THE FONTAINE-MAZUR CONJECTURE FOR GL_2

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Abstract. We prove new cases of the Fontaine-Mazur conjecture, that a two dimensional p-adic representation ρ of $G_{\mathbb{Q},S}$ which is potentially semi-stable at p with distinct Hodge-Tate weights arises from a twist of a modular eigenform of weight $k \geq 2$. Our approach is via the Breuil-Mézard conjecture, which we prove (many cases of) by combining a global argument with recent results of Colmez and Berger-Breuil on the p-adic local Langlands correspondence.

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Introduction

In [FM] Fontaine and Mazur made a remarkable conjecture, predicting that global p-adic Galois representations which are potentially semi-stable at prime dividing p and unramified outside finitely many places, ought to come from algebraic geometry. For two dimensional representations, the conjecture asserts that potentially semi-stable representations with odd determinant come from modular forms. The purpose of these notes is to prove that this is so in many cases. Our methods reveal an intimate connection between modularity lifting theorems, the Breuil-Mézard conjecture, and Breuil's p-adic local Langlands correspondence.

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To state our main theorem, let $p > 2$, S a set of primes containing $\{p, \infty\}$, $G_{\mathbb{Q},S}$ the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S, and $G_{\mathbb{Q}_p} \subset G_{\mathbb{Q},S}$ a decomposition group at p. We prove the following

Theorem. *Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$, having residue field $\mathbb{F}$, and*

$$\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathcal{O})$$

a continuous representation. Suppose that

(1) *$\rho|_{G_{\mathbb{Q}_p}}$ is potentially semi-stable with distinct Hodge-Tate weights.*
(2) *ρ becomes semi-stable over an abelian extension of $\mathbb{Q}_p$.*
(3) *$\bar{\rho} : G_{\mathbb{Q},S} \xrightarrow{\rho} \mathrm{GL}_2(\mathcal{O}) \to \mathrm{GL}_2(\mathbb{F})$ is modular, and $\bar{\rho}|_{\mathbb{Q}(\zeta_p)}$ is absolutely irreducible.*
(4) *$\bar{\rho}|_{G_{\mathbb{Q}_p}} \nsim \begin{pmatrix} \chi & * \\ 0 & \chi \end{pmatrix}, \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ for any character $\chi : G_{F_v} \to \mathbb{F}^\times$, where ω denotes the mod p cyclotomic character*

Then (up to a twist) ρ is modular.

The condition (2) in the theorem can be removed, assuming a compatibility between the p-adic and classical local Langlands correspondences, which describes the locally algebraic vectors in the p-adic unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to a de Rham representation. (The precise statement is given in §1.2). This result should, hopefully, soon be proved by Colmez. Assuming (2) it is a result of Colmez and Berger-Breuil [Co], [BB 1]. What we prove here is the theorem assuming (1), (3), (4) and this compatibility.

The restrictions in (4) are almost certainly not intrinsic to our method, and should be removed in a later version of the paper. They require some extra arguments which we have not included here. The restriction that $p > 2$ is also likely to be unnecessary, at least in many cases (for example $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ irreducible) since the p-adic Langlands correspondence, is available in this situation, unlike the usual difficulties encountered in integral p-adic Hodge theory when $p = 2$.

In fact we prove the theorem in somewhat greater generality, where $\mathbb{Q}$ is replaced by any totally real field in which p splits completely. Let us also remind the reader that the hypothesis $\bar{\rho}$-modular is now not so serious thanks to the work of Khare-Wintenberger [KW] on Serre's conjecture. For example it holds for odd $\bar{\rho}$ with odd conductor.

One consequence of the theorem (using only the case when ρ becomes semi-stable over an abelian extension) is a conjecture made in [Ki 4, 11.8] which gives a construction of the eigencurve of Coleman-Mazur in purely Galois theoretic terms.

We now explain how the Breuil-Mézard conjecture and the p-adic local Langlands correspondence enter the proof of the theorem. The first fundamental breakthrough in the direction of the Fontaine-Mazur conjecture was made by Wiles and Taylor-Wiles [Wi], [TW] a little over 10 years ago. They showed how one could deduce the modularity of certain p-adic Galois representations, assuming the mod p reduction was modular. Subsequently a number of authors established modularity lifting theorems for (2-dimensional) potentially Barosotti-Tate representations, and more generally representations of small Hodge-Tate weights [Di 2], [CDT], [BCDT], [DFG], [Ta 2]. There was also work of Skinner-Wiles establishing the conjecture for ordinary representations [SW 1], [SW 2].

One of the themes in these papers is that in order to prove a modularity lifting theorem one needs to show a certain local deformation ring is formally smooth (i.e. a power series ring). In [BCDT] the authors considered potentially Barsotti-Tate representations, and they made a conjecture predicting when one could expect this formal smoothness. This conjecture was later generalized by Breuil-Mézard [BM] who predicted that μ_{Gal}, the Hilbert-Samuel multiplicity of the mod p reduction of the local deformation ring, should be given by a certain invariant μ_{Aut} which could be computed representation theoretically.

