\xi(1-s,\overline{\chi}), \end{aligned} \tag{2}$$

where $a=0$ if χ is even and $a=1$ if χ is odd, and $*****$.

The Dedekind zeta function[66] of a number field K satisfies the functional equation

$$\begin{aligned} \xi_K(s) &= \left(\frac{\sqrt{|d_K|}}{2^{r_2}\pi^{n/2}}\right)^s \Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\zeta_K(s)) \\ &= \xi_K(1-s). \end{aligned} \tag{3}$$

Here r_1 and $2r_2$ are the number of real and complex conjugate embeddings $K\subset\mathbb{C}$, d_K is the discriminant, and $n=[K,\mathbb{Q}]$ is the degree of $K/\mathbb{Q}$.

L-functions associated with a newform[13] $f\in S_k(\Gamma_0(N)$ satisfy the functional equation

$$\begin{aligned} \xi(s,f) &= \left(\frac{\pi}{N}\right)^{-s}\Gamma\left(s+\frac{k-1}{2}\right)L(s,f) \\ &= \varepsilon\xi(1-s,f), \end{aligned} \tag{4}$$

where $a=0$ if χ is even and $a=1$ if χ is odd.

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[11]page 11, *The Selberg class*
[12]page 28, *Iwaniec' approach*
[4]page 7, *The Riemann zeta function*
[9]page 8, *Dirichlet L-functions*
[66]page 8, *Dedekind zeta functions*
[13]page 8, *Dirichlet series associated with holomorphic cusp forms*

L-functions associated with a Maass newform[14] with eigenvalue $\lambda = \frac{1}{4} + R^2$ on $\Gamma_0(N)$ satisfy the functional equation

$$\begin{aligned}\xi(s,f) &= \left(\frac{N}{\pi}\right)^s \Gamma\left(\frac{s+iR+a}{2}\right)\Gamma\left(\frac{s-iR+a}{2}\right)L(s,f)\\ &= \varepsilon\xi(1-s,f),\end{aligned} \tag{5}$$

where $a = 0$ if f is even and $a = 1$ if f is odd.

$GL(r)$ L-functions[44] satisfy functional equations of the form

$$\Phi(s) := \left(\frac{N}{\pi^r}\right)^{s/2} \prod_{j=1}^{r} \Gamma\left(\frac{s+r_j}{2}\right) F(s) = \varepsilon\Phi(1-s).$$

[This section needs a bit of work]

A.1.b Euler product. An *Euler product* is a representation of an L-function as a convergent infinite product over the primes p, where each factor (called the "local factor at p") is a Dirichlet series supported only at the powers of p.

The Riemann ζ-function[4] has Euler product

$$\zeta(s) = \prod_p \left(1-p^{-s}\right)^{-1}.$$

A Dirichlet L-function[9] has Euler product

$$L(s,\chi) = \prod_p \left(1-\chi(p)p^{-s}\right)^{-1}.$$

The Dedekind zeta function[66] of a number field K has Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1-N\mathfrak{p}^{-s}\right)^{-1},$$

where the product is over the prime ideals of $\mathcal{O}_K$.

An L-functions associated with a newform[13] $f \in S_k(\Gamma_0(N))$ or a Maass newform[14] $f(z)$ on $\Gamma_0(N)$ has Euler product

$$L(s,f) = \prod_{p|N} \left(1-a_p p^{-s}\right)^{-1} \prod_{p\nmid N} \left(1-a_p p^{-s} + \chi(p)p^{-2s+1}\right)^{-1}.$$

$GL(r)$ L-functions[44] have Euler products where almost all of the local factors are (reciprocals of) polynomials in p^{-s} of degree r.

[14]page 11, *Dirichlet series associated with Maass forms*
[44]page 11, *Higher rank L-functions*
[4]page 7, *The Riemann zeta function*
[9]page 8, *Dirichlet L-functions*
[66]page 8, *Dedekind zeta functions*
[13]page 8, *Dirichlet series associated with holomorphic cusp forms*
[14]page 11, *Dirichlet series associated with Maass forms*
[44]page 11, *Higher rank L-functions*

A.1.c ξ and Z functions. The functional equation[69] can be written in a form which is more symmetric:

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{\frac{1}{2}s}\Gamma(s/2)\zeta(s) = \xi(1-s).$$

Here $\xi(s)$ is known as the Riemann ξ-function. It is an entire function of order 1, and all of its zeros lie in the critical strip.

The ξ-function associated to a general L-function is similar, except that the factor $\frac{1}{2}s(s-1)$ is omitted, since its only purpose was to cancel the pole at $s = 1$.

The Ξ function just involves a change of variables: $\Xi(z) = \xi(\frac{1}{2} + iz)$. The functional equation now asserts that $\Xi(z) = \Xi(-z)$.

The Hardy Z-function is defined as follows. Let

$$\vartheta = \vartheta(t) = \frac{1}{2}\arg(\chi(\frac{1}{2} + it)),$$

and define

$$Z(t) = e^{i\vartheta}\zeta(\frac{1}{2} + it) = \chi(\frac{1}{2} + it)^{-\frac{1}{2}}\zeta(\frac{1}{2} + it).$$

Then $Z(t)$ is real for real t, and $|Z(t)| = |\zeta(\frac{1}{2} + it)$.

Plots of $Z(t)$ are a nice way to picture the ζ-function on the critical line. $Z(t)$ is called `RiemannSiegelZ[t]` in Mathematica.

A.1.d Critical line and critical strip. The *critical line* is the line of symmetry in the functional equation[69] of the L-function. In the usual normalization the functional equation associates s to $1 - s$, so the critical line is $\sigma = \frac{1}{2}$.

In the usual normalization the Dirichlet series and the Euler product converge absolutely for The functional equation maps $\sigma > 1$ to $\sigma < 0$. The remaining region, $0 < \sigma < 1$ is known as the *critical strip*.

By the Euler product there are no zeros in $\sigma > 1$, and by the functional equation there are only trivial zeros in $\sigma < 0$. So all of the nontrivial zeros are in the critical strip, and the Riemann Hypothesis asserts that the nontrivial zeros are actually on the critical line.

A.1.e Trivial zeros. The trivial zeros of the ζ-function are at $s = -2, -4, -6, ...$

The trivial zeros correspond to the poles of the associated Γ-factor.

A.1.f Zero counting functions. Below we present the standard notation for the functions which count zeros of the zeta-function.

Zeros of the zeta-function in the critical strip are denoted

$$\rho = \beta + i\gamma.$$

It is common to list the zeros with $\gamma > 0$ in order of increasing imaginary part as $\rho_1 = \beta_1 + i\gamma_1$, $\rho_2 = \beta_2 + i\gamma_2$,.... Here zeros are repeated according to their multiplicity.

We have the zero counting function

$$N(T) = \#\{\rho = \beta + i\gamma \ : \ 0 < \gamma \le T\}.$$

[69] page 4, *Functional equation of an L-function*

[69] page 4, *Functional equation of an L-function*

In other words, $N(T)$ counts the number of zeros in the critical strip, up to height T. By the functional equation and the argument principle,

$$N(T) = \frac{1}{2\pi} T \log\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + S(T) + O(1/T),$$

where

$$S(T) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + it\right),$$

with the argument obtained by continuous variation along the straight lines from 2 to $2+iT$ to $\frac{1}{2}+iT$. Von Mangoldt proved that $S(T) = O(\log T)$, so we have a fairly precise estimate of the number of zeros of the zeta-function with height less than T. Note that Von Mangoldt's estimate implies that a zero at height T has multiplicity $O(\log T)$. That is still the best known result on the multiplicity of zeros. It is widely believed that all of the zeros are simple.

A number of related zero counting functions have been introduced. The two most common ones are:

$$N_0(T) = \#\{\rho = \frac{1}{2} + i\gamma \ : \ 0 < \gamma \le T\},$$

which counts zeros on the critical line up to height T. The Riemann Hypothesis is equivalent to the assertion $N(T) = N_0(T)$ for all T. Selberg proved that $N_0(T) \gg N(T)$. At present the best result of this kind is due to Conrey [90g:11120], who proved that

$$N_0(T) \ge 0.40219\, N(T)$$

if T is sufficiently large.

And,

$$N(\sigma, T) = \#\{\rho = \beta + i\gamma \ : \ \beta > \sigma \textit{ and } 0 < \gamma \le T\},$$

which counts the number of zeros in the critical strip up to height T, to the right of the σ-line. Riemann Hypothesis is equivalent to the assertion $N(\frac{1}{2}, T) = 0$ for all T.

For more information on $N(\sigma, T)$, see the article on the density hypothesis[25].

A.2 Arithmetic L-functions

Loosely speaking, arithmetic L-functions are those Dirichlet series with appropriate functional equations and Euler products which should satisfy a Riemann Hypothesis. Selberg has given specific requirements[11] which seem likely to make this definition precise. Arithmetic L-functions arise in many situations: from the representation theory of groups associated with number fields, from automorphic forms on arithmetic groups acting on symmetric spaces, and from the harmonic analysis on these spaces.

