Working Groups
Conference on Small Gaps Between Primes
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De Polignac Numbers. De Polignac conjectured that every even number is a difference of two primes in infinitely many ways. Goldston proposed the problem of proving that the set of De Polignac numbers has positive density. The working group, lead by K. Soundarajaran, was able to prove this on the assumption of the Elliott-Halberstam conjecture. A fairly simple argument shows that the density is at least 1/75 of all even numbers. With a more elaborate argument, the group was able to improve this to 8/75. The proof uses the conditional 7-tuple result of Goldston, Pintz, and Yıldırım. For E_2 numbers, the corresponding result is true unconditionally for 3-tuples. In this case, one can easily prove that at least 1/3 of all even numbers are de Polignac numbers.

Ratios of shifted primes. P. Elliott proved that for a fixed a, there exists some k such that $a^k = (p+1)/(q+1)$ has infinitely many solutions in primes p, q. K. Ford asked if the GPY machinery could prove this result with a good bound on k. He lead a group that attempted a proof of Elliott's result with $a = 2, k < \epsilon \log p$. The group quickly identified the primary issue, which turns out to be understanding the singular series associated to linear forms $2n + 1, 4n + 1, 8n + 1, \ldots$. The group made some progress, but identified some difficult issues that have to be resolved. The problem appears difficult but not hopeless.

An analog of GPY for polynomials. C. David and A. Granville led a large group, consisting mostly of postdocs and graduate students, that investigated analogs of GPY for polynomials. Instead of considering k-tuples of the form $\{p+h_1, \ldots, p+h_k\}$, the group considered k-tuples of polynomials $\{f_1(p), \ldots, f_k(p)\}$. Their objective was to work through the heuristics for evaluating the sums that come up when the GPY machinery is applied to the polynomial k-tuples. The results are highly speculative, as they require deep unproved results about the distribution of primes in polynomials and corresponding analogs of the Bombieri-Vinogradov theorem. Nonetheless, the group did succeed in getting some heuristic arguments, and they identified some interesting unexpected behavior on the associated singular series.

Strings of Consecutive Divisors. A group led by C. Elstholtz and S. Graham looked at the issues that would need to be resolved to get a proof that $d(n) = d(n+1) = d(n+2)$ infinitely often. If one assumes the general Hardy-Littlewood conjecture for prime k-tuples, there are numerous constructions that lead to a proof of this result. The group looked at a number of these constructions, and they speculated on how they might be combined with possible results on E_2's to lead to an unconditional proof. The group also speculated on how new result on E_2's might lead to new results on the consecutive values of the Liouville function $\lambda(n)$.

Bombieri-Vinogradov averaged over residue classes. One idea proposed for proving bounded gaps between primes was to prove a version of the Bombieri-Vingradov theorem with extra averaging over a set of residue classes. A group led by R. Vaughan looked at this problem. They were able to develop a line of attack that appeared likely to give a Bombieri-Vinogradov theorem of the desired type. However, they later realized that their proposed result would not lead to any improvements on gaps between primes.

Motohashi's idea for bounded gaps. Motohashi proposed another idea for proving bounded gaps between primes. His idea was to examine the terms that give a negative contribution in the basic sum considered by Goldston, Pintz, and Yıldırım. A. Granville and K. Soundarajaran made some preliminary calculations on these terms. They focused on estimating the contribution of terms with small prime divisors, and they were able to show that those will not contribute enough to successfully solve the problem. In other words, this approach will not succeed unless one can take account of terms with large prime divisors; "large" meaning on the order of $N^{1/10}$ or larger.