In [Ki 2] we showed how to modify the Taylor-Wiles argument, so that it applied when the local deformation was not formally smooth. This was used to establish a fairly general modularity lifting theorem for potentially Barsotti-Tate Galois representations. However, another consequence of this modification was that one could use a global argument to show that $\mu_{\mathrm{Gal}} \geq \mu_{\mathrm{Aut}}$, and that establishing a modularity lifting theorem was essentially equivalent to proving the reverse equality. This is explained in §2 of this paper.

The tool which enables us to prove the reverse inequality is the p-adic local Langlands correspondence, whose study was initiated by Breuil [Br 1], [Br 2], and developed by Breuil, Berger and Colmez [BB 1], [BB 2], [Co]. A key insight, due to Colmez, is that one can construct instances of this correspondence using Fontaine's theory of φ, Γ-modules. The papers just cited show how to construct unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representations starting with a local Galois representation which Colmez terms *trianguline*. For de Rham representations, this means that the representation becomes semi-stable over an abelian extension of $\mathbb{Q}_p$. In September 2005, at the Montreal conference on p-adic representations, Colmez explained a quite general construction which associated a local Galois representation to a p-adic unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation satisfying a mild restriction. This association works integrally, and using it we show that the local deformation rings we wish to study act faithfully on certain $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. This leads to the required inequality.

We first announced these results at the Montreal conference for ρ which become crystalline over an abelian extension in $\mathbb{Q}_p$, and $\bar{\rho}$ absolutely irreducible at p. The previous day Colmez had outlined his theory, attaching local Galois representations to certain $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. The ad hoc arguments we had in mind at that time for proving the inequality $\mu_{\mathrm{Gal}} \leqslant \mu_{\mathrm{Aut}}$, immediately suggested that one should formulate Colmez's correspondence on the level of deformation rings for representations of $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$\Theta : R_{G_{\mathbb{Q}_p}} \to R_{\mathrm{GL}_2(\mathbb{Q}_p)}.$$

The advantage of this was that, thanks to the previous work of Colmez and Berger-Breuil, one knew that the image of $\operatorname{Spec}\Theta$ contained all trianguline points. A local analogue of an argument of Gouvêa-Mazur [GM] and Böckle [Bö] then showed that these points were Zariski dense in $\operatorname{Spec} R_{G_{\mathbb{Q}_p}}[1/p]$. This showed that Θ was injective, and its surjectivity was reduced to a calculation involving a map of Ext groups. Colmez was soon able to carry out this calculation.

This allowed the association of a unitary $\mathrm{GL}_2(\mathbb{Q}_p)$ representation to each $G_{\mathbb{Q}_p}$-representation, however this was not yet useful since one could not say much about the locally algebraic vectors in the $\mathrm{GL}_2(\mathbb{Q}_p)$ representation attached to a de Rham representation of $G_{\mathbb{Q}_p}$. On the other hand just the existence of Colmez's functor made possible the application of our method to cases where $\bar{\rho}$ was reducible at p, and greatly simplified the arguments. Then, to our surprise, about a month after the conference Colmez informed us that it ought to be possible to prove that the locally algebraic vectors were of the right kind.

Finally, let us mention that using Colmez's correspondence, and especially the isomorphism Θ, Emerton has found an alternative approach to the Fontaine-Mazur conjecture (at least in many cases). His method has as a consequence a stronger version of the conjecture made in [Ki 4, 11.8], which we only dared raise as a question [Ki 4, 11.7(2)]. Namely that a two dimensional representation of $G_{\mathbb{Q},S}$ which is trianguline at p arises (up to twist) from an overconvergent modular eigenform.

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§1 Breuil-Mézard conjecture and the p-adic local Langlands.

(1.0) Notation: Throughout p will denote an odd prime. We denote by $\bar{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ and we write $G_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$ for the inertia subgroup. We will write $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ for the cyclotomic character.

We denote by $\bar{\mathbb{Z}}_p$ the ring of integers of $\bar{\mathbb{Q}}_p$, and by $\bar{\mathbb{F}}_p$ the residue field of $\bar{\mathbb{Z}}_p$. Let $\mathbb{Q}_p^{\mathrm{ab}} \subset \bar{\mathbb{Q}}_p$ denote the maximal abelian extension of $\mathbb{Q}_p$. Local class field theory gives an inclusion $\mathbb{Q}_p^\times \subset \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p)$ normalized to take uniformizers to geometric Frobenius. This allows us to consider characters of $G_{\mathbb{Q}_p}$ as characters of $\mathbb{Q}_p^\times$.

(1.1) The Breuil-Mézard conjecture: Let $E/\mathbb{Q}_p$ be a finite extension, and V a finite dimensional E-vector of dimension d, equipped with a continuous action of $G_{\mathbb{Q}_p}$.