A.2.a The Riemann zeta function. The Riemann zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1},$$

where $s = \sigma + it$, the product is over the primes, and the series and product converge absolutely for $\sigma > 1$.

[25] page 16, *The Density Hypothesis*

[11] page 11, *The Selberg class*

The use of $s = \sigma + it$ as a complex variable in the theory of the Riemann ζ-function has been standard since Riemann's original paper.

A.2.b Dirichlet L-functions. Dirichlet L-functions are Dirichlet series of the form

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

where χ is a primitive Dirichlet character to a modulus q. Equivalently, χ is a function from the natural numbers to the complex numbers, which is periodic with period q (i.e. $\chi(n+q) = \chi(n)$ for all $n \geq 1$), completely multiplicative (i.e. $\chi(mn) = \chi(m)\chi(n)$ for all natural numbers m and n), which vanishes at natural numbers which have a factor > 1 in common with q and which do not satisfy $\chi(m+q_1) = \chi(m)$ for all numbers m, q_1 with $q_1 < q$ and $(m, q_1) = 1$. The last condition gives the primitivity. Also, we do not consider the function which is identically 0 to be a character.

If the modulus $q = 1$ then $\chi(n) = 1$ for all n and $L(s,\chi) = \zeta(s)$. If $q > 1$, then the series converges for all s with $\sigma > 0$ and converges absolutely for $\sigma > 0$.

$L(s,\chi)$ has an Euler product

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi)p}{p^s}\right)^{-1}.$$

A.2.c Dedekind zeta functions. Let K be a number field (ie, a finite extension of the rationals $\mathbb{Q}$), with ring of integers $\mathcal{O}_K$. The Dedekind zeta function of K is given by

$$\zeta_K(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s},$$

for $\sigma > 1$, where the sum is over all integral ideals of $\mathcal{O}_K$, and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$.

A.2.d GL(2) L-functions. We call GL(2) L-functions those Dirichlet series with functional equations and Euler products each of whose factors is the reciprocal of a degree two polynomial in p^{-s}. These are associated with (i.e. their coefficients are the Fourier coefficients) of cusp forms on congruence subgroups of $SL(2,Z)$ which are eigenfunctions of the Hecke operators and of the Atkin-Lehner operators (newforms).

A.2.d.i Dirichlet series associated with holomorphic cusp forms. **Level one modular forms.** A cusp form of weight k for the full modular group is a holomorphic function f on the upper half-plane which satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all integers a, b, c, d with $ad - bc = 1$ and also has the property that $\lim_{y\to\infty} f(iy) = 0$. Cusp forms for the whole modular group exist only for even integers $k = 12$ and $k \geq 16$. The cusp forms of a given weight k of this form make a complex vector space S_k of dimension $[k/12]$ if $k \neq 2$ mod 12 and of dimension $[k/12] - 1$ if $k = 2$ mod 12. Each such vector space has a special basis H_k of Hecke eigenforms which consist of functions $f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz)$ for which

$$a_f(m) a_f(n) = \sum_{d|(m,n)} d^{k-1} a_f(mn/d^2).$$

The Fourier coefficients $a_f(n)$ are real algebraic integers of degree equal to the dimension of the vector space $= \#H_k$. Thus, when $k = 12, 16, 18, 20, 22, 26$ the spaces are one dimensional and the coefficients are ordinary integers. The L-function associated with a Hecke form f of weight k is given by

$$L_f(s) = \sum_{n=1}^{\infty} a_f(n)/n^{(k-1)/2} n^s = \prod_p \left(1 - \frac{a_f(p)/p^{(k-1)/2}}{p^s} + \frac{1}{p^{2s}}\right)^{-1}.$$

By Deligne's theorem $a_f(p)/p^{(k-1)/2} = 2\cos\theta_f(p)$ for a real $\theta_f(p)$. It is conjectured (Sato-Tate) that for each f the $\{\theta_f(p) : p \text{ prime}\}$ is uniformly distributed on $[0, \pi)$ with respect to the measure $\frac{2}{\pi}\sin^2\theta d\theta$. We write $\cos\theta_f(p) = \alpha_f(p) + \overline{\alpha_f(p)}$ where $\alpha_f(p) = e^{i\theta_f(p)}$; then

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha_f(p)}}{p^s}\right)^{-1}.$$

The functional equation satisfied by $L_f(s)$ is

$$\xi_f(s) = (2\pi)^{-s}\Gamma(s + (k-1)/2)L_f(s) = (-1)^{k/2}\xi_f(1-s).$$

Higher level forms. Let $\Gamma_0(q)$ denote the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integers a, b, c, d satisfying $ad - bc = 1$ and $q \mid c$. This group is called the *Hecke congruence group.* A function f holomorphic on the upper half plane satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all matrices in $\Gamma_0(q)$ and $\lim_{y\to\infty} f(iy) = 0$ is called a cusp form for $\Gamma_0(q)$; the space of these is a finite dimensional vector space $S_k(q)$. The space S_k above is the same as $S_k(1)$. Again, these spaces are empty unless k is an even integer. If k is an even integer, then

$$\dim S_k(q) = \frac{(k-1)}{12}\nu(q) + \left(\left[\frac{k}{4}\right] - \frac{k-1}{4}\right)\nu_2(q) + \left(\left[\frac{k}{3}\right] - \frac{k-1}{3}\right)\nu_3(q) - \frac{\nu_\infty(q)}{2}$$

where $\nu(q)$ is the index of the subgroup $\Gamma_0(q)$ in the full modular group $\Gamma_0(1)$:

$$\nu(q) = q\prod_{p|q}\left(1 + \frac{1}{p}\right);$$

$\nu_\infty(q)$ is the number of *cusps* of $\Gamma_0(q)$:

$$\nu_\infty(q) = \sum_{d|q} \phi((d, q/d));$$

$\nu_2(q)$ is the number of inequivalent *elliptic points* of order 2:

$$\nu_2(q) = \begin{cases} 0 & \text{if } 4 \mid q \\ \prod_{p|q}(1 + \chi_{-4}(p)) & \text{otherwise} \end{cases}$$

and $\nu_3(q)$ is the number of inequivalent *elliptic points* of order 3:

$$\nu_3(q) = \begin{cases} 0 & \text{if } 9 \mid q \\ \prod_{p|q}(1 + \chi_{-3}(p)) & \text{otherwise.} \end{cases}$$

It is clear from this formula that the dimension of $S_k(q)$ grows approximately linearly with q and k.

For the spaces $S_k(q)$ the issue of primitive forms and imprimitive forms arise, much as the situation with characters. In fact, one should think of the Fourier coefficients of cusp forms as being a generalization of characters. They are not periodic, but they act as harmonic detectors, much as characters do, through their orthogonality relations (below). Imprimitive cusp forms arise in two ways. Firstly , if $f(z) \in S_k(q)$, then $f(z) \in S_k(dq)$ for any integer $d > 1$. Secondly, if $f(z) \in S_k(q)$, then $f(dz) \in S_k(\Gamma_0(dq))$ for any $d > 1$. The dimension of the subspace of primitive forms is given by

$$\dim S_k^{\mathrm{new}}(q) = \sum_{d|q} \mu_2(d) \dim S_k(q/d)$$

where $\mu_2(n)$ is the multiplicative function defined for prime powers by $\mu_2(p^e) = -2$ if $e = 1$, $= 1$ if $e = 2$, and $= 0$ if $e > 2$. The subspace of newforms has a Hecke basis $H_k(q)$ consisting of primitive forms, or newforms, or Hecke forms. These can be identified as those f which have a Fourier series

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz)$$

where the $a_f(n)$ have the property that the associated L-function has an Euler product

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_f(n)/n^{(k-1)/2}}{n^s} = \prod_{p\nmid q} \left(1 - \frac{a_f(p)/p^{(k-1)/2}}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \prod_{p|q} \left(1 - \frac{a_f(p)/p^{(k-1)/2}}{p^s}\right)^{-1}.$$

The functional equation satisfied by $L_f(s)$ is

$$\xi_f(s) = (2\pi/\sqrt{q})^{-s} \Gamma(s + (k-1)/2) L_f(s) = \pm(-1)^{k/2} \xi_f(1-s).$$

A.2.d.i.A Examples. See the website[1] for many specific examples.

Ramanujan's tau-function defined implicitly by

$$x \prod_{n=1}^{\infty} (1 - x^n)^{24} = \sum_{n=1}^{\infty} \tau(n) x^n$$

also yields the simplest cusp form. The associated Fourier series $\Delta(z) := \sum_{n=1}^{\infty} \tau(n) \exp(2\pi i n z)$ satisfies

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12} \Delta(z)$$

for all integers a, b, c, d with $ad - bc = 1$ which means that it is a cusp form of weight 12 for the full modular group.

