Introductory lecture for the workshop

"p-adic representations, modularity and beyond"

I] A preliminary conjecture for $GL_{d+1}(K)$ and de Rham representations (joint with P. Schneider)

II] Survey of known cases of the conjecture

III] Possible refinements for $GL_2(\mathbb{Q}_p)$.

I/ In this talk, I fix L and K two finite extensions of $\mathbb{Q}_p$ such that $[L:\mathbb{Q}_p] = |\mathrm{Hom}_{\mathbb{Q}_p}(L,K)|$.

The basic motivation underlying the "p-adic Langlands programme" is to wonder if there are natural correspondences:

$$\left\{\begin{matrix} d+1\ \text{dim}^l\ \text{p-adic repr.} \\ \text{of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/L) \end{matrix}\right\} \overset{?}{\longleftrightarrow} \left\{\begin{matrix} \text{p-adic Banach spaces} \\ +\ \text{continuous unitary action of } GL_{d+1}(L) \end{matrix}\right\}$$

$$V \longmapsto B(V)$$

[this is the first naïve question one asks; it is highly probable that things won't be so simple minded ...]

Of course, it works for $d=0$ by LCF (taking 1-dim^l Banach spaces on RHS).

I would like in this lecture to give a preliminary conjecture dealing with $d+1$-dim^l de Rham repres. of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ and to sketch the known cases of this conjecture (the most important cases being so far for $GL_2(\mathbb{Q}_p)$). This conjecture can be seen as a first step relating the above two worlds (for de Rham representations). A twin talk by L. Berger will go further in the $GL_2(\mathbb{Q}_p)$-case.

maximal unramified subfield ... L'_0 is ... $\mathbb{Q}_p$ ...
again (increase K if necessary). First, some preliminaries:

ⓐ Consider the following two categories:

- the category $WD_{L'/L}$ of representations (r, N, V) of the Weil-Deligne group of L on a K-vector space V of finite dimension such that $r|_{W(\overline{\mathbb{Q}}_p/L')}$ is unramified.

- the category $MOD_{L'/L}$ of quadruples $(\varphi, N, Gal(L'/L), D)$ where D is a free $L'_0 \otimes_{\mathbb{Q}_p} K$-module of finite rank endowed with:

  $\varphi : D \to D$ (Frobenius) bijective semi-linear / L'_0, linear / K

  $N : D \to D$ (monodromy) linear st. $N\varphi = p\varphi N$

  $Gal(L'/L) \curvearrowright D$ semi-linear / L'_0, linear / K, commuting with φ and N

Fix an embedding $\sigma'_0 : L'_0 \hookrightarrow K$, then Fontaine has defined a functor $WD : MOD_{L'/L} \to WD_{L'/L}$ (depending strictly speaking on σ'_0) as follows:

let $V := D \otimes_{L'_0 \otimes K,\, \sigma'_0 \otimes Id} K$ with the induced $N : V \to V$ (as N is linear).

let $w \in W(\overline{\mathbb{Q}}_p/L)$ act on V as $\overline{w} \circ \varphi^{-\alpha(w)}$ where $\overline{w} \in Gal(L'/L)$, $w \mapsto (\text{abs. arith. Frob.})^{\alpha(w)} \in Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ $(\alpha(w) \in \mathbb{Z})$

then $(r, N, V) \in WD_{L'/L}$ and in fact the functor WD is an equivalence of categories (up to isom. (r,N,V) doesn't depend on the choice of σ'_0).

ⓑ We now modify the local Langlands correspondence as follows: for an

π := (unitary) local Langlands (r, N, V) ...

by Harris-Taylor and Henniart.

If π^{unit} is generic, define $\pi := \pi^{unit} \otimes_{\overline{\mathbb{Q}}_p} |\det|_L^{-d/2}$ $\left(|p|_L = \frac{1}{q_L^{e_L}}\right)$

If π^{unit} is not generic, then π^{unit} is the unique irreducible quotient of a normalized parabolic induction $\mathrm{Ind}_P^{GL_{d+1}} \pi_1^{un} \otimes \cdots \otimes \pi_s^{un}$ where the π_i^{unit} are generalized Steinberg such that $\pi_i^{u} \underset{un.L.L.}{\longleftrightarrow} (r_i, N_i, V_i)$ where $(r, N, V) = \bigoplus_i (r_i, N_i, V_i)$ with (r_i, N_i, V_i) indecomposable. One has to assume that the π_i^{unit} in the parabolic induction are written in a certain way (so that the "does not precede" condition holds). Define:

$$\pi := \left(\mathrm{Ind}_P^{GL_{d+1}} \pi_1^{u} \otimes \cdots \otimes \pi_s^{u}\right) \otimes_{\overline{\mathbb{Q}}_p} |\det|_L^{-d/2}.$$

One can then prove that π always admits a unique model over K.