Suppose that V is potentially semi-stable in the sense of Fontaine [Fo]. Attached to V is d-dimensional $\bar{\mathbb{Q}}_p$-representation of the Weil-Deligne group $\mathrm{WD}_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. Given a representation $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_d(\bar{\mathbb{Q}}_p)$ with open kernel, we say that V is of *type* τ if the restriction to $I_{\mathbb{Q}_p}$ of the associated Weil-Deligne group representation is equivalent to τ. This is possible only if τ extends to a representation of the Weil group of $\mathbb{Q}_p$. Such τ are said to be of *Galois type.*

Now let $\mathbb{F} \subset \bar{\mathbb{F}}_p$ be a subfield. Fix a continuous representation

$$\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F}),$$

and fix τ as above, with $d = 2$. We also fix an integer $k \geq 2$. When $\mathrm{End}_{\mathbb{F}[G_K]}\bar{\rho} = \mathbb{F}\,\bar{\rho}$ admits a universal deformation ring $R(\bar{\rho})$. In [BM] Breuil-Mézard conjectured that the deformations of $\bar{\rho}$ to characteristic 0 which are of type τ and with Hodge-Tate weights 0 and $k-1$ are parameterized by a quotient $R(k, \tau, \bar{\rho})$. Moreover, they gave a conjectural formula for the Hilbert-Samuel multiplicity of $R(k, \tau, \bar{\rho})/pR(k, \tau, \bar{\rho})$ in terms of certain representation theoretic data attached to the triple $(k, \tau, \bar{\rho})$.

We will recall this conjecture below. In fact we will define the corresponding invariant in all cases, not just those when $\bar{\rho}$ has trivial endomorphisms. Before giving this definition, we recall a result from [Ki 1], which establishes the existence and basic properties of the ring $R(k, \tau, \bar{\rho})$. In fact it will be more convenient to work with representations of fixed determinant.

Let $V_{\mathbb{F}}$ denote the underlying $\mathbb{F}$-vector space of $\bar{\rho}$. Recall that the universal framed deformation ring $R^{\square}(\bar{\rho})$ of $\bar{\rho}$ is the ring representing the functor which to a local Artin ring A with residue field $\mathbb{F}$, attaches the set of isomorphism classes of a deformation V_A of $\bar{\rho}$ to A, together with a lifting to V_A of some fixed choice of basis for $V_{\mathbb{F}}$.

We also fix a finite, totally ramified extension $E/W(\mathbb{F})[1/p]$ with ring of integers $\mathcal{O}$ and a uniformizer $\pi \in \mathcal{O}$ such that τ factors through $\mathrm{GL}_2(E)$, and a character $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$. For E'/E a finite extension we will denote by $\mathcal{O}_{E'}$ the ring of integers of E', and by $\pi_{E'}$ a uniformizer of E'.

Proposition (1.1.1). *There exists a unique (possibly trivial) quotient $R^{\square,\psi}(k,\tau,\bar{\rho})$ of $R^{\square}(\bar{\rho}) \otimes_{W(\mathbb{F})} \mathcal{O}$ with the following properties.*

1. *$R^{\square,\psi}(k,\tau,\bar{\rho})$ is p-torsion free, $R^{\square,\psi}(k,\tau,\bar{\rho})[1/p]$ is reduced and all its components are 5-dimensional.*
2. *If E'/E is a finite extension, then a map $x : R^{\square}(\bar{\rho}) \to E'$ factors through $R^{\square,\psi}(k,\tau,\bar{\rho})$ if and only if the corresponding E'-representation V_x is potentially semi-stable of type τ, with Hodge-Tate weights 0 and $k-1$ and determinant $\psi\chi$ where χ denotes the p-adic cyclotomic character.*

If $\bar{\rho}$ has only scalar endomorphisms, then there exists a quotient $R^{\psi}(k,\tau,\bar{\rho})$ of $R(\bar{\rho})$ with analogous properties, except that the dimension in (1) is 2 rather than 5.

(1.1.2) If E is not a finite extension of $\mathbb{Q}_p$, then the meaning of the condition that V_x is potentially semi-stable of type τ may not be completely clear. There are two ways to address this problem. The first is to extend the usual constructions of p-adic Hodge theory to representations over finite extensions of $W(\mathbb{F})[1/p]$. When taking tensor product of V_x with B_{cris} one should then take completed tensor product

$$V_x \widehat{\otimes}_{\mathbb{Q}_p} B_{\mathrm{cris}} := \cup_{i\geq 0} V_x \widehat{\otimes}_{\mathbb{Q}_p} t^{-i} B^+_{\mathrm{cris}},$$

while the tensor product of V_x with $B_{\mathrm{st}} = B_{\mathrm{cris}}[\ell_u]$ should be defined by tensoring the above by $\otimes_{B_{\mathrm{cris}}} B_{\mathrm{st}}$.

The second way, is to note that $\bar{\rho}$ is defined over a finite subfield $\mathbb{F}'' \subset \mathbb{F}$, and that τ and ψ are defined over a finite extension E'' of $W(\mathbb{F}'')[1/p]$. Let $\mathcal{O}_{E''}$ be the ring of integers of E''. Applying the above proposition one obtains a complete local $\mathcal{O}_{E''}$ algebra $R^{\psi,\square}(k,\tau,\bar{\rho})''$. Then one can *define* V_x in (2) to be potentially semi-stable of type τ if and only if x induces an E' valued point of $R^{\psi,\square}(k,\tau,\bar{\rho})''$.

Using the results of [Ki 1] one can show that these two definitions give equivalent notions of potentially semi-stable representations of type τ. The cautious reader can simply adopt the second definition here.