The unique cusp forms of weights 16, 18, 20, 22, and 26 for the full modular group can be given explicitly in terms of (the Eisenstein series)

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e(nz)$$

[1] http://www.math.okstate.edu/~loriw/degree2/degree2hm/degree2hm.html

and

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) e(nz)$$

where $\sigma_r(n)$ is the sum of the rth powers of the positive divisors of n:

$$\sigma_r(n) = \sum_{d|n} d^r.$$

Then, $\Delta(z)E_4(z)$ gives the unique Hecke form of weight 16; $\Delta(z)E_6(z)$ gives the unique Hecke form of weight 18; $\Delta(z)E_4(z)^2$ is the Hecke form of weight 20; $\Delta(z)E_4(z)E_6(z)$ is the Hecke form of weight 22; and $\Delta(z)E_4(z)^2E_6(z)$ is the Hecke form of weight 26. The two Hecke forms of weight 24 are given by

$$\Delta(z)E_4(z)^3 + x\Delta(z)^2$$

where $x = -156 \pm 12\sqrt{144169}$.

An example is the L-function associated to an elliptic curve $E : y^2 = x^3 + Ax + B$ where A, B are integers. The associated L-function, called the Hasse-Weil L-function, is

$$L_E(s) = \sum_{n=1}^{\infty} \frac{a(n)/n^{1/2}}{n^s} = \prod_{p \nmid N} \left(1 - \frac{a(p)/p^{1/2}}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \prod_{p|N} \left(1 - \frac{a(p)/p^{1/2}}{p^s}\right)^{-1}$$

where N is the conductor of the curve. The coefficients a_n are constructed easily from a_p for prime p; in turn the a_p are given by $a_p = p - N_p$ where N_p is the number of solutions of E when considered modulo p. The work of Wiles and others proved that these L-functions are associated to modular forms of weight 2.

A.2.d.j Dirichlet series associated with Maass forms.

A.2.e Higher rank L-functions.

A.3 The Selberg class

Selberg [94f:11085] has given an elegant a set of axioms which presumably describes exactly the set of arithmetic L-functions. He also made two deep conjectures[82] about these L-functions which have far reaching consequences.

The collection of Dirichlet series satisfying Selberg's axioms is called "The Selberg Class." This set has many nice properties. For example, it is closed under products. The elements which cannot be written as a nontrivial product are called "primitive," and every member can be factored uniquely into a product of primitive elements.

In some cases it is useful to slightly relax the axioms so that the set is closed under the operation

$$F(s) \mapsto F(s + iy)$$

for real y.

Some of the important problems concerning the Selberg Class are:

1. Show that the members of the Selberg Class are arithmetic L-functions.
2. Prove a prime number theorem[26] for members of the Selberg class.

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[82]page 12, *Selberg Conjectures*
[26]page 16, *Zeros on the $\sigma = 1$ line*

See Perelli and Kaczorowski [MR 2001g:11141], Conrey and Ghosh [95f:11064] and Chapter 7 of Murty and Murty [98h:11106] for more details and some additional consequences of Selberg's conjectures.

A.3.a Axiom 1: Dirichlet series. For $\Re(s) > 1$,

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

A.3.b Axiom 2: Analytic Continuation. $F(s)$ extends to a meromorphic function such that, for some integer m, $(s-1)^m F(s)$ is an entire function of finite order.

A.3.c Axiom 3: Functional Equation. There exist numbers $Q > 0$, $\alpha_j > 0$, and $\Re(r_j) \geq 0$, such that

$$\Phi(s) := Q^s \prod_{j=1}^{d} \Gamma(\alpha_j + r_j) F(s)$$

satisfies

$$\Phi(s) = \varepsilon \overline{\Phi}(1-s).$$

Here $|\varepsilon| = 1$ and $\overline{\Phi}(z) = \overline{\Phi(\overline{z})}$.

A.3.d Axiom 4: Euler Product.

$$F(s) = \prod_{p} F_p(s),$$

where the product is over the rational primes. Here

$$F_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}\right)$$

with $b_n \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

Note that this implies $a_1 = 1$, so $F(s) = 1$ is the only constant function in the Selberg class.

A.3.e Axiom 5: Ramanujan Hypothesis. We have

$$a_n \ll n^{\varepsilon}$$

for all $\varepsilon > 0$.

A.3.f Selberg Conjectures. Selberg has made two conjectures concerning the Dirichlet series in the Selberg class $\mathcal{S}$:

Conjecture A. *For each $F \in \mathcal{S}$ there exists an integer n_F such that*

$$\sum_{p \leq X} \frac{|a_p(F)|^2}{p} = n_F \log\log x + O(1).$$

Conjecture A follows from

Conjecture B. *If $F \in \mathcal{S}$ is primitive, then $n_F = 1$, and if $F, F' \in \mathcal{S}$ are distinct and primitive, then*

$$\sum_{p \le X} \frac{a_p(F)\overline{a_p(F'}}{p} = O(1).$$

The above sums are over p prime.

Conjecture B can be interpreted as saying that the primitive functions form an orthonormal system. This conjecture is very deep. It implies, among other things, Artin's conjecture on the holomorphy of non-abelian L-functions [98h:11106], and that the factorization of elements into primitives is unique [95f:11064].

If you extend the Selberg Class to include $G(s) = F(s+iy)$ for $F \in \mathcal{S}$ and y real, then Conjecture B with $F'(s) = \zeta(s-iy)$ is equivalent to a prime number theorem[26] for $F(s)$.

A.4 Analogues of zeta-functions

A.4.a Dynamical zeta-functions. For several decades now there has been interest amongst physicists and mathematicians to study the statistics of the eigenvalue spectra of physical systems - especially those with a classical counterpart which displays chaotic behaviour. One particular goal is to try to detect in the statistical distribution of a set of energy eigenvalues an indication of whether the corresponding classical system behaves chaoticly or integrably. The answer was suggested by Berry and Tabor in 1977, and examined in depth by Bohigas, Giannoni and Schmit [85f:58034], and reveals that while classically integrable (non-chaotic) systems have a spectrum of uncorrelated eigenvalues, the spectra of classically chaotic systems show the characteristic correlations seen in the spectra of ensembles of random matrices. In particular, the semiclassical spectra of physical systems which are not symmetric under time-reversal have the same local statistics (those of the ensemble of unitary matrices, $U(N)$, with Haar measure) as the zeros high on the critical line of the Riemann zeta function and other L-functions.

Thus for a given physical system which possesses the correct symmetries, we can construct a zeta function which will have zeros correlated on a local scale like those of the Riemann zeta function. For a system with a Hamiltonian operator H and a set of eigenvalues E_n, satisfying

$$H\psi_n(\mathbf{r}) = \mathbf{E_n}\psi_\mathbf{n}(\mathbf{r}),$$

with wavefunctions ψ_n, a natural function to study is the spectral determinant

$$\Delta(E) \equiv \det[A(E,H)(E-H)] = \prod_j [A(E,E_j)(E-E_j)],$$

where A has no real zeros and is included so as to make the product converge.

A semiclassical expression for the spectral determinant can be given using only classical attributes of the system in question. Following Berry and Keating [92m:81053], we can write

$$\Delta(E) \sim B(E)\exp(-i\pi\overline{N}(E)) \prod_p \exp\left(-\sum_{k=1}^{\infty} \frac{\exp(ikS_p/\hbar)}{k\sqrt{|\det(M_p^k - 1)|}} \right).$$

[26] page 16, *Zeros on the $\sigma = 1$ line*

Here $B(E)$ is a real function with no zeros and

$$\frac{d\overline{N}(E)}{dE} = \overline{d}(E),$$

the mean density of eigenvalues. The sum is over the periodic orbits of the classical system, S_p is the action of the orbit p, and M_p is the monodromy matrix which describes flow linearized around the orbit.

We can now define the dynamical zeta function

$$Z(E) = \prod_p \exp\left(-\sum_{k=1}^{\infty} \frac{\exp(ikS_p/\hbar)}{k\sqrt{|\det(M_p^k - 1)|}}\right).$$

Since $B(E)$ has no zeros, the zeros of $Z(E)$ are the eigenvalues of the system under investigation.

A.4.b Spectral zeta functions.

Chapter B: Riemann Hypotheses

B.1 The Riemann Hypothesis and its generalizations

The Riemann Hypothesis for $L(s)$ is the assertion that the nontrivial zeros[64] of $L(s)$ lie on the critical line[64]. For historical reasons there are names given to the Riemann hypothesis for various sets of L-functions. For example, the Generalized Riemann Hypothesis (GRH)[17] is the Riemann Hypothesis for all Dirichlet L-functions[9]. More examples collected below.

In certain applications there is a fundamental distinction between nontrivial zeros on the real axis and nontrivial zeros with a positive imaginary part. Here we use the adjective *modified* to indicate a Riemann Hypothesis except for the possibility of nontrivial zeros on the real axis. Thus, the Modified Generalized Riemann Hypothesis (MGRH)[17] is the assertion that all nontrivial zeros of Dirichlet L-functions[9] lie either on the critical line or on the real axis.