© Now fix: $\begin{cases} \bullet\ (r, N, V) \in WD_{L'/L} \text{ with } r \text{ semi-simple} \\ \bullet\ \text{for each } \sigma : L \hookrightarrow K, \text{ a list of integers } i_{j,\sigma} \in \mathbb{Z} \text{ s.t.} \\ \qquad i_{1,\sigma} < \cdots < i_{d+1,\sigma}. \end{cases}$

Define $\rho := \bigotimes_{\sigma : L \hookrightarrow K} \rho_\sigma$ where ρ_σ = K-rational algebraic representation of GL_{d+1} of highest weight $-i_{d+1,\sigma} \leq -i_{d,\sigma} - 1 \cdots \leq -i_{1,\sigma} - d$ where $GL_{d+1}(L)$ acts via $L \overset{\sigma}{\hookrightarrow} K$. Define π as above.

Then our main conjecture is:

(i) There is an invariant norm on $\rho \otimes_K \Pi$;

(ii) There is an object $(\varphi, N, \mathrm{Gal}(L'/L), D) \in \mathrm{MOD}_{L'/L}$ such that $WD(\varphi, N, \mathrm{Gal}(L'/L), D)^{F\text{-ss}} \simeq (r, N, V)$ and a (weakly) admissible filtration preserved by $\mathrm{Gal}(L'/L)$ on

$$D_{L'} := L' \otimes_{L_0} D = \prod_{\sigma : L \hookrightarrow K} \underbrace{D_{L'} \otimes_{(L' \otimes_{\mathbb{Q}_p} K)} (L' \otimes_{L,\sigma} K)}_{D_{L',\sigma}}$$

such that:

$$\frac{\mathrm{Fil}^i D_{L',\sigma}}{\mathrm{Fil}^{i+1} D_{L',\sigma}} \neq 0 \iff i \in \{ i_{1,\sigma}, \dots, i_{d+1,\sigma} \} \quad (*) .$$

Transparency on w. a. filtrations (note that $\mathrm{Fil}^i D_{L',\sigma}$ = free over $L' \otimes_{L,\sigma} K$).

$\boxed{\mathrm{II}}$ The following proposition is easy:

<u>prop.</u>: The central character of $\rho \otimes_K \Pi$ is integral (ie admits an invariant norm) iff for any filtration satisfying $(*)$, one has $t_H(D_{L'}) = t_N(D)$.

<u>Corollary</u>: Conjecture holds if r is abs. irreducible (equiv. if Π is supercuspidal).

<u>proof</u>: . $\rho \otimes_K \Pi$ admits an invariant norm iff its central character does

. a filtration is weakly admissible iff it satisfies $t_H(D_{L'}) = t_N(D)$

hence the result follows from the proposition. $\square$

Steinberg), then a filtration as in the conjecture is (weakly) admissible iff $t_H(D_{L'}) = t_N(D)$.

<u>Conj</u>: If π is a generalized Steinberg, then $\rho \otimes_k \pi$ admits an invariant norm iff its central character does.

For instance, this conj. holds for $GL_2(\mathbb{Q}_p)$ (π = Steinberg up to twist / superc.)

see Teitelbaum, Grosse-Kloenne, Emerton

<u>Thm</u> (Schneider, Teit): Assume that (r, N, V) is a direct sum of unramified characters, then (i) $\Rightarrow$ (ii).

The proof uses an examination of the p-adic Satake isomorphism. The converse (ii) $\Rightarrow$ (i) is much more difficult in general.

<u>Thm</u> (Berger - B)

utilise une stratégie due à Colmez

: Assume $GL_{d+1}(L) = GL_2(\mathbb{Q}_p)$ and r is the direct sum of 2 different characters (and $N=0$), then the conjecture holds and moreover there is a unique (up to equivalence) invariant norm on $\rho \otimes_k \pi$.