(1.1.3) Suppose that $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ is of Galois type. In the appendix to [BM] Henniart shows that there is a unique finite dimensional $\bar{\mathbb{Q}}_p$-representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathbb{Z}_p)$, with open kernel, such that if $\tilde{\tau}$ is any extension of $I_{\mathbb{Q}_p}$ to a representation of $\mathrm{WD}_{\mathbb{Q}_p}$, and π is the smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $\tilde{\tau}$ by the local Langlands correspondence, then $\pi|_{\mathrm{GL}_2(\mathbb{Z}_p)}$ contains $\sigma(\tau)$.

We may assume that $\sigma(\tau)$ is defined over E, (increasing E if necessary). Following [BM], we set $\sigma(k,\tau) = \sigma(\tau) \otimes_E \mathrm{Sym}^{k-2} E^2$. This is a finite dimensional representation of the compact group $\mathrm{GL}_2(\mathbb{Z}_p)$, and hence it contains a $\mathrm{GL}_2(\mathbb{Z}_p)$-stable $\mathcal{O}$-lattice $L_{k,\tau}$.

Now any irreducible, finite dimensional representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ on an $\mathbb{F}$-vector space is isomorphic to $\sigma_{n,m} = \mathrm{Sym}^n \bar{\mathbb{F}} \otimes \det^m$ where $n \in \{0,1,\dots,p-1\}$ and $m \in \{0,1,\dots p-2\}$. (Note that such a representation necessarily factors through $\mathrm{GL}_2(\mathbb{F}_p)$, since the normal subgroup $\ker(\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{F}_p))$ is a pro-p group, and hence has a fixed vector). Then we have

$$(L_{k,\tau})^{\mathrm{ss}} \otimes_{\mathcal{O}} \mathbb{F} \xrightarrow{\sim} \oplus_{n,m} \sigma_{n,m}^{a(n,m)}$$

where n and m run over the same ranges explained above.

We set

$$\mu_{\mathrm{Aut}} = \mu_{\mathrm{Aut}}(k, \tau, \bar{\rho}) = \sum_{n,m} a(n,m)\mu_{n,m}(\bar{\rho})$$

where $\mu_{n,m}(\bar{\rho}) \in \{0, 1, 2\}$ will be defined below.

(1.1.4) For i a positive integer, we denote by $\omega_i : I_{\mathbb{Q}_p} \to \bar{\mathbb{F}}_p^\times$ the fundamental character of level i, and we write $\omega = \omega_1$. Recall that if $\mathbb{Q}_{p^i}$ denotes the unramified extension of $\mathbb{Q}_p$, of degree i, and $\mathbb{Z}_{p^i}$ denotes the ring of integers of $\mathbb{Q}_{p^i}$, then ω_i is obtained by composing the maps

$$I_{\mathbb{Q}_p} \xrightarrow{\sim} I_{\mathbb{Q}_{p^i}} \xrightarrow{\sim} \mathbb{Z}_{p^i}^\times \to \bar{\mathbb{F}}_p^\times$$

where the second map is given by local class field theory normalized as in (1.0). We extend the map $\mathbb{Z}_{p^i}^\times \to \bar{\mathbb{F}}_p^\times$ to $\mathbb{Q}_{p^i}^\times$, by sending p to 1, and view ω_i as a character if $G_{\mathbb{Q}_p}$ via the class field theory isomorphism. In particular $\omega = \omega_1$ is then the mod p cyclotomic character.

Suppose first that $\bar{\rho}$ is absolutely irreducible. For $(n, m) \in \{0, 1, \dots, p-1\} \times \{0, 1, \dots p-2\}$ we set $\mu_{n,m}(\bar{\rho}) = 1$ if

$$\bar{\rho}|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega_2^{p(n+1)} \end{pmatrix} \otimes \omega^m$$

and $\mu_{n,m}(\bar{\rho}) = 0$ otherwise. Note that for a given $\bar{\rho}$, there are exactly two pairs (n, m) such that $\mu_{n,m}(\bar{\rho}) \neq 0$.

Suppose now that $\bar{\rho}$ is reducible. For $\lambda \in \bar{\mathbb{F}}_p^\times$, we denote by $\mu_\lambda : G_{\mathbb{Q}_p} \to \bar{\mathbb{F}}_p^\times$ the unramified character sending the geometric Frobenius to λ. We set $\mu_{n,m}(\bar{\rho}) = 0$ unless

$$\bar{\rho} \sim \begin{pmatrix} \omega^{n+1}\mu_\lambda & * \\ 0 & \mu_{\lambda'} \end{pmatrix} \otimes \omega^m$$

for $\lambda, \lambda' \in \bar{\mathbb{F}}_p^\times$, in which case we set

1. $\mu_{n,m}(\bar{\rho}) = 2$ if $\lambda = \lambda'$, $*$ is peu ramifié (including the case $*$ trivial) and $n = p - 1$.
2. $\mu_{n,m}(\bar{\rho}) = 0$ if $\lambda = \lambda'$, $*$ is très ramifié, and $n = 0$.
3. $\mu_{n,m}(\bar{\rho}) = 1$ otherwise

The following conjecture generalizes the Breuil-Mézard conjecture to the case when $\bar{\rho}$ has non-trivial endomorphisms. It is the crux of out approach to the Fontaine-Mazur conjecture, explained in the introduction, and we will prove most cases of it.