Nontrivial zeros which are very close to the point $s = 1$ are called Landau-Siegel zeros[36].

B.1.a The Riemann Hypothesis. The Riemann Hypothesis is the assertion that the nontrivial zeros[64] of the Riemann zeta-function[4] lie on the critical line[64] $\sigma = \frac{1}{2}$.

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[64] page 4, *Terminology and basic properties*
[64] page 4, *Terminology and basic properties*
[17] page 14, *The Generalized Riemann Hypothesis*
[9] page 8, *Dirichlet L-functions*
[17] page 14, *The Generalized Riemann Hypothesis*
[9] page 8, *Dirichlet L-functions*
[36] page 17, *Landau-Siegel zeros*
[64] page 4, *Terminology and basic properties*
[4] page 7, *The Riemann zeta function*
[64] page 4, *Terminology and basic properties*

B.1.b The Generalized Riemann Hypothesis. The Generalized Riemann Hypothesis(GRH) is the assertion that the Riemann Hypothesis[16] is true, and in addition the nontrivial zeros[64] of all Dirichlet L-functions[9] lie on the critical line[64] $\sigma = \frac{1}{2}$.

Equivalently, GRH asserts that the nontrivial zeros of all degree 1[11] L-functions lie on the critical line.

The Modified Generalized Riemann Hypothesis(MGRH) is the assertion that the Riemann Hypothesis[16] is true, and in addition the nontrivial zeros[64] of all Dirichlet L-functions[9] lie either on the critical line[64] $\sigma = \frac{1}{2}$ or on the real axis.

B.1.c The Extended Riemann Hypothesis. The Extended Riemann Hypothesis(ERH) is the assertion that the nontrivial zeros of the Dedekind zeta function[66] of any algebraic number field lie on the critical line.

Note that ERH includes RH because the Riemann zeta function is the Dedekind zeta function of the rationals.

[[more needs to be said here and also in the section where the Dedekind zeta function is defined]]

B.1.d The Grand Riemann Hypothesis. The Grand Riemann Hypothesis is the assertion that the nontrivial zeros of all automorphic L-functions[12] lie on the critical line.

The Modified Grand Riemann Hypothesis is the assertion that the nontrivial zeros of all automorphic L-functions[12] lie on the critical line or the real line.

It is widely believed that all global L-functions[53] are automorphic L-functions[12]. Presumably this also coincides with the Selberg class[11].

B.2 Other statements about the zeros of L-functions

The Riemann Hypothesis[55] is the strongest possible statement about the horizontal distribution of the nontrivial zeros[64] of an L-function. In this section we collect together various weaker assertions. Each of these statements arises in a natural way, usually due to a relationship with the prime numbers.

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[16] page 14, *The Riemann Hypothesis*
[64] page 4, *Terminology and basic properties*
[9] page 8, *Dirichlet L-functions*
[64] page 4, *Terminology and basic properties*
[11] page 11, *The Selberg class*
[16] page 14, *The Riemann Hypothesis*
[64] page 4, *Terminology and basic properties*
[9] page 8, *Dirichlet L-functions*
[64] page 4, *Terminology and basic properties*
[66] page 8, *Dedekind zeta functions*
[12] page 28, *Iwaniec' approach*
[12] page 28, *Iwaniec' approach*
[53] page 7, *Arithmetic L-functions*
[12] page 28, *Iwaniec' approach*
[11] page 11, *The Selberg class*
[55] page 14, *Riemann Hypotheses*
[64] page 4, *Terminology and basic properties*

Examples include zeros on[26] or near[25] the $\sigma = 1$ line, zeros on[35] or near[25] the critical line, and zeros on the real axis[36].

B.2.a Quasi Riemann Hypothesis. The term "Quasi Riemann Hypothesis" for $L(s)$ is sometimes used to mean that $L(s)$ has no zeros in a half-plane $\sigma > \sigma_0$, for some $\sigma_0 < 1$.

B.2.b 100 percent hypothesis. The 100% Hypothesis for $L(s)$ asserts that

$$N_0(T) \sim N(T).$$

Here $N(T)$ is the zero counting function[64] for $L(s)$, and $N_0(T)$ counts only the zeros on the critical line[64]. In other words, 100 percent of the nontrivial zeros (in the sense of density) are on the critical line. An equivalent assertion is

$$N(T) - N_0(T) = o(T \log T),$$

which makes it clear that the 100% Hypothesis still allows quite a few zeros off the critical line.

The term "100% Hypothesis" is not standard.

In contrast to most of the other conjectures in this section, the 100% Hypothesis is not motivated by applications to the prime numbers. Indeed, at present there are no known consequences of this hypothesis.

B.2.c The Density Hypothesis. The Density Hypothesis is the assertion

$$N(\sigma, T) = O(T^{2(1-\sigma)+\varepsilon})$$

for all $\varepsilon > 0$. Note that this is nontrivial only when $\sigma > \frac{1}{2}$.

The Density Hypothesis follows from the Lindelöf Hypothesis. The importance of the Density Hypothesis is that, in terms of bounding the gaps between consecutive primes, the density hypothesis appears to be as strong as the Riemann Hypothesis.

Results on $N(\sigma, T)$ are generally obtained from mean values of the zeta-function. Further progress in this direction, particularly for σ close to $\frac{1}{2}$, appears to be hampered by the great difficulty in estimating the moments of the zeta-function on the critical line.

See Titchmarsh [88c:11049], Chapter 9, for an extensive discussion.

B.2.d Zeros on the $\sigma = 1$ line. By the Euler Product[70], the L-function $L(s)$ does not vanish in the half-plane $\sigma > 1$. Thus, the simplest nontrivial assertion about the zeros of $L(s)$ is that $L(s)$ does not vanish on the $\sigma = 1$ line. Such a result is known as a *Prime Number Theorem* for $L(s)$. The name arises as follows. The classical Prime Number Theorem(PNT):

$$\pi(x) := \sum_{p \le X} 1$$

[26]page 16, *Zeros on the $\sigma = 1$ line*
[25]page 16, *The Density Hypothesis*
[35]page 16, *100 percent hypothesis*
[25]page 16, *The Density Hypothesis*
[36]page 17, *Landau-Siegel zeros*
[64]page 4, *Terminology and basic properties*
[64]page 4, *Terminology and basic properties*
[70]page 5, *Euler product*

$$\sim \quad \frac{X}{\log X},$$

where the sum is over the primes p, is equivalent to the assertion that $\zeta(s) \neq 0$ when $\sigma = 1$. The deduction of the PNT from the nonvanishing involves applying a Tauberian theorem to ζ'/ζ. The Tauberian Theorem requires that ζ'/ζ be regular on $\sigma = 1$, except for the pole at $s = 1$.

The Prime Number Theorem for $\zeta(s)$ was proven by Hadamard and de la Valee Poissin in 1896. Jacquet and Shalika [55 #5583] proved the corresponding result for L-functions associated to automorphic representations on $GL(n)$. It would be significant to prove such a result for the Selberg Class[11].

B.2.e Landau-Siegel zeros. A zero of $L(s,\chi)$ which is very close to $s = 1$ is called a *Landau-Siegel zero*, often shortened to "Siegel zero." [this section will be expanded]

B.2.f The vertical distribution of zeros of L-functions. The Riemann Hypothesis[55] is an assertion about the horizontal distribution of zeros of L-functions[2]. The question of the vertical distribution of the zeros may be even more subtle. Questions of interest include the neighbor spacing of zeros, the very large and very small gaps between zeros, correlations among zeros, etc.

The current view is that Random Matrix Theory provides the description of these and many other properties of L-functions and their zeros. This is discussed on the web site L-functions and Random Matrix Theory[2].

B.3 The Lindelof hypothesis and breaking convexity

B.4 Perspectives on the Riemann Hypothesis

An attraction of the Riemann Hypothesis is that it arises naturally in many different contexts. Several examples are given below.

B.4.a Analytic number theory. The motivation for studying the zeros of the zeta-function is the precise relationship between the zeros of $\zeta(s)$ and the errot term in the prime number theorem[26]. Define the Von Mangoldt function by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & otherwise, \end{cases}$$

where p is a prime number. Then

$$\begin{aligned} \psi(X) := & \quad \sum_{n \leq X} \Lambda(n) \\ = & \quad Li(X) + O(X^{\sigma_0+\varepsilon}) \end{aligned}$$

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[11] page 11, *The Selberg class*

[55] page 14, *Riemann Hypotheses*

[2] page 4, *What is an L-function?*

[2] http://aimath.org/WWN/lrmt/

[26] page 16, *Zeros on the $\sigma = 1$ line*

if and only if $\zeta(s)$ does not vanish for $\sigma > \sigma_0$. There $Li(X)$ is the logarithmic integral

$$Li(X) = \int_2^X \frac{dt}{\log t}.$$

In particular, the best possible error term in the prime number theorem is $O(X^{\frac{1}{2}+\varepsilon})$, which is equivalent to the Riemann Hypothesis.

