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IV (φ,Γ)-modules of p-adic representations

In Chr.'s talk, a functor $V \mapsto D(V) = (\widehat{\mathcal{E}^{nr}} \otimes_{\mathbb{Q}_p} V)^{Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_{p^\infty}))}$ was defined. This functor is an equivalence of categories:

$$\left\{\begin{matrix} p\text{-adic reps} \\ \text{of } G_{\mathbb{Q}_p} \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \text{étale } (\varphi,\Gamma)\text{-modules} \\ \text{over } \mathcal{E} \end{matrix}\right\}$$

where $\mathcal{E} = \mathcal{O}_K[[X]][\frac{1}{X}]^\wedge[\frac{1}{p}] = \{ f(X) = \sum_{i \in \mathbb{Z}} f_i X^i$ with $f_i \in K$, $\{f_i\}_{i\in\mathbb{Z}}$ bounded and $f_i \to 0$ as $i \to -\infty \}$

let $\mathcal{E}^{\dagger, r} = \{ f(X) = \sum_{i\in\mathbb{Z}} f_i X^i$ such that $f(X)$ is convergent and bounded on the annulus $p^{-1/r} \leq |X|_p < 1\}$

$\mathcal{R}^{\dagger, r} = \{$ — — — is convergent — — — $\}$

let $\mathcal{E}^\dagger = \bigcup_{r>0} \mathcal{E}^{\dagger, r}$ $\quad \mathcal{R} = \bigcup_{r>0} \mathcal{R}^{\dagger, r}$. Note that $\mathcal{E}^\dagger$ is a field, and that $\left.\begin{matrix}\mathcal{E}^\dagger \subset \mathcal{E} \\ \mathcal{E}^\dagger \subset \mathcal{R}\end{matrix}\right\}$

If $D^\dagger$ is a (φ,Γ)-module over $\mathcal{E}^\dagger$, then $D = \mathcal{E} \otimes_{\mathcal{E}^\dagger} D^\dagger$ is a (φ,Γ)-module over $\mathcal{E}$

$D_{rig} = \mathcal{R} \otimes_{\mathcal{E}^\dagger} D^\dagger$ — — — — — — over $\mathcal{R}$

$$\begin{array}{ccc} & \left\{\begin{matrix}\text{étale } (\varphi,\Gamma)\text{-modules} \\ \text{over } \mathcal{E}^\dagger\end{matrix}\right\} & \\ \swarrow & & \searrow \\ \left\{\begin{matrix}\text{étale } (\varphi,\Gamma)\text{-modules} \\ \text{over } \mathcal{E}\end{matrix}\right\} & & \left\{\begin{matrix}\text{slope } 0 \\ (\varphi,\Gamma)\text{-modules over } \mathcal{R}\end{matrix}\right\} \end{array}$$

both arrows are equivalences of categories ($D^\dagger \mapsto D$: Ch + Co
$D^\dagger \mapsto D_{rig}$: Ked)

Therefore, to a p-adic representation V, one can attach $D(V)$, $D^\dagger(V)$ and $D_{rig}(V)$ and one can recover V from any of them.

A p-adic rep. V is said to be trianguline if $D_{rig}(V)$ is a successive extension of rank 1 (φ,Γ)-modules over $\mathcal{R}$ ($\not\Rightarrow$ over $\mathcal{E}^\dagger$!!)

The (φ,Γ)-mods of rank 1 over $\mathcal{R}$ are classified by ~~characters~~ continuous characters $\delta : \mathbb{Q}_p^\times \to L^\times$, to which corresponds the (φ,Γ)-mod. $\mathcal{R}(\delta) = \mathcal{R} \cdot e_\delta$ where $\left\{\begin{matrix}\varphi(e_\delta) = \delta(p) e_\delta \\ \gamma(e_\delta) = \delta(\varepsilon(\gamma)) e_\delta\end{matrix}\right.$

Which reps of $G_{\mathbb{Q}_p}$ are trianguline? Here are some examples:

(1) semistable reps
(2) crystabelline reps (those reps which become crys. over an ~~finite~~ abelian ext. of $\mathbb{Q}_p$)
(3) reps V_f attached to an overconvergent modular ~~form~~ eigenform f (with U_p eigvl $\neq 0$)

I will now explain the ideas behind (1) and (2). As for (3), it follows from similar ideas, starting from a theorem of Kisin which states that $D_{cris}(V_f) \neq 0$.