Conjecture (1.1.5). *The Hilbert-Samuel multiplicity of $R^{\square,\psi}(k, \tau, \bar{\rho})/(\pi)$ is equal to μ_{Aut}.*

(1.2) Review of Colmez's functor: We review some results of Colmez which allow one to attach a Galois representations to certain representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We begin by recalling the definition of some mod p $\mathrm{GL}_2(\mathbb{Q}_p)$ representations studied by Barthel-Livne and Breuil.

From now on we assume that $\mathbb{F} = \bar{\mathbb{F}}_p$.

(1.2.1) Write $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$ and denote by Z the center of $\mathrm{GL}_2(\mathbb{Q}_p)$. If σ is any representation of KZ on a finite dimensional $\mathbb{F}$-vector space V_σ, then we denote by $I(\sigma) = \mathrm{Ind}_{KZ}^G \sigma$ the compact induction of σ.

Recall [BL, Prop. 5] that $I(\sigma)$ has a natural action by the algebra of KZ-bi-invariant functions $\varphi : G \to \mathrm{End}_{\mathbb{F}} V_\sigma$. That is, the functions φ satisfying $\varphi(h_1 g h_2) = \sigma(h_1)\varphi(g_1)\sigma(h_2)$, for all $g \in G$, and $h_1, h_2 \in KZ$ acts on $I(\sigma)$. Explicitly, if $f \in I(\sigma)$ then this action is given by [BL, Prop. 5].

$$\varphi(f)(g) = \sum_{KZy \in KZ\backslash G} \varphi(gy^{-1})f(y) = \sum_{yKZ \in G/KZ} \varphi(y)f(y^{-1}g).$$

Next we regard $\mathbb{F}^2$ as a representation of KZ with $\mathrm{GL}_2(\mathbb{Z}_p)$ acting in the natural way via then map $\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{F}_p)$, and the element $p \in Z$ acting trivially. Let $r \in [0, p-1]$ be a non-negative integer, and set $\sigma = \mathrm{Sym}^r \mathbb{F}$. Denote by T the endomorphism of $I(\sigma)$ corresponding to the KZ-bi-invariant function which is supported on the double coset $KZ \begin{bmatrix} 1 & 0 \\ 0 & p^{-1} \end{bmatrix} KZ$ and takes $\begin{bmatrix} 1 & 0 \\ 0 & p^{-1} \end{bmatrix}$ to the endomorphism $\mathrm{Sym}^r \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. According to [BL, Prop. 8] $\mathbb{F}[T]$ is the full endomorphism algebra of $I(\sigma)$.

Let $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ be a character, and $\lambda \in \mathbb{F}$. For $x \in \mathbb{F}$ we denote by $\mu_x : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ the unramified character sending $p \in \mathbb{Q}_p^\times$ to x.

We set $\pi(r, \lambda, \chi) = I(\sigma)/(T - \lambda)I(\sigma) \otimes \chi \circ \det$. The structure of these representations is given by the following result [BL, Thm. 30, Cor. 36], [Br 1, Thm. 1.1, 1.3], where Sp denotes the space of $\mathbb{F}$-valued, locally constant functions on $\mathbb{P}^1(\mathbb{Q}_p)$, modulo the space of constant functions.

Proposition (1.2.2).

1. *$\pi(r, \lambda, \chi)$ is irreducible unless $(r, \lambda) \in \{(0, \pm 1), (p-1, \pm 1)\}$.*
2. *If $(r, \lambda) = (0, \pm 1)$ then $\pi(r, \lambda, \chi)$ is a non-trivial extension of $\chi\mu_{\pm 1} \circ \det$ by $\chi\mu_{\pm 1} \circ \det \otimes \mathrm{Sp}$.*
3. *If $(r, \lambda) = (p-1, \pm 1)$ then $\pi(p-1, \lambda, \chi)$ is a non-trivial extension of $\chi\mu_{\pm 1} \circ \det \otimes \mathrm{Sp}$ by $\chi\mu_{\pm 1} \circ \det$.*
4. *If (r, χ, λ) and (r', χ', λ') are two such triples then there exists an isomorphism*
$$\pi(r, \lambda, \chi) \xrightarrow{\sim} \pi(r', \lambda', \chi')$$
exactly in the following cases:
   - *(i) $r = r'$, or and $\{\chi', \lambda'\}$ is $\{\chi, \lambda\}$ or $\{\chi\mu_{-1}, -\lambda\}$.*
   - *(ii) $\lambda = 0$, $r' = p - 1 - r$ and $\chi' \in \{\chi\omega^r, \chi\omega^r\mu_{-1}\}$.*
   - *(iii) $\{r, r'\} = \{0, p-1\}$, $\lambda \neq \pm 1$, and $\{\chi', \lambda'\}$ is $\{\chi, \lambda\}$ or $\{\chi\mu_{-1}, -\lambda\}$.*

(1.2.3) Let Π be a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a $W(\mathbb{F})$-module. If Π has finite length, we say that Π is *admissible* if each of its Jordan-Hölder factors has a central character.