Similarly, the Generalized Riemann Hypothesis[17] is equivalent to the best possible error term for the counting function of primes in arithmetic progressions.

Riemann Hypotheses for other L-functions have not yet been shown to be strongly connected to the distribution of the prime numbers.

B.4.b Physics and zeros of the zeta-function. The local statistics of the imaginary parts γ_n of the complex zeros of the Riemann zeta function show the characteristic distribution seen in the eigenvalues of a matrix pulled at random from the unitary group endowed with Haar measure. This collection of matrices is called by physicists the CUE: circular unitary ensemble. For more details on the statistics of the zeros of the zeta function and other L-functions, see L-functions and Random Matrix Theory[3].

These very same CUE statistics are also seen on a local scale when one studies the distribution of the semiclassical eigenvalues E_n of quantum systems having classical analogues that display chaotic behaviour and are not symmetric under time-reversal.

This suggests that the γ_n can be construed as the eigenvalues of some Hermitean operator which is itself obtained by quantizing a classical dynamical system sharing the properties mentioned above: chaoticity and no time-reversal symmetry.

Furthermore, the long-range statistics of γ_n for the Riemann zeta function depend on the prime numbers in a manner which is very accurately predicted by formulae with analogues in semiclassical periodic orbit theory. This suggests that the periodic orbits of the hypothetical system underlying the Riemann zeta function would be determined by the positions of the prime numbers.

Clearly the identification of a dynamical system which when quantized produced a Hermitean operator with eigenvalues E_n related to the complex Riemann zeros ρ_n by $iE_n = \rho_n - 1/2$ would lead to a proof of the Riemann Hypothesis. In the field of quantum chaos studies are made of the very systems which are relevant in such a search, and results from this field suggest what many of the characteristics of such a system should be. For a detailed review of these issues and further references see Berry and Keating [2000f:11107].

B.4.c Probability.

B.4.d Fractal geometry.

Chapter C: Equivalences to the Riemann Hypothesis

The Riemann Hypothesis has been shown to be equivalent to an astounding variety of statements in several different areas of mathematics. Some of those equivalences are nearly trivial. For example, RH is equivalent to the nonvanishing of $\zeta(s)$ in the half-plane $\sigma > \frac{1}{2}$. Other equivalences appear surprizing and deep. Examples of both kinds are collected below.

[17] page 14, *The Generalized Riemann Hypothesis*
[3] http://aimath.org/WWN/lrmt/

The results in the following articles will eventually find their way here:
[96g:11111] [98f:11113] [96a:11085] [95c:11105] [94i:58155] [89j:15029] [87b:11084]

C.1 Equivalences involving primes

The main point of Riemann's original paper[4] of 1859 is that the two sequences, of prime numbers on the one hand, and of zeros of ζ on the other hand, are in duality. A precise mathematical formulation of this fact is given by the so-called explicit formulas of prime number theory (Riemann, von Mangoldt, Guinand, Weil). Therefore, any statement about one of these two sequences must have a translation in terms of the other. This is the case for the Riemann hypothesis.

C.1.a The error term in the Prime Number Theorem. The Riemann hypothesis is equivalent to the following statement.

For every positive ϵ, the number $\pi(x)$ of prime numbers $\leq x$ is

$$Li(x) + O(x^{1/2+\epsilon}).$$

Here Li is the "Logarithmic integral" function, defined by

$$Li(x) := \int_0^x \frac{dt}{\log t},$$

the integral being evaluated in principal value in the neighbourhood of $x = 1$.

Roughly speaking, it means that the first half of the digits of the n-th prime are those of $Li^{-1}(n)$.

C.1.b More accurate estimates. The Riemann hypothesis is equivalent to the following statement.

$$\pi(x) - li(x) = O(\sqrt{x}\log x),$$

(von Koch, Acta Mathematica 24 (1901), 159-182).

L.Schoenfeld [56 #15581b] gave in 1976 a numerically explicit version of this equivalent form :

$$|\pi(x) - li(x)| \leq \frac{\sqrt{x}\log x}{8\pi}, \quad x \geq 2657.$$

C.2 Equivalent forms involving arithmetic functions

The Riemann hypothesis is equivalent to several statements involving average or extreme values of arithmetic functions. These are complex-valued functions, defined on the set of positive integers, often related to the factorization of the variable into a product of prime numbers.

C.2.a Averages. These equivalent statements have the following shape :

$$\sum_{n\leq x} f(n) = F(x) + O(x^{\alpha+\epsilon}), \quad x \to +\infty,$$

where f is an arithmetic function, $F(x)$ a smooth approximation to $\sum_{n\leq x} f(n)$, and α a real number.

[4] http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Zeta

C.2.a.i The von Mangoldt function. The von Mangoldt function $\Lambda(n)$ is defined as $\log p$ if n is a power of a prime p, and 0 in the other cases. Define :

$$\psi(x) := \sum_{n \leq x} \Lambda(n).$$

Then RH is equivalent to each of the following statements

$$\psi(x) = x + O(x^{1/2+\epsilon}),$$

for every $\epsilon > 0$;

$$\psi(x) = x + O(x^{1/2} \log^2 x);$$

and

$$|\psi(x) - x| \leq \frac{x^{1/2} \log^2 x}{8\pi}, \quad x > 73.2.$$

(see L. Schoenfeld [56 #15581b]).

C.2.a.j The Möbius function. The Möbius function $\mu(n)$ is defined as $(-1)^r$ if n is a product of r distinct primes, and as 0 if the square of a prime divides n. Define :

$$M(x) := \sum_{n \leq x} \mu(n).$$

Then RH is equivalent to each of the following statements

$$M(x) \ll x^{1/2+\epsilon},$$

for every positive ϵ ;

$$M(x) \ll x^{1/2} \exp(A \log x / \log\log x);$$

for some positive A.

Both results are due to Littlewood.

C.2.b Large values. The RH is equivalent to several inequalities of the following type :

$$f(n) < F(n),$$

where f is an "arithmetic", "irregular" function, and F an "analytic", "regular" function.

C.2.b.i The sum of divisors of n. Let

$$\sigma(n) = \sum_{d|n} d$$

denote the sum of the divisors of n.

G. Robin [86f:11069] showed that the Riemann Hypothesis is equivalent to

$$\sigma(n) < e^{\gamma} n \log\log n$$

for all $n \geq 5041$, where γ is Euler's constant. That inequality does not leave much to spare, for Gronwall showed

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma},$$

and Robin showed unconditionally that

$$\sigma(n) < e^{\gamma} n \log\log n + 0.6482 \frac{n}{\log\log n},$$

for $n \geq 3$.

J. Lagarias [arXiv:math.NT/0008177] elaborated on Robin's work and showed that the Riemann Hypothesis is equivalent to

$$\sigma(n) < H_n + \exp(H_n)\log(H_n)$$

for all $n \geq 2$, where H_n is the harmonic number

$$H_n = \sum_{j=1}^{n} \frac{1}{j}.$$

By definition,

$$\gamma = \lim_{n\to\infty} H_n - \log n,$$

so Lagarias' and Robin's inequalities are the same to leading order.

C.2.b.j The Euler function. The Euler function $\phi(n)$ is defined as the number of positive integers not exceeding n and coprime with n. Also, let N_k be the product of the first k prime numbers, and γ be Euler's constant.

Then RH is equivalent to each of the following statements :

$$\frac{N_k}{\phi(N_k)} > e^{\gamma} \log\log N_k,$$

for all k's ;

$$\frac{N_k}{\phi(N_k)} > e^{\gamma} \log\log N_k,$$

for all but finitely many k's.

This is due to Nicolas [85h:11053].

C.2.b.k The maximal order of an element in the symmetric group. Let $g(n)$ be the maximal order of a permutation of n objects, $\omega(k)$ be the number of distinct prime divisors of the integer k and Li be the integral logarithm.

Then RH is equivalent to each of the following statements :

$$\log g(n) < \sqrt{Li^{-1}(n)} \quad \text{for } n \text{ large enough;}$$

$$\omega(g(n)) < Li(\sqrt{Li^{-1}(n)}) \quad \text{for } n \text{ large enough.}$$

This is due to Massias, Nicolas and Robin [89i:11108].

C.3 Equivalences involving the Farey series

Equidistribution of Farey sequence: Let r_v be the elements of the Farey sequence of order N, $v = 1, 2, \ldots \Phi(N)$ where $\Phi(N) = \sum_{n=1}^{N} \phi(n)$. Let $\delta_v = r_v - v/\Phi(N)$. Then RH if and only if

$$\sum_{v=1}^{\Phi(N)} \delta_v^2 \ll N^{-1+\epsilon}.$$

Also, RH if and only if

$$\sum_{v=1}^{\Phi(N)} |\delta_v| \ll N^{1/2+\epsilon}.$$

Here is a good bibliography[5] on this subject.