The proof uses the theory of (φ, Γ)-modules, except in the "ordinary" cases, where an invariant norm on $\rho \otimes_k \pi$ is obvious. I want now to sketch the proof of the theorem, that is to say the proof of (ii) $\Rightarrow$ (i) of the conjecture. We have to use the Galois representⁿ.

So let V be a potentially crystalline representation of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of dim 2 over k that becomes crystalline over an abelian extension L' of $\mathbb{Q}_p$.

that $(\varphi, \mathrm{Gal}(L/\mathbb{Q}_p), D_{\mathrm{pst}}(V))$

is an object of $\mathrm{MOD}_{L/\mathbb{Q}_p}$ (if we forget the filtration).

- If V is reducible, then it is easy to see that $\rho \otimes_K \pi = \rho(V) \otimes_K \pi(V)$ is contained in a locally analytic parabolic induction $\mathrm{ind}_{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}^{GL_2(\mathbb{Q}_p)} \chi$ with χ integral, hence there is a unique invariant norm on $\rho \otimes_K \pi$. No need of V here to prove this.
- Assume V is irreducible.

Let $B(V) :=$ p-adic completion of $\rho(V) \otimes_K \pi(V)$ with respect to any generating $\mathcal{O}_K[GL_2(\mathbb{Q}_p)]$-submodule of finite type.

Such generating submodules always exist, but it can happen that they are not lattices, i.e. one can have $B(V) = 0$. In fact:

<u>$\exists$ an invariant norm on $\rho(V) \otimes_K \pi(V) \Longleftrightarrow B(V) \neq 0$.</u>

The point is that $B(V)$ will admit a description in terms of the (φ, Γ)-module of V.

Transparency on (φ, Γ)-modules.

<u>Th.:</u> Let V be as before and assume that φ on $D_{\mathrm{pst}}(V)$ is <u>semi-simple</u>, then there is a unique (topological) isomorphism:

$$\left(\varprojlim_{\psi} D(V)\right)^b \simeq B(V)^\vee \quad \text{(enlarging } K \text{ if necessary)}$$

where $\left(\varprojlim_{\psi} D(V)\right)^b \simeq \left\{ (v_n)_{n \in \mathbb{Z}_{\geq 0}}, v_n \in D(T), \psi(v_{n+1}) = v_n, \right\} \otimes K$
(v_n) bounded for weak topology on $D(T)$

such that:

the action of $\begin{pmatrix} 1 & 0 \\ 0 & \mathbb{Z}_p^\times \end{pmatrix}$ on $B(V)^\vee$ corresponds to $(v_n) \mapsto (\gamma \cdot v_n)$

$\gamma \in \Gamma \xrightarrow[\varepsilon^{-1}]{\sim} \mathbb{Z}_p^\times$

the action of $\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ on $B(V)^\vee$ corresponds to $(v_n) \mapsto \left(\varphi^n\left((1+X)^{\mathbb{Z}_p}\right) v_n\right)$.

In particular $B(V) \neq 0$, but gives much more: V irred. $\Rightarrow D(V)$ irreduc. (as (φ,Γ)-module) $\Rightarrow B(V)$ top. irred. (as repr. of the Borel). One also gets that $B(V)$ is admissible.

<u>Sketch of prf of thm.</u>: twisting (and enlarging K if necessary), can assume
$$\rho \otimes_K \pi = \mathrm{Sym}^{k-2} K^2 \otimes_K \left(\mathrm{ind}_{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}^{GL_2(\mathbb{Q}_p)} \alpha \otimes \beta |\,|^{-1}\right)$$

where $k \geq 2$, $\alpha, \beta : \mathbb{Q}_p^\times \to K^\times$ = smooth characters such that:
$$\begin{cases} 0 \leq \mathrm{val}(\alpha(p)^{-1}) \leq k-1 \\ 0 \leq \mathrm{val}(\beta(p)^{-1}) \leq k-1 \end{cases} \text{ and } \underbrace{\mathrm{val}(\alpha(p)^{-1})}_{\overset{\shortparallel}{\alpha_p}} + \underbrace{\mathrm{val}(\beta(p)^{-1})}_{\overset{\shortparallel}{\beta_p}} = k-1.$$

Assume π generic <u>for simplicity</u> in the sequel.