If Π has finite length then it is a $W_n(\mathbb{F})$-module for some $n \geq 1$, and the admissibility condition implies that the Jordan-Hölder factors of Π are either 1-dimensional, or an infinite dimensional subquotient of some $\pi(r, \lambda, \chi)$ [Br 1, 1.2].

We have the following result of Colmez.

Theorem (1.2.4). *There exists an exact contravariant functor V^* from the category finite length, admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-representations to the category of finite length representations of $W(\mathbb{F})[G_{\mathbb{Q}_p}]$. Moreover, we have*

1. *$V^*(\Pi) = 0$ if Π is 1-dimensional*

(2) $V^*(\pi(r,\lambda,\chi)) = \chi\mu_{\lambda^{-1}}$ *if* $\lambda \neq 0$.
(3) $V^*(\pi(r,0,\chi)) = \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1} \otimes \chi$.

(1.2.5) It will often be more convenient to use a covariant functor. For this, suppose we fix a character $G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ as before, which we regard as a character of $\mathbb{Q}_p^\times$ via local class field theory. Suppose that Π is a finite length $\mathcal{O}[\mathrm{GL}_2(\mathbb{Q}_p)]$-module, which is admissible as a $W(\mathbb{F})[1/p]$-module. Then we define $V_\psi(\Pi) = (V^*(\Pi))^*(\chi_{\mathrm{cyc}}\psi)$ where $V^*(\Pi)^*$ denotes the Pontryagin dual of the finite length $\mathcal{O}$-module $V^*(\Pi)$. We will typically only use this functor when Π has central character ψ. The formulas (2) and (3) of (1.2.4) then become

(1) $V_\psi(\pi(r,\lambda,\chi)) = \omega^{r+1}\mu_\lambda\chi$ if $\lambda \neq 0$.
(2) $V_\psi(\pi(r,0,\chi)) = \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1} \otimes \chi$.

where in each case $\psi = \omega^r\chi^2$ is the central character of the representation to which the functor V_ψ is being applied.

Suppose now that Π is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a $W(\mathbb{F})$-module and set $\Pi_n = \Pi \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$. Suppose that Π is p-adically complete and separated, so that $\Pi = \varprojlim \Pi_n$, and that for each n Π_n is of finite length and admissible. We set $V_\psi(\Pi) = \varprojlim V_\psi(\Pi_n)$. Since admissible representations have finite length inverse limits in this category are exact, so one sees that $V_\psi(\Pi)/pV_\psi(\Pi) = V_\psi(\Pi_1)$, and in particular that $V_\psi(\Pi)$ is a finitely generated $W(\mathbb{F})$-module, since it is p-adically separated. We call such a representation Π an admissible lattice. If it carries the structure of an $\mathcal{O}$-module, we call it an admissible $\mathcal{O}$-lattice.

The following result should, hopefully, soon be proved by Colmez. For trianguline representations it is proved in [Co] and [BB 1]. In the rest of the paper, we proceed as if (1.2.6) is known. The reader who wishes to remain on completely firm ground can assume that we deal only with representations which become semi-stable over an abelian extension of $\mathbb{Q}_p$. This corresponds to the representation τ in (1.1) being abelian.

Expectation/Theorem (1.2.6). *Let E'/E be a finite extension and V a 2-dimensional E'-vector space equipped with a continuous action of $G_{\mathbb{Q}_p}$. Suppose that V is potentially semi-stable of type τ with Hodge-Tate weights $0, k-1$ ($k \geq 2$) and that $\det V = \psi\chi$.*

Then there exists an admissible $\mathcal{O}_{E'}$-lattice Π with central character ψ such that $V_\psi(\Pi) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} V$. If Π' is another such lattice, then there exists a continuous isomorphism of $E'[\mathrm{GL}_2(\mathbb{Q}_p)]$-modules $\Pi' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \Pi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Moreover, there exists a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant inclusion $\sigma(k,\tau) \hookrightarrow \Pi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

(1.3) Hilbert-Samuel multiplicities: Suppose that A is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finite A-module. There is a polynomial $P_M^A(X)$ such that $P_M^A(n)$ is equal to the length of $M/\mathfrak{m}^{n+1}M$ for sufficiently large integers n.

If A has dimension d, then P_M^A has degree at most d, and the Hilbert-Samuel multiplicity $e(M,A)$ of M is defined to be $d!$ times the coefficient of X^d in P_M^A.

Suppose now that G is a group, and that M is equipped with an action of G. Let α be a collection of irreducible representations of G on finite dimensional $A/\mathfrak{m}$-vector spaces. Then instead of considering the length of $M/\mathfrak{m}^{n+1}M$ one can consider the number of Jordan-Hölder factors of $M/\mathfrak{m}^{n+1}M$ as an $A[G]$-module, which are isomorphic to an element of α. We denote this number by $\chi^A_{M,\alpha}(n)$.