C.3.a Mikolas functions.

C.3.b Amoroso's criterion. Amoroso [MR 98f:11113] has proven the following interesting equivalent to the Riemann Hypothesis. Let $\Phi_n(z)$ be the nth cyclotomic polynomial and let $F_N(z) = \prod_{n \le N} \Phi_n(z)$. Let

$$\tilde{h}(F_N) = (2\pi)^{-1} \int_{-\pi}^{\pi} \log^+ |F(e^{i\theta})| \, d\theta.$$

Then, $\tilde{h}(F_n) \ll N^{\lambda+\epsilon}$ is equivalent to the assertion that the Riemann zeta function does not vanish for $\mathrm{Re} z \ge \lambda + \epsilon$.

C.4 Weil's positivity criterion

André Weil [MR 14,727e] proved the following explicit formula (see also A. P. Guinand [MR 10,104g] which specifically illustrates the dependence between primes and zeros. Let h be an even function which is holomorphic in the strip $|\Im t| \le 1/2 + \delta$ and satisfying $h(t) = O((1 + |t|)^{-2-\delta})$ for some $\delta > 0$, and let

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-iur} \, dr.$$

Then we have the following duality between primes and zeros:

$$\sum_{\gamma} h(\gamma) = 2h(\tfrac{i}{2}) - g(0) \log \pi + \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(\tfrac{1}{4} + \tfrac{1}{2} ir) \, dr - 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} g(\log n).$$

In this formula, a zero is written as $\rho = 1/2 + i\gamma$ where $\gamma \in \mathbb{C}$; of course RH is the assertion that all of the γ are real. Using this duality Weil gave a criterion[76] for RH.

C.4.a Bombieri's refinement. Bombieri [1 841 692]has given the following version of Weil's criterion: The Riemann Hypothesis holds if and only if

$$\sum_{\rho} \hat{g}(\rho) \hat{\bar{g}}(1 - \rho) > 0$$

for every complex-valued $g(x) \in C_0^{\infty}(0, \infty)$ which is not identically 0, where

$$\hat{g}(s) = \int_0^{\infty} g(x) x^{s-1} \, dx$$

[5] http://www.math.jussieu.fr/~miw/telecom/biblio-Amoroso.html
[76] page 22, *Bombieri's refinement*

C.4.b Li's criterion. Xian-Jin Li [98d:11101] proved the following assertion: The Riemann Hypothesis is true if and only if $\lambda_n \geq 0$ for each $n = 1, 2, \ldots$ where

$$\lambda_n = \sum_\rho (1 - (1 - 1/rho)^n)$$

Another expression for λ_n is given by

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} (s^{n-1} \log \xi(s))|_{s=1}$$

and

$$\xi(s) = \frac{1}{2} s(s-1)\Gamma(s/2)\zeta(s)$$

C.5 Equivalences involving function-theoretic properties of ζ

C.5.a Speiser's criterion. A. Speiser (Math Annahlen 110 (1934) 514-521) proved that the Riemann Hypothesis is equivalent to the non-vanishing of the derivative $\zeta'(s)$ in the left-half of the critical strip $0 < \sigma < 1/2$. Levinson and Montgomery [MR 54 #5135] gave an alternative, quantitative version of this result which led to Levinson's [MR 58 #27837] discovery of his method for counting zeros on the critical line. He then proceeded to prove that at least 1/3 of the zeros of $\zeta(s)$ are on the critical line.

C.5.b Logarithmic integrals. V. V. Volchkov [MR 96g:11111] has recently proved that the truth of the Riemann hypothesis is equivalent to the equality $\int_0^\infty \int_{1/2}^\infty \frac{1-12y^2}{)1+4y^2)^3} \log(|\zeta(x + iy)|) \, dx \, dy = \pi \frac{3-\gamma}{32}$

C.5.c An inequality for the logarithmic derivative of xi. The Riemann Hypothesis is true if and only if

$$\Re \frac{\xi'(s)}{\xi(s)} > 0$$

for $\Re s > 1/2$ (see Hinkkanen [MR 98d:30047]).

See also Lagarias' paper (to appear in Acta Arithmetica) at www.research.att.com/~jcl/doc/positivity

C.6 Equivalences involving function spaces

Beginning with Wiener's paper, "Tauberian Theorems" in the Annals of Math (1932) a number of functional analytic equivalences of RH have been proven. These involve the completeness of various spaces. M. Balazard has recently written an excellent survey on these developments (See Surveys in Number Theory, Papers from the Millenial Conference on Number Theory, A. K. Peters, 2003.)

C.6.a The Beurling-Nyman Criterion. In his 1950 thesis [MR 12,108g], B. Nyman, a student of A. Beurling, proved that the Riemann Hypothesis is equaivalent to the assertion that $\mathcal{N}_{(0,1)}$ is dense in $L^2(0,1)$. Here, $\mathcal{N}_{(0,1)}$ is the space of functions

$$f(t) = \sum_{k=1}^n c_k \rho(\theta_k/t)$$

for which $\theta_k \in (0,1)$ and such that $\sum_{k=1}^n c_k = 0$.

Beurling [MR 17,15a] proved that the following three statements regarding a number p with $1 < p < \infty$ are equivalent:

(1) $\zeta(s)$ has nozeros in $\sigma > 1/p$

(2) $\mathcal{N}_{(0,1)}$ is dense in $L^p(0,1)$ (3) The characteristic function $\chi_{(0,1)}$ is in the closure of $\mathcal{N}_{(0,1)}$ in $L^p(0,1)$

C.6.a.i A mollification problem. Baez-Duarte [arXiv:math.NT/0202141] has recently proven that the Riemann Hypothesis is equivalent to the assertion that

$$\inf_{A_N(s)} \int_{-\infty}^{\infty} |1 - A_N(1/2+it)\zeta(1/2+it|^2) \ \frac{dt}{\frac{1}{4}+t^2}$$

tends to 0 as $N \to \infty$ where $A_N(s)$ can be any Dirichlet polynomial of length N:

$$A_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}.$$

Previously, Baez-Duarte, Balazard, Landreaux, and Saias had noted that as a consequence of the Nyman-Beurling criterion, the above assertion implies RH.

C.6.b Salem's criterion. R. Salem [MR 14,727a] proved that the non-vanishing of $\zeta(s)$ on the σ –line is equivalent to the completeness in $L^1(0,\infty)$ of $\{k_\sigma(\lambda x),\ \lambda > 0\}$ where

$$k_\sigma(x) = \frac{x^{\sigma-1}}{e^x+1}.$$

C.7 Other analytic estimates

C.7.a M. Riesz series. M. Riesz (Sur l'hypothèse de Riemann, Acta Math. 40 (1916), 185-190) proved that the Riemann Hypothesis is equivalent to the assertion that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{(k-1)!\zeta(2k)} \ll x^{1/2+\epsilon}$$

C.7.b Hardy-Littlewood series. Hardy and Littlewood (Acta Mathematica 41 (1918), 119 - 196) showed that the Riemann Hypothesis is equivalent to the estimate

$$\sum_{k=1}^{\infty} \frac{(-x)^k}{k!\zeta(2k+1)} \ll x^{-1/4}$$

as $x \to$.

C.7.c Polya's integral criterion. Polya (see Collected Works, Volume 2, Paper 102, section 7) gave a number of integral criteria for Fourier transforms to have only real zeros. One of these, applied to the Riemann ξ-function, is as follows.

The Riemann Hypothesis is true if and only if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\alpha)\Phi(\beta)e^{i(\alpha+\beta)x}e^{(\alpha-\beta)y}(\alpha-\beta)^2 \ d\alpha \ d\beta \geq 0$$

for all real x and y where

$$\Phi(u) = 2\sum_{n=1}^{\infty}(2n^4\pi^2 e^{\frac{9}{2}u} - 3n^2\pi e^{\frac{5}{2}u})e^{-n^2\pi e^{2u}}$$

C.7.d Newman's criterion. Charles Newman [MR 55 #7944], building on work of deBruijn [MR 12,250] defined

$$\Xi_\lambda(z) = \int_{-\infty}^{\infty} \Phi(t)e^{-\lambda t^2} e^{iz}\ dt$$

where

$$\Phi(t) = 2\sum_{n=1}^{\infty}(2n^4\pi^2 e^{\frac{9}{2}t} - 3n^2\pi e^{\frac{5}{2}t})e^{-n^2\pi e^{2t}}.$$

Note that $\Xi_0(z) = \Xi(z)$.

He proved that there exists a constant Λ (with $-1/8 \le \Lambda < \infty$) such that $\Xi_\lambda(z)$ has only real zeros if and only if $\lambda \ge \Lambda$. RH is equivalent to the assertion that $\Lambda \le 0$.