Say we are in case (2). If $F \subset \mathbb{Q}_p(\zeta_{p^\infty})$, then $D_{cris}(V_{|F}) = (D_{rig}(V)[\frac{1}{t}])^{\Gamma_F}$ and if $V_{|F}$ is crystalline, then $\mathcal{R}[\frac{1}{t}] \otimes_K D_{cris}(V_{|L}) = \mathcal{R}[\frac{1}{t}] \otimes_{\mathcal{R}} D_{rig}(V)$. So if V is crystabelline, then $\mathcal{R}[\frac{1}{t}] \otimes D_{rig}(V) = \mathcal{R}[\frac{1}{t}] \otimes_K D$, D = fin-diml K-ev w. φ, Γ. (∃ cotrigo) It is then an exercise in the semilinear algebra of (φ,Γ)-modules over $\mathcal{R}$ to show that this $\Rightarrow D_{rig}(V)$ is a successive extension of 1-diml (φ,Γ)-modules.

V The Colmez isomorphism

If D is a (φ,Γ)-mod. over $\mathcal{E}$, then it is equipped with:

(1) an operator ψ (see Chr.'s lecture)
(2) a « weak topology », which one should think of as the (p,X)-adic topo.

The space $(\varprojlim_\psi D)^b$ is then the set of sequences $(y_i)_{i\geq 0}$ such that

Choose a continuous character χ of $\mathbb{Q}_p^\times$. We make $(\varprojlim_\psi D)^b$ into a rep^n of $B(\mathbb{Q}_p) = \begin{pmatrix} * & * \\ & * \end{pmatrix}$ as follows: if $y \in (y_i)_{i \geq 0} \in (\varprojlim_\psi D)^b$, then:

$$\left[\begin{pmatrix} x & \\ & x \end{pmatrix} y\right]_i = \chi(x)^{-1} y_i \qquad \left[\begin{pmatrix} 1 & \\ & a \end{pmatrix} y\right]_i = \gamma(y_i) \quad \text{where} \quad \varepsilon(\gamma) = a^{-1} \in \mathbb{Z}_p^\times$$

$$\left[\begin{pmatrix} 1 & \\ & p^j \end{pmatrix} y\right]_i = \psi^j(y_i) \qquad \left[\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} y\right]_i = (1+X)^{p^i z} y_i$$

Suppose that V is a trianguline rep of dim. 2: then one can construct a p-adic Banach rep^n of $GL_2(\mathbb{Q}_p)$, $B(V)$ (the one constr. in Chr's lecture if V is crystab. or sst), such that the "Colmez isomorphism" holds:

$$[(\varprojlim_\psi D(V))^b]^* \simeq B(V) \quad \text{as reps of } B(\mathbb{Q}_p) \quad (\text{for a suitable choice of } \chi)$$

The point of showing this isom. is that:

(1) it proves that $B(V) \neq 0$, that it is topo. irred, and admissible.

(2) it allows us to compute the reduction mod. p of $B(V)$

I will now sketch how one proves the CI for crystalline reps. Recall that in his lecture, Chr. has already explained part of the proof: there is a map $B(V)^\vee = (B(\alpha)/L(\alpha))^\vee$ to $(\varprojlim_\psi D(V))^b$ which sends μ_α to $(w_{\alpha,n} \otimes e_\alpha + w_{\beta,n} \otimes e_\beta)_{n \geq 0}$, with $w_{\alpha,n}, w_{\beta,n} \in \mathcal{R}^+$.

I will expand on ~~this~~ comment that "any element in $(\varprojlim_\psi D(V))^b$ can in fact be described like this". Let V be the one in Chr's talk. Since V is crystalline, we have $\mathcal{R}[\frac{1}{t}] \otimes_{\mathcal{E}^\dagger} D^\dagger(V) = \mathcal{R}[\frac{1}{t}] \otimes_K D_{cris}(V)$ and (b/c the weights are ≥ 0) we have $D^\dagger(V) \subset \mathcal{R} \otimes_K D_{cris}(V)$. Given $y = y_\alpha \otimes e_\alpha + y_\beta \otimes e_\beta \in \mathcal{R}^{t,r} \otimes_K D_{cris}(V)$, how can we tell if $y \in D^\dagger(V)$? For $n \geq n(r)$, we have a map

$$\iota_n : \mathcal{R}^{t,r} \to K_n[[t]] \quad \text{which sends } f(X) \text{ to } f(\zeta_{p^n} e^{t/p^n} - 1)$$

and one can show that $y \in D^\dagger(V) \Leftrightarrow$

(1) $\mathrm{ord}(y) = 0$, i.e. $\mathrm{ord}(y_{\substack{\alpha \\ \beta}}) = \mathrm{val}(\substack{\alpha_p \\ \beta_p})$

(2) $\iota_n(y) \in \mathrm{Fil}^0(K_n((t)) \otimes D_{cris}(V)) \quad \forall n \geq 0$

Now let $M(V) = \{ y \in \mathcal{R}^+ \otimes_K D_{cris}(V)$ such that $\mathrm{ord}(y) = 0$ and $\iota_n(y) \in \mathrm{Fil}^0(K_n((t)) \otimes D_{cris}(V)) \quad \forall n \geq m(V) \}$, $m(V) = \mathrm{cond}(\alpha\beta^{-1})$

Using the theory of Wach modules, we can prove:

(1) $M(V)$ is a $K \otimes \mathcal{O}_K[[X]]$-module of finite type, stable under ψ, which contains a basis of $D^\dagger(V)$

(2) the map $(\varprojlim_\psi M(V))^{\text{bdd } p\text{-adic}} \to (\varprojlim_\psi D(V))^b$ is a topo. isom.