Proposition (1.3.1). *There is a polynomial $P^A_{M,\alpha}$ of degree at most d such that for sufficiently large n such that $\chi^A_{M,\alpha}(n) = P^A_{M,\alpha}(n)$ for sufficiently large positive integers n. Moreover the coefficient of X^d in $P^A_{M,\alpha}$ has the form $e_\alpha(M,A)/d!$ where $e_\alpha(M,A)$ is a non-negative integer.*

Proof. The proof is identical to the standard result for G trivial [Ma, §13]. Note that one only has to show that $P^A_{M,\alpha}$ as above, of some degree exists, since the bound on the degree follows from the case when G is trivial. □

Proposition (1.3.2). *If*

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of $A[G]$-modules which are finite over A, then we have

$$e_\alpha(M,A) = e_\alpha(M',A) + e_\alpha(M'',A)$$

Proof. The proof with G trivial goes over unchanged [Ma, Thm. 14.6] □

Proposition (1.3.3). *Let $f : M \to M'$ be an inclusion of A-finite $A[G]$-modules, and $x \in A$ such that M and M' have no x-torsion.*

(1) *If f is an inclusion then*

$$e_\alpha(M/xM, A/xA) \leqslant e_\alpha(M'/xM', A/xA).$$

(2) *If f is an isomorphism at all the generic points of* $\operatorname{Spec} A$, *then*

$$e_\alpha(M/xM, A/xA) = e_\alpha(M'/xM', A/xA).$$

Proof. Let $P = \ker(f)$. If $\mathfrak{p} \in \operatorname{Spec} A/x \subset \operatorname{Spec} A$ is a minimal prime of A/x, such that $A/\mathfrak{p}$ has dimension $d-1$, then then $P_\mathfrak{p} = 0$, for otherwise $\mathfrak{p}$ would be an associated prime of P, [Ma, Thm. 6.5] and $x \in \mathfrak{p}$ would be a zero divisor of M. Hence $e_\alpha(P/xP, A/xA) = 0$, and we may replace M by its image in M' in (2).

Next let $Q \subset M'/M$ be the submodule consisting of elements which are killed by some power of x. Choose $i > 0$ so that x^i kills Q. The sequence

$$0 \to Q[x] \to Q \xrightarrow{x} Q \to Q/xQ \to 0$$

and (1.3.2) shows that

$$e_\alpha(Q[x], A/xA) = e_\alpha(Q[x], A/x^iA) = e_\alpha(Q/xQ, A/x^iA) = e_\alpha(Q/xQ, A/xA).$$

Hence, if M'' denotes the preimage of Q in M'', then using (1.3.2) we see that $e_\alpha(M/xM, A/xA) = e_\alpha(M''/xM'', A/xA)$. Hence we may replace M by M'' and assume that M'/M is x-torsion free.

Now (1) follows from (1.3.2), and the same argument as in the first paragraph shows that under the hypothesis of (2), $e_\alpha(M'/(M+xM'), A/xA) = 0$, so (2) also follows □

(1.3.4) We now return to the situation without the action of a group. If $\mathfrak{q} \subset A$ is any $\mathfrak{m}$-primary ideal, and M is a finite A-module, then there is a polynomial $P_\mathfrak{q}$ of degree at most d such that the length of $M/\mathfrak{q}^{n+1}M$ is given by $P_\mathfrak{q}(n)$. As above, we write $e_\mathfrak{q}(M,A)$ for $d!$ times the leading coefficient of $P_\mathfrak{q}$. If $M = A$ we write simply $e_\mathfrak{q}(A)$ for $e_\mathfrak{q}(A,A)$. If $\mathfrak{q} = \mathfrak{m}$ we sometimes abbreviate $e_\mathfrak{q}(A)$ to $e(A)$.

Proposition (1.3.5). *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local map of Noetherian complete local rings such that*

$$\dim B = \dim A + \dim B/\mathfrak{m}B. \tag{1.3.6}$$

Then

$$e_{\mathfrak{n}}(B) \leqslant e_{\mathfrak{m}}(A) e_{\mathfrak{n}/\mathfrak{m}B}(B/\mathfrak{m}B). \tag{1.3.7}$$

Proof. We first reduce to the case where A has infinite residue field. Suppose $A/\mathfrak{m}$ is finite. Let $B_0 \subset B$ be a coefficient ring for B [Ma, Thm. 29.3]. So $B_0 \xrightarrow{\sim} B/\mathfrak{n}$ if B has equal characteristic $p > 0$, and B_0 is a discrete valuation ring with uniformizer p, and $B_0/pB_0 \xrightarrow{\sim} B/\mathfrak{n}$, if B has mixed characteristic. Since $A/\mathfrak{m}$ is finite A contains a unique coefficient ring A_0, which maps to B_0. One checks easily that replacing A by the $\mathfrak{m}$-adic completion of $A \otimes_{A_0} B_0$ does not change either side of (1.3.7). In particular, we may assume that A and B have the same residue field.

If B_0/pB_0 is a finite field, let κ' be an algebraic closure of B_0/pB_0. Again, one sees easily that replacing A by the $\mathfrak{m}$-adic completion of $A \otimes_{A_0} W(\kappa')$ and B by the $\mathfrak{n}$-adic completion of $B \otimes_{B_0} W(\kappa')$ does not change either side (1.3.7). Thus, we may assume that A and B have infinite residue fields.