The constant Λ (which Newman conjectured is equal to 0) is now called the deBruijn-Newman constant. A. Odlyzko [MR 2002a:30046] has recently proven that $-2.710^{-9} < \Lambda$.

C.8 Grommer inequalities

Let

$$-\frac{\Xi'}{\Xi}(t) = s_1 + s_2 t + s_3 t^2 + \dots .$$

Let M_n be the matrix whose i,j entry is s_{i+j}. J. Grommer (J. Reine Angew. Math. 144 (1914), 114–165) proved that necessary and sufficient conditions for the truth of the Riemann Hypothesis are that

$$\det M_n > 0$$

for all $n \ge 1$.

See also the paper of R. Alter [MR 36 #1399].

C.9 Redheffer's matrix

The Redheffer matrix $A(n)$ is an $n \times n$ matrix of 0's and 1's defined by $A(i,j) = 1$ if $j = 1$ or if i divides j, and $A(i,j) = 0$ otherwise. Redheffer proved that $A(n)$ has $n - [n \log 2] - 1$ eigenvalues equal to 1. Also, A has a real eigenvalue (the spectral radius) which is approximately $\sqrt(n)$, a negative eigenvalue which is approximately $-\sqrt{n}$ and the remaining eigenvalues are small. The Riemann hypothesis is true if and only if $\det(A) = O(n^{1/2+\epsilon})$ for every $\epsilon > 0$.

Barrett Forcade, Rodney, and Pollington [MR 89j:15029] give an easy proof of Redheffer's theorem. They also prove that the spectral radius of $A(n)$ is $= n^{1/2} + \frac{1}{2}\log n + O(1)$. See also the paper of Roesleren [MR 87i:11111].

Vaughan [MR 94b:11086] and [MR 96m:11073] determines the dominant eigenvalues with an error term $O(n^{-2/3})$ and shows that the nontrivial eigenvalues are $\ll (\log n)^{2/5}$ (unconditionally), and $\ll \log\log(2+n)$ on Riemann's hypothesis.

It is possible that the nontrivial eivenvalues lie in the unit disc.

C.10 Equivalences involving dynamical systems

The topics in this section are generally of the form "A implies RH," where there is some reason to hope that "A" could be attacked.

D.1 Non-commutative number theory: around ideas of A. Connes

D.1.a Dynamical system problem studied by Bost–Connes. We state the problem solved by Bost and Connes in [BC] (J-B. Bost, A. Connes, Selecta Math. (New Series), **1**, (1995) 411–457) and its analogue for number fields, considered in [HL](D. Harari, E. Leichtnam, Selecta Mathematica, New Series 3 (1997), 205-243), [ALR](J. Arledge, M. Laca, I. Raeburn, Doc. Mathematica **2** (1997) 115–138) and [Coh](Paula B Cohen, J. Théorie des Nombres de Bordeaux, **11** (1999), 15–30). A (unital) C^*-algebra B is an (unital) algebra over the complex numbers $\mathbb{C}$ with an adjoint $x \mapsto x^*$, $x \in B$, that is, an anti-linear map with $x^{**} = x$, $(xy)^* = y^*x^*$, $x, y \in B$, and a norm $\|\,.\,\|$ with respect to which B is complete and addition and multiplication are continuous operations. One requires in addition that $\|xx^*\| = \|x\|^2$ for all $x \in B$. A C^* dynamical system is a pair (B, σ_t), where σ_t is a 1-parameter group of C^*-automorphisms $\sigma : \mathbb{R} \mapsto \mathrm{Aut}(B)$. A state φ on a C^*-algebra B is a positive linear functional on B satisfying $\varphi(1) = 1$. The definition of Kubo-Martin-Schwinger (KMS) of an equilibrium state at inverse temperature β is as follows.

Definition: *Let (B, σ_t) be a dynamical system, and φ a state on B. Then φ is an equilibrium state at inverse temperature β, or KMS_β-state, if for each $x, y \in B$ there is a function $F_{x,y}(z)$, bounded and holomorphic in the band $0 < \mathrm{Im}(z) < \beta$ and continuous on its closure, such that for all $t \in \mathbb{R}$,*

$$F_{x,y}(t) = \varphi(x\sigma_t(y)), \qquad F_{x,y}(t + \sqrt{-1}\beta) = \varphi(\sigma_t(y)x). \tag{6}$$

A symmetry group G of the dynamical system (B, σ_t) is a subgroup of $\mathrm{Aut}(B)$ commuting with σ:

$$g \circ \sigma_t = \sigma_t \circ g, \qquad g \in G, t \in \mathbb{R}.$$

Consider now a system (B, σ_t) with interaction. Then, guided by quantum statistical mechanics, we expect that at a critical temperature β_0 a phase transition occurs and the symmetry is broken. The symmetry group G then permutes transitively a family of extremal KMS_β-states generating the possible states of the system after phase transition: the KMS_β-state is no longer unique. This phase transition phenomenon is known as spontaneous symmetry breaking at the critical inverse temperature β_0. We state the problem related to the Riemann zeta function[4] and solved by Bost and Connes.

Problem 1: *Construct a dynamical system (B, σ_t) with partition function the zeta function $\zeta(\beta)$ of Riemann, where $\beta > 0$ is the inverse temperature, having spontaneous symmetry breaking at the pole $\beta = 1$ of the zeta function with respect to a natural symmetry group.*

The symmetry group turns out to be the unit group of the ideles, given by $W = \prod_p \mathbb{Z}_p^*$ where the product is over the primes p and $\mathbb{Z}_p^* = \{u_p \in \mathbb{Q}_p : |u_p|_p = 1\}$. We use here the normalisation $|p|_p = p^{-1}$. This is the same as the Galois group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$, where $\mathbb{Q}^{ab}$ is the maximal abelian extension of the rational number field $\mathbb{Q}$. The interaction detected in the

[4] page 7, *The Riemann zeta function*

phase transition comes about from the interaction between the primes coming from considering at once all the embeddings of the non-zero rational numbers $\mathbb{Q}^*$ into the completions $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the prime valuations $|.|_p$. The following natural generalisation of this problem to the number field case and the Dedekind zeta function[66] was solved in [Coh] (see also [HL], [ALR]).

Problem 2: *Given a number field K, construct a dynamical system (B, σ_t) with partition function the Dedekind zeta function $\zeta_K(\beta)$, where $\beta > 0$ is the inverse temperature, having spontaneous symmetry breaking at the pole $\beta = 1$ of the Dedekind function with respect to a natural symmetry group.*

D.1.b The C*-algebra of Bost–Connes. We give a different construction of the C^*-algebra of Bost–Connes to that found in their original paper. It is directly inspired by work of Arledge-Laca-Raeburn. Let A_f denote the ring of finite adeles of $\mathbb{Q}$, that is the restricted product of $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$ as p ranges over the finite primes. Recall that this restricted product consists of the infinite vectors $(a_p)_p$, indexed by the primes p, such that $a_p \in \mathbb{Q}_p$ with $a_p \in \mathbb{Z}_p$ for almost all primes p. The group of (finite) ideles $\mathcal{J}$ consists of the invertible elements of the adeles. Let $\mathbb{Z}_p^*$ be those elements of $u_p \in \mathbb{Z}_p$ with $|u_p|_p = 1$. Notice that an idele $(u_p)_p$ has $u_p \in \mathbb{Q}_p^*$ with $u_p \in \mathbb{Z}_p^*$ for almost all primes p. Let

$$\mathcal{R} = \prod_p \mathbb{Z}_p, \qquad I = \mathcal{J} \cap \mathcal{R}, \qquad W = \prod_p \mathbb{Z}_p^*.$$

Further, let $\mathbf{I}$ denoted the semigroup of integral ideals of $\mathbb{Z}$, which are of the form $m\mathbb{Z}$ where $m \in \mathbb{Z}$. Notice that I as above is also a semigroup. We have a natural short exact sequence,

$$1 \to W \to I \to \mathbf{I} \to 1. \tag{7}$$

The map $I \to \mathbf{I}$ in this short exact sequence is given as follows. To $(u_p)_p \in I$ associate the ideal $\prod_p p^{\mathrm{ord}_p(u_p)}$ where $\mathrm{ord}_p(u_p)$ is determined by the formula $|u_p|_p = p^{-\mathrm{ord}_p(u_p)}$. By the Strong Approximation Theorem we have

$$\mathbb{Q}/\mathbb{Z} \simeq A_f/\mathcal{R} \simeq \oplus_p \mathbb{Q}_p/\mathbb{Z}_p \tag{8}$$

and we have therefore a natural action of I on $\mathbb{Q}/\mathbb{Z}$ by multiplication in $A_f/\mathcal{R}$ and transport of structure. We have the following straightforward Lemmata

Lemma 1. *For $a = (a_p)_p \in I$ and $y \in A_f/\mathcal{R}$, the equation*

$$ax = y$$

has $n(a) =: \prod_p p^{\mathrm{ord}_p(a_p)}$ solutions in $x \in A_f/\mathcal{R}$. Denote these solutions by $[x : ax = y]$.

