LIST OF PROBLEMS FROM THE MAY 2007 AIM WORKSHOP ON RATIONAL CURVES ON ALGEBRAIC VARIETIES

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Question 0.1. (Campana, Peternell) If $-K_X$ is strictly nef (meaning that $-K_X.C > 0$ for all $C \subset X$), then is X Fano?

More generally:

Question 0.2. (Serrano): Suppose X is a smooth projective variety over $\mathbb{C}$. If L is strictly nef, then $K_X + tL$ is ample for $t > \dim X + 1$.

Question 0.3. Suppose X is a smooth, projective, and rationally connected. What is the subset f the (closure of the) cone of curve generated (as a cone) by the rational curves? Or by the free rational curves?

Question 0.4. If $-K_X$ is strictly nef, is it possible to have infinitely many extremal rational curves?

Question 0.5. Characterize rationally connected varieties whose Mori cone is polyhedral. (it was asked whether maybe log Fano might be an answer, but this turned out not to work. An example of Miyaoka, a resolution of $x^3 + y^3 + z^3 + txyz = 0$, turned out to be a counter-example since it is not log fano)

Question 0.6. In a related question, it was suggested that finite generation of the total coordinate ring might perhaps be equivalent to $q(X) = 0$ and $\mathrm{NE}^1(X)$ being polyhedral. This turned out to be false by an example of Hassett and Tschinkel.

Question 0.7. In a smooth family of $X \to B$ of Fano varieties, is the Mori cone locally constant? Here with think of the Mori cone of X_t as a subset of $(H^2(X_t, \mathbb{R}))^\vee$ which we identify with $(H^2(X_s, \mathbb{R}))^\vee$ for some t, s in a small neighborhood.

Question 0.8. If X is a rationally connected smooth projective variety over $\mathbb{C}$, is the map

$$A_1(X) \to H_2(X, \mathbb{Z})$$

surjective? (This is known for 3-folds by work of Voisin)

Question 0.9. (Hassett, Tschinkel) Weak approximation for families of rationally connected varieties over a curve. That is, if $\pi : X \to B$ is a proper family of rationally connected varieties over a curve, and given $x_1, \dots, x_N$ smooth points on distinct fibers, does there exist a section through all of them? This is an open question when $X \to B$ is a family of cubic surfaces over a curve (or even a pencil of cubic surfaces).

Question 0.10. If you have a smooth complete nonprojective threefold, the the complements of the maximal quasi-projective open sets are curves. Are these curves always rational?

Question 0.11. Is the fundamental group of the smooth locus of a log Fano variety always finite? Here we mean the smooth locus of the variety itself, not the smooth locus of the pair. This may be related to Ramachandran.

Question 0.12. Is the smooth locus of a log fano variety rationally connected (that is, can two general points be connected by a complete rational curve)? This is known in the case of surfaces by work of Keel and Mckernan. One can also ask the same question of any two points.

Question 0.13. Suppose X is smooth over $\mathbb{C}$ and not projective. We assume it is rationally connected in the sense that any two general points can be conected by a complete rational curve.

Let $U \subset X$ be the largest open set where any finite numbers of points can be joined by a very free curve.

Fact: Every rational curve meeting U is contained in U.

Is every free curve in U?

Question 0.14. In the setup of the previous question, does $X = U$?

Question 0.15. Does every smooth proper, but not projective algebraic space contain a rational curve? (This is known in characteristic zero)

Question 0.16. Suppose X is a smooth projective variety and $f : C \to X$ is a map from a complete curve. There is a map (a Kuranishi map)

$$H^0(C, f^*T_X) \to H^1(C, f^*T_X)$$

If we are given a subsheaf $\mathscr{F} \subset T_X$ and an infinitesimal deformation lying in $H^0(X, f^*\mathscr{F})$, when is the obstruction in $H^1(C, f^*\mathscr{F})$? Is being closed under Lie-bracket enough?

Question 0.17. Come up with a definition of a "rationally simply connected variety". This definition should capture something like "some space of rational curves on X is itself rationally connected".

Question 0.18. Study varieties with the property that, through a general point, there passes a rational surface.

This is known to hold for quartic threefolds. It is unknown for a quintic four-fold and unknown for three-folds that are double covers of $\mathbb{P}^3$ branched over a sextic.

One might even ask if these varities mentioned above are unirational.

Question 0.19. Consider the following theorem:

Theorem 0.20. *A smooth projective variety is uniruled if $\Omega^1_X|_Y$ is non-nef (where Y is a general complete intersection curve).*

It was then asked, is there a variant that only depends on "birational properties of X"? The following theorem was suggested.

Theorem 0.21. *(Campana, Peternell) X is not uniruled if and only if for all N and all torsion free quotients $\mathscr{F}$ of $(\Omega^1_X)^{\otimes N}$,*

$$(\Omega^1_X)^{\otimes N} \to \mathscr{F} \to 0,$$

we have that $\det \mathscr{F}$ is pseudo-effective.

Question 0.22. Are rationally connected varieties potentially dense. For example: If X over $\mathbb{Q}$ is rationally connected and smooth projective, then does there exist a finite extension $\mathbb{Q} \subset L$ so that $X(L)$ is Zariski dense?

Furthermore, if $X(L) \neq \emptyset$, is $X(L)$ Zariski dense? We could even ask this question for X a del Pezzo surface of degree 1 or X a conic bundle over $\mathbb{P}^2$. A negative answer to this in the case of a conic bundle would disprove the generalized twin-prime conjecture.

Question 0.23. What sort of infinite fields satisfy the type of conditions that appear in the previous question?

Question 0.24. Suppose (X, D) is a log fano variety (where D is integral and thus (X, D) is log canonical or dlt, or possibly even plt). Is a log fano variety *log rationally connected*? Here log rationally connected means that, given any two points $p, q \in X$ (not necessarily general), does there exist a rational curve going through those two points and intersecting D at a single point?

Question 0.25. Suppose Y is uniruled with Du Val singularities. Further suppose that D is a divisor on Y with $D \subset Y_{\text{reg}}$ and D a simple normal crossings divisor. Finally suppose $K_Y + D \sim 0$. Is $Y \backslash (D \cup Y_{\text{sing}})$ dominated by images of $\mathbb{C}^*$?

Question 0.26. Let C over $\mathbb{R}$ be a geometrically connected curve such that $C(\mathbb{R}) = \emptyset$. Is $\mathbb{R}(C)$ a C_1-field? More generally, let $\pi : \mathcal{X} \to C$ be a projective morphism whose geometric generic fiber is rationally connected. Does there exist an $\mathbb{R}$-morphism $\sigma : C \to \mathcal{X}$?