($D(V) = \mathcal{E} \otimes \mathbb{N}(V) + \text{bdd}(X)$)

As a corollary, we get that indeed $(\varprojlim_\psi D(V))^b \simeq \{(y_n = w_{\alpha,n} \otimes e_\alpha + w_{\beta,n} \otimes e_\beta)_{n \geq 0}$ satisfying:

(1) $\psi(w_{\alpha,n+1}) = w_{\alpha,n}$ $n \geq 0$, idem β. $\to \psi$-compatible

(2) $\mathrm{ord}(w_{\alpha,n}) = \mathrm{val}(\alpha_p)$, idem β $\to$ comes from $(B(\alpha)/L(\alpha))^\vee$ $(B(\beta)/L(\beta))^\vee$

(3) $\iota_m(y_n) \in \mathrm{Fil}^0(K_m[[t]] \otimes D_{cris}(V))$ $\forall m \geq m(V)$ $\forall n \geq 0$ $\to$ compatible wrt interchanging

(4) $\{ \|w_{\alpha,n}\|_{\mathrm{ord}(w_{\alpha,n})} \}$ bounded, idem β

This is how one proves the CI.

VI. Reduction mod. p

One can easily prove that if $[(\varprojlim_\psi D(V))^b]^* \simeq B(V) \Rightarrow [(\varprojlim_\psi D(\overline{V}))^b]^* \underset{B(\mathbb{Q}_p)}{\sim} \overline{B(V)}$

Note that if V is crystalline irred. and $k \leq p+1$, then we know both $\overline{V}$ (it is $\mathrm{ind}(\omega_2^{k-1})$) and $\overline{B(V)}$ (it is $\pi(k-2, 0, 1)$).

So if $\overline{V} = \mathrm{ind}(\omega_2^{k-1}) \Rightarrow \overline{B(V)} \sim \pi(k-2, 0, 1)$

$D(W) = k_K((X))\cdot e$ where $\varphi e = d e$ and $\gamma(e) = \omega^r(\gamma) e$

One can show that $(\varprojlim_{\psi} D(W))^{\natural} = \varprojlim_{\psi} \frac{1}{X} k_K[[X]]\cdot e$ is a compact rep. of $B(\mathbb{Q}_p)$

it contains the topo. irreducible subspace $\varprojlim_{\psi} k_K[[X]]\cdot e$ and we set $\Omega_\chi(W) = (\varprojlim_{\psi} k_K[[X]]\cdot e)^*$, it is a smooth admissible irreducible rep of $B(\mathbb{Q}_p)$, and:

$$0 \to \chi\omega\eta_W^{-1}\otimes\omega^{-1}\eta_W \to [(\varprojlim_{\psi} D(W))^{\natural}]^* \to \Omega_\chi(W) \to 0$$

What is $\Omega_\chi(W)$?

If χ_1 and χ_2 are two C^0 chars. of $\mathbb{Q}_p^\times$, let $\mathrm{Ind}_B^G(\chi_1\otimes\chi_2)$ be the parabolic induction of $\chi_1\otimes\chi_2$, seen as a rep of $B(\mathbb{Q}_p)$. We have

$$0 \to \mathrm{Ind}_B^G(\chi_1\otimes\chi_2)_0 \to \mathrm{Ind}_B^G(\chi_1\otimes\chi_2) \to \chi_1\otimes\chi_2 \to 0$$
$$\sigma \mapsto \sigma(Id)$$

and the main result is that $\Omega_\chi(W) \simeq \mathrm{Ind}_B^G(\eta_W\otimes\chi\eta_W^{-1})_0$

sketch of proof: let $LC_0(\mathbb{Q}_p, k_K) = \{f : \mathbb{Q}_p \to k_K$, locally constant with compact support$\}$

if $\sigma \in \mathrm{Ind}_B^G(\chi_1\otimes\chi_2)_0 \rightsquigarrow f_\sigma \in LC_0(\mathbb{Q}_p, k_K)$ $\quad f_\sigma(z) = \sigma\left(\begin{pmatrix}0&1\\-1&z\end{pmatrix}\right)$

the map $\sigma \mapsto f_\sigma$ is a bijection (by the Bruhat decomposition)

if ν is a measure on $\mathbb{Z}_p$ ie $\nu : LC(\mathbb{Z}_p, k_K) \to k_K$

we define $\mathcal{A}(\nu) = \sum_{n\geq 0} \nu(\{z\mapsto \binom{z}{n}\})\cdot X^n \quad (= \nu(z\mapsto(1+X)^z)) \quad \in k_K[[X]]$

and this defines a bijection between measures and $k_K[[X]]$. The action of ψ, Γ, etc... on $k_K[[X]]$ gives an action on {measures}, e.g. $\int_{\mathbb{Z}_p} f(z)\, d\psi\nu = \int_{p\mathbb{Z}_p} f(p^{-1}z)\, d\nu$.