Let $\mathbb{C}[A_f/\mathcal{R}] =: \mathrm{span}\{\delta_x : x \in A_f/\mathcal{R}\}$ be the group algebra of $A_f/\mathcal{R}$ over $\mathbb{C}$, so that $\delta_x \delta_{x'} = \delta_{x+x'}$ for $x, x' \in A_f/\mathcal{R}$. We have,

Lemma 2. *The formula*

$$\alpha_a(\delta_y) = \frac{1}{n(a)} \sum_{[x:ax=y]} \delta_x$$

for $a \in I$ defines an action of I by endomorphisms of $C^*(A_f/\mathcal{R})$.

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[66] page 8, *Dedekind zeta functions*

We now appeal to the notion of semigroup crossed product developed by Laca and Raeburn, applying it to our situation. A covariant representation of $(C^*(A_f/\mathcal{R}), I, \alpha)$ is a pair (π, V) where

$$\pi : C^*(A_f/\mathcal{R}) \to B(\mathcal{H})$$

is a unital representation and

$$V : I \to B(\mathcal{H})$$

is an isometric representation in the bounded operators in a Hilbert space $\mathcal{H}$. The pair (π, V) is required to satisfy,

$$\pi(\alpha_a(f)) = V_a \pi(f) V_a^*, \qquad a \in I, \quad f \in C^*(A_f/\mathcal{R}).$$

Such a representation is given by (λ, L) on $l^2(A_f/\mathcal{R})$ with orthonormal basis $\{e_x : x \in A_f/\mathcal{R}\}$ where λ is the left regular representation of $C^*(A_f/\mathcal{R})$ on $l^2(A_f/\mathcal{R})$ and

$$L_a e_y = \frac{1}{\sqrt{n(a)}} \sum_{[x:ax=y]} e_x.$$

The universal covariant representation, through which all other covariant representations factor, is called the (semigroup) crossed product $C^*(A_f/\mathcal{R}) \rtimes_\alpha I$. This algebra is the universal C^*-algebra generated by the symbols $\{e(x) : x \in A_f/\mathcal{R}\}$ and $\{\mu_a : a \in I\}$ subject to the relations

$$\mu_a^* \mu_a = 1, \quad \mu_a \mu_b = \mu_{ab}, \qquad a, b \in I, \tag{9}$$

$$e(0) = 1, \quad e(x)^* = e(-x), \quad e(x)e(y) = e(x+y), \qquad x, y \in A_f/\mathcal{R}, \tag{10}$$

$$\frac{1}{n(a)} \sum_{[x:ax=y]} e(x) = \mu_a e(y) \mu_a^*, \qquad a \in I, y \in A_f/\mathcal{R}. \tag{11}$$

When $u \in W$ then μ_u is unitary, so that $\mu_u^* \mu_u = \mu_u \mu_u^* = 1$ and we have for all $x \in A_f/\mathcal{R}$,

$$\mu_u e(x) \mu_u^* = e(u^{-1}x), \qquad \mu_u^* e(x) \mu_u = e(ux). \tag{12}$$

Therefore we have a natural action of W as inner automorphisms of $C^*(A_f/\mathcal{R}) \rtimes_\alpha I$.

To recover the C^*-algebra of Bost–Connes we must split the above short exact sequence. Let $m\mathbb{Z}$, $m \in \mathbb{Z}$, be an ideal in $\mathbb{Z}$. This generator m is determined up to sign. Consider the image of $|m|$ in I under the diagonal embedding $q \mapsto (q)_p$ of $\mathbb{Q}^*$ into I, where the p-th component of $(q)_p$ is the image of q in $\mathbb{Q}_p^*$ under the natural embedding of $\mathbb{Q}^*$ into $\mathbb{Q}_p^*$. The map

$$+ : m\mathbb{Z} \mapsto (|m|)_p \tag{13}$$

defines a splitting of the short exact sequence. Let I_+ denote the image and define B to be the semigroup crossed product $C^*(A_f/\mathcal{R}) \rtimes_\alpha I_+$ with the restricted action α from I to I_+. By transport of structure, this algebra is easily seen to be isomorphic to a semigroup crossed product of $C^*(\mathbb{Q}/\mathbb{Z})$ by $\mathbb{N}_+$, where $\mathbb{N}_+$ denotes the positive natural numbers. This is the algebra of Bost–Connes. The replacement of I by I_+ now means that the group W acts by outer automorphisms. For $x \in B$, one has that $\mu_u^* x \mu_u$ is still in B (computing in the larger algebra $C^*(A_f/\mathcal{R}) \rtimes_\alpha I$), but now this defines an outer action of W. This coincides with the definition of W as the symmetry group as in the paper of Bost–Connes.

D.2 Iwaniec' approach

D.2.a families of rank 2 elliptic curves. Henryk Iwaniec during the conference on the Riemann Hypothesis, New York, 2003.

An exposition is being prepared.

D.3 Unsuccessful attacks on the Riemann Hypothesis

The results in this section are generally of the form “A implies RH,” where A has been shown to be false.

D.3.a Zeros of Dirichlet polynomials. Turan showed that if for all sufficiently large N, the Nth partial sum of $\zeta(s)$ does not vanish in $\sigma > 1$ then the Riemann Hypothesis follows.

He [MR 10,286a] strengthened this criterion by showing that for every $\epsilon > 0$ there is an $N_0(\epsilon)$ such that if the Nth partial sum

$$\sum_{n=1}^{N} n^{-s}$$

of the zeta-function has no zeros in $\sigma > 1 + N^{-1/2+\epsilon}$ for all $N > N_0(\epsilon)$ then the Riemann Hypothesis holds.

H. Montgomery [MR 87a:11081] proved that this approach cannot work because for any positive number $c < 4/\pi - 1$ the Nth partial sum of $\zeta(s)$ has zeros in the half-plane

$$\sigma > 1 + c(\log\log N)/\log N.$$

D.3.b de Branges' positivity condition. Let $E(z)$ be an entire function satisfying $|E(\bar{z})| < |E(z)|$ for z in the upper half-plane. Define a Hilbert space of entire functions $\mathcal{H}(E)$ to be the set of all entire functions $F(z)$ such that $F(z)/E(z)$ is square integrable on the real axis and such that

$$|F(z)|^2 \leqslant \|F\|^2_{\mathcal{H}(E)} K(z,z)$$

for all complex z, where the inner product of the space is given by

$$\langle F(z), G(z)\rangle_{\mathcal{H}(E)} = \int_{-\infty}^{\infty} \frac{F(x)\bar{G}(x)}{|E(x)|^2} dx$$

for all elements $F, G \in \mathcal{H}(E)$ and where

$$K(w,z) = \frac{E(z)\bar{E}(w) - \bar{E}(\bar{z})E(\bar{w})}{2\pi i(\bar{w} - z)}$$

is the reproducing kernel function of the space $\mathcal{H}(E)$, that is, the identity

$$F(w) = \langle F(z), K(w,z)\rangle_{\mathcal{H}(E)}$$

holds for every complex w and for every element $F \in \mathcal{H}(E)$.

de Branges [MR 87m:11050] and [MR 93f:46032] proved the following beautiful

Theorem . *Let $E(z)$ be an entire function having no real zeros such that $|E(\bar{z})| < |E(z)|$ for $\Im z > 0$, such that $\bar{E}(\bar{z}) = \epsilon E(z - i)$ for a constant ϵ of absolute value one, and such that $|E(x+iy)|$ is a strictly increasing function of $y > 0$ for each fixed real x. If $\Re\langle F(z), F(z+i)\rangle_{\mathcal{H}(E)} \geqslant 0$ for every element $F(z) \in \mathcal{H}(E)$ with $F(z+i) \in \mathcal{H}(E)$, then the zeros of $E(z)$ lie on the line $\Im z = -1/2$, and $\Re\{\bar{E}'(w)E(w+i)/2\pi i\} \geqslant 0$ when w is a zero of $E(z)$.*

Let $E(z) = \xi(1 - iz)$. Then it can be shown that $|E(x - iy)| < |E(x + iy)|$ for $y > 0$, and that $|E(x + iy)|$ is a strictly increasing function of y on $(0, \infty)$ for each fixed real x.

Therefore, it is natural to ask whether the Hilbert space of entire functions $\mathcal{H}(E)$ satisfies the condition that

$$\Re\langle F(z), F(z+i)\rangle_{\mathcal{H}(E)} \geqslant 0$$

for every element $F(z)$ of $\mathcal{H}(E)$ such that $F(z+i) \in \mathcal{H}(E)$, because if so, then the Riemann Hypothesis would follow.

It is shown in [MR 2001h:11114] that this condition is not satisfied.

Chapter E: Zeta Gallery

Chapter F: Anecdotes about the Riemann Hypothesis