<u>Step 1:</u> $B(V) \simeq B(\alpha)/L(\alpha) \simeq B(\beta)/L(\beta)$

Transparency for $B(\alpha)/L(\alpha)$

<u>Step 2:</u> $\left(B(\alpha)/L(\alpha)\right)^\vee \xrightarrow{\sim} \left(\varprojlim_{\psi} D(V)\right)^b$

$\wr\shortparallel$

$\{\mu \in B(\alpha)^\vee \mid \langle \mu, L(\alpha)\rangle = 0\}$

The map is essentially "p-adic Fourier transform".

let $\mathcal{R}_K = \{ \sum_{n \geq 0} a_n X^n \in K[[X]]$ converging on the open unit disc $\}$,

the map $\mu \longmapsto \sum_{n=0}^{+\infty} \langle \mu, \binom{z}{n} \rangle X^n$ induces an isomorphism between $C^{an}(\mathbb{Z}_p, K)^\vee$ (= locally analytic distributions on $\mathbb{Z}_p$) and $\mathcal{R}_L^+$.

Take $\mu_\alpha \in B(\alpha)^\vee$ and define $\mu_{\alpha,n} \in C^{val(\alpha_p)}(\mathbb{Z}_p, K)^\vee \hookrightarrow C^{an}(\mathbb{Z}_p, K)^\vee$ as:

$$\langle \mu_{\alpha,n}, f(z) \rangle := \langle \mu_\alpha, \underbrace{\mathbb{1}_{\frac{1}{p^n}\mathbb{Z}_p} \cdot f(p^n z)}_{\in B(\alpha)} \rangle$$

and let $w_{\alpha,n} \in \mathcal{R}_K^+$ such that $\alpha_p^{-n} \mu_{\alpha,n} \longmapsto w_{\alpha,n}$.

If $\mu_\alpha \in \left(B(\alpha)/L(\alpha)\right)^\vee$, then $\mu_\alpha \mapsto \mu_\beta \in \left(B(\beta)/L(\beta)\right)^\vee \rightsquigarrow w_{\beta,n} \in \mathcal{R}_K^+$ analogously.

Enlarging K if necessary, one can write $D_{cris}(V) \simeq K e_\alpha \oplus K e_\beta$

$$\begin{cases} \varphi(e_\alpha) = \alpha_p^{-1} e_\alpha \\ \varphi(e_\beta) = \beta_p^{-1} e_\beta \end{cases} \quad \begin{cases} \gamma e_\alpha = \alpha(\varepsilon(\gamma)) e_\alpha \\ \gamma e_\beta = \beta(\varepsilon(\gamma)) e_\beta \end{cases} \quad \gamma \in \Gamma$$

then the map is: $\mu_\alpha \longmapsto (w_{\alpha,n} \otimes e_\alpha \oplus w_{\beta,n} \otimes e_\beta)_n \in \varprojlim_{\psi} (\mathcal{R}_K^+ \otimes_K D_{cris}(V))$

in fact sits in $\left(\varprojlim_{\psi} D(V)\right)^b$.

Any element in $\left(\varprojlim_{\psi} D(V)\right)^b$ can in fact be described like this.

III) One looks for a strong refinement of the previous conjecture, in the spirit of p-adic local Langlands. For instance, as there is an invariant norm iff there is a (weakly) admissible filtration, it suggests:

$$\left\{ \text{equiv. classes of irred. admissible invariant norms on } \rho\otimes\Pi \right\}$$
$$\updownarrow ?$$
$$\left\{ \text{irreducible (weakly) admissible filtr}^{\circ} \text{ as in the conjecture} \right\}$$

Probably not so simple minded in general. However, for $GL_2(\mathbb{Q}_p)$, that could be roughly the picture. At least, in the case Π = Steinberg (up to twist), Colmez has shown that the map:

can be described explicitly

$$\begin{array}{ccccccc} \text{irred. (weakly) admissible} & \mapsto & V \text{ semi-stable} & \mapsto & \left(\varprojlim_{\psi} D(V)\right)^{b} & \xrightarrow{\sim} & \text{completion} \\ \text{filtr}^{\circ} \text{ on } (\varphi, N, D) & & \text{(non crystalline)} & & & & \text{of } \rho\otimes_{K}\Pi \\ \text{as in conjecture} & & \text{irreducible} & & & & \text{with respect to} \\ & & & & & & \text{an invariant} \\ & & & & & & \text{norm} \end{array}$$

creates equivalence classes of invariant norms really depending on the weakly admissible filtrations.

<u>Question:</u> Does this still work for Π supercuspidal (for $GL_2(\mathbb{Q}_p)$)?

T = continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on a free $\mathcal{O}_K$-module of finite rank

$$D(T) := \left(\mathcal{O}_{\widehat{\mathcal{E}^{unr}}} \otimes_{\mathbb{Z}_p} T\right)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_{p^\infty}))}$$

$\varphi \circlearrowleft$ $\qquad$ $\circlearrowleft \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) = \Gamma$

$D(T)$ = free over $\mathcal{O}_\mathcal{E} \simeq \mathcal{O}_K[[X]][\frac{1}{X}]^\wedge$ $\quad$ rk = $\mathrm{rk}_{\mathcal{O}_K}(T)$

$\varphi : D(T) \to D(T)$ inj. semi-linear

$(\varphi(X) = (1+X)^p - 1)$

$\gamma \in \Gamma : D(T) \to D(T)$ bij. semi-linear

$(\gamma(X) = (1+X)^{\varepsilon(\gamma)} - 1$ where ε = p-adic cyclo$)$

Any $v \in D(T)$ can be written uniquely:

$$v = \sum_{i=0}^{p-1} (1+X)^i \varphi(v_i)$$

Define $\psi : D(T) \to D(T)$ by $\psi(v) := v_0$.

Then $\psi\varphi = \mathrm{Id}$.

If $V := T \otimes \mathbb{Q}_p$, $D(V) := D(T) \otimes \mathbb{Q}_p$.

$D := L'_0 e_1 \oplus \dots \oplus L'_0 e_n$, $N : D \to D$ linear

$\varphi : D \to D$ bij. semi-linear , $N\varphi = p\varphi N$

$(\mathrm{Fil}^i D_{L'})_{i\in\mathbb{Z}}$ = decreasing exhaustive separated filtration on $D_{L'} := L' \otimes_{L'_0} D$

$$t_N(D) = t_N\left(\Lambda^n_{L'_0} D\right) = \mathrm{val}_{\mathbb{Q}_p}\left(\frac{\varphi(e_1 \wedge \dots \wedge e_n)}{e_1 \wedge \dots \wedge e_n}\right)$$

$$t_H(D_{L'}) = t_H\left(\Lambda^n_{L'} D_{L'}\right) = \sum_{i\in\mathbb{Z}} i \dim_{L'} \frac{\mathrm{Fil}^i D_{L'}}{\mathrm{Fil}^{i+1} D_{L'}}$$

<u>Def</u> (Fontaine): The Filtration is (weakly) admissible if $t_N(D) = t_H(D_{L'})$ and for any $D' \subseteq D$ preserved by φ, N with the induced Filtration on $D'_{L'}$, one has $t_H(D'_{L'}) \leq t_N(D')$.

"Hodge polygon under Newton polygon"

- For $r \in \mathbb{R}_{\geq 0}$, a function $f : \mathbb{Z}_p \to K$ is C^r if its Mahler expansion $f(z) = \sum_{n=0}^{+\infty} a_n \binom{z}{n}$ satisfies

$$n^r |a_n| \xrightarrow[n \to +\infty]{} 0$$

$\rightsquigarrow C^r(\mathbb{Z}_p, K) =$ Banach space $\|f\|_r = \sup_n \{n^r |a_n|\}$

- $B(\alpha) :=$ Banach space of functions $f : \mathbb{Q}_p \to K$ s.t.

$f|_{\mathbb{Z}_p}$ is $C^{\mathrm{val}(\alpha_p)}$ and $\beta\alpha^{-1}(z)|z|^{-1} z^{k-2} f\left(\frac{1}{z}\right)\Big|_{\mathbb{Z}_p - \{0\}}$ extends as a function $C^{\mathrm{val}(\alpha_p)}$

- $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(z) = \alpha(ad-bc)\, \beta\alpha^{-1}(-cz+a)\, |-cz+a|^{-1} (-cz+a)^{k-2} \cdot f\left(\frac{dz-b}{-cz+a}\right)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p)$

- $L(\alpha) =$ closure of subspace generated by

$z \longmapsto z^j$

$z \longmapsto \beta\alpha^{-1}(z-a)\, |z-a|^{-1} (z-a)^{k-2-j}$

for $a \in \mathbb{Q}_p$ and $0 \leq j < \mathrm{val}(\alpha_p)$

Then $B(\alpha) / L(\alpha) =$ unitary $GL_2(\mathbb{Q}_p)$-Banach space