If $y \in \Omega_\chi(W)$, then $y = (f_i(X)\cdot e)_{i\geq 0}$ with $\psi(d^{-i}f_i) = d^{-(i-1)}f_{i-1}$

Define a measure $\nu_{y,i}$ on $\mathbb{Z}_p$ by $\mathcal{A}(\nu_{y,i}) = d^{-i}f_i$

a measure ν_y on $\mathbb{Q}_p$ by $\int_{\mathbb{Q}_p} f(z)\, d\nu_y = \int_{\mathbb{Z}_p} f(p^{-i}z)\, d\nu_{y,i}$ if $\mathrm{supp}(f) \subset p^{-i}\mathbb{Z}_p$

this does not depend on i by the ψ compatibility.

Now one can check that if $\sigma \in \mathrm{Ind}_B^G(\eta_W\otimes\chi\eta_W^{-1})_0$, $y \in \varprojlim_{\psi} k_L[[X]]\cdot e$, $g \in B(\mathbb{Q}_p)$,

then $\int_{\mathbb{Q}_p} f_{g\cdot\sigma}(z)\, d\nu_{g\cdot y} = \int_{\mathbb{Q}_p} f_\sigma(z)\, d\nu_y$ so that

$$(\varprojlim_{\psi} k_L[[X]])^* \underset{B(\mathbb{Q}_p)}{\simeq} \mathrm{Ind}_B^G(\eta_W\otimes\chi\eta_W^{-1})_0 .$$

corollary: if $W = \mu_\lambda\omega^{r+1}\oplus\mu_{\lambda^{-1}}$ and $\chi = \omega^r$, then:

$$[(\varprojlim_{\psi} D(W))^{\natural}]^* \underset{B(\mathbb{Q}_p)}{\simeq} (\mu_{\lambda^{-1}}\otimes\omega^r\mu_\lambda) \oplus \mathrm{Ind}_B^G(\omega^{r+1}\mu_\lambda\otimes\omega^{-1}\mu_{\lambda^{-1}})_0 \to \mathrm{Ind}$$
$$\oplus\, (\omega^{r+1}\mu_\lambda\otimes\omega^{-1}\mu_{\lambda^{-1}}) \oplus \mathrm{Ind}_B^G(\mu_{\lambda^{-1}}\otimes\omega^r\mu_\lambda)_0 \to \mathrm{Ind}$$

the last step is to go from $B(\mathbb{Q}_p)$ to $GL_2(\mathbb{Q}_p)$

smooth irred reps of $GL_2(\mathbb{Q}_p)$ (with cent char)	restricted to $B(\mathbb{Q}_p)$ (and ssed):
(1) dim. 1	$\to$ dim 1
(2) twist of special: $Sp\otimes\chi = \mathrm{Ind}_B^G(\chi\otimes\chi)/\dim 1$	$\to \mathrm{Ind}_B^G(\chi\otimes\chi)_0$ = irred by (φ,Γ)-mod.
(3) parabolic inductions $\mathrm{Ind}_B^G(\chi_1\otimes\chi_2)$, $\chi_1\neq\chi_2$	$\to (\chi_1\otimes\chi_2)\oplus\mathrm{Ind}_B^G(\chi_1\otimes\chi_2)_0$
(4) supersingular $\pi(r,0,\chi)$	$\to \pi(r,0,\chi)$ = irred by (φ,Γ)-mod.

If Π_1 and Π_2 are two k_L-reps of $GL_2(\mathbb{Q}_p)$ smooth, semisimple, of finite length, ~~where~~ such that the semisimplification of their restriction to $B(\mathbb{Q}_p)$ are $\simeq$ $\Rightarrow$ they were $\simeq$ as reps of $GL_2(\mathbb{Q}_p)$.

corollary: If V is trianguline, and if $B(V)$ is the associated rep of $GL_2(\mathbb{Q}_p)$, then:

$\bar{V} = \mathrm{ind}(\omega_2^{r+1})\otimes\chi \iff \overline{B(V)} = \pi(r,0,\chi)$

$\bar{V} = (\mu_\lambda\omega^{r+1} \quad)\otimes\chi \iff \overline{B(V)} = [\ldots]$