We now prove the proposition by induction on $\dim A$. Suppose first that $\dim A = 0$, so that A is an Artin ring, and $e_{\mathfrak{m}}(A)$ is its length. We prove the inequality by induction on the length of A. If this is 1, then A is a field, and there is noting to prove. Let $I \subset A$ be an ideal such that $\mathfrak{m}I = 0$. Then using the induction hypothesis, and (1.3.2), we have

$$\begin{aligned} e_{\mathfrak{n}}(B) &\leqslant e_{\mathfrak{n}}(I \otimes_A B, B) + e_{\mathfrak{n}}(B/I, B) = e_{\mathfrak{n}}(I \otimes_{A/\mathfrak{m}} B/\mathfrak{m}, B) + e_{\mathfrak{n}}(B/I, B) \\ &\leqslant (\dim_{A/\mathfrak{m}} I + e_{\mathfrak{m}/I}(A/I)) e_{\mathfrak{n}/\mathfrak{m}B}(B/\mathfrak{m}) = e_{\mathfrak{m}}(A) e_{\mathfrak{n}/\mathfrak{m}B}(B/\mathfrak{m}). \end{aligned}$$

Suppose that $\dim A > 0$. Let $0 = \cap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition of $\{0\} \subset A$, where the radicals $\mathfrak{p}_i = r(\mathfrak{q}_i)$ satisfy $\dim A/\mathfrak{p}_i = \dim A$ if and only if $1 \leqslant i \leqslant m$ for some $m \leqslant n$. Let $I = \cap_{i=1}^m \mathfrak{q}_i$. By (1.3.3) (applied with $G = 1$) replacing A by A/I does not change the right hand side of (1.3.7), and (1.3.6) implies that replacing B by B/IB does not change the left hand side of (1.3.7) [Ma, Thm. 15.1]. Thus we may assume that A has no embedded primes, and that for each minimal prime $\mathfrak{p}_i$ of A the quotient $A/\mathfrak{p}_i$ has dimension equal to $\dim A$.

By [Ma, Thm. 14.14], since $A/\mathfrak{m}$ is infinite, there is a $\mathfrak{m}$-primary ideal $\mathfrak{q} \subset A$ such that $\mathfrak{m}^{r+1} = \mathfrak{q}\mathfrak{m}^r$ for some $r > 0$, (an ideal $\mathfrak{q}$ with this property is called a reduction of $\mathfrak{m}$) and $\mathfrak{q}$ is generated by a sequence of parameters $x_1, \dots x_d$ for A. The condition (1.3.6) implies that $x_1, \dots, x_d$ extends to a sequence of parameters of B, which implies that

$$e_{\mathfrak{n}}(B) \leqslant e_{\mathfrak{n}/x_1 B}(B/x_1 B) \tag{1.3.8}$$

by [Ma, Thm. 14.9].

None of the x_j can be a zero-divisor, since otherwise we would have $x_j \in \mathfrak{p}_i$ for a minimal prime $\mathfrak{p}_i$, and $\dim A/x_j A \geq \dim A/\mathfrak{p}_i = \dim A$, which is impossible [Ma, Thm. 13.6]. Hence if we set $x = x_1$, then

$$e_{\mathfrak{m}}(A) = e_{\mathfrak{q}}(A) = e_{\mathfrak{q}/xA}(A/xA) = e_{\mathfrak{m}/xA}(A/xA)$$

where the second equality follows from [Ma, Thm. 14.11] and the fact that x is not a zero divisor, while the other two equalities follow from [Ma, Thm. 14.13] since $\mathfrak{q}$ and $\mathfrak{q}/xA$ are reductions of $\mathfrak{m}$ and $\mathfrak{m}/xA$ respectively. Finally, we have

$$e_{\mathfrak{n}}(B) \leqslant e_{\mathfrak{n}/xB}(B/xB) \leqslant e_{\mathfrak{m}/xA}(A/xA)e_{\mathfrak{n}/\mathfrak{m}B}(B/\mathfrak{m}B) = e_{\mathfrak{m}}(A)e_{\mathfrak{n}/\mathfrak{m}B}(B/\mathfrak{m}B),$$

where in the second inequality we have used the induction hypothesis applied to A/xA, which we saw above has dimension $< \dim A$. $\square$

Proposition (1.3.9). *Let κ be a field and $(A_1, \mathfrak{m}_1)$ and $(A_2, \mathfrak{m}_2)$ Noetherian, complete local κ-algebras with residue field κ. Write $\mathfrak{n}$ for the radical of $B = A_1 \widehat{\otimes}_\kappa A_2$. Then*

$$e_{\mathfrak{n}}(B) = e_{\mathfrak{m}_1}(A_1)e_{\mathfrak{m}_2}(A_2).$$

Proof. We repeat the proof of (1.3.5) with $A = A_1$. As in (1.3.5), we may assume that κ is infinite. If $\dim A_1 = 0$, then since B is flat over A_1, one sees by induction on the length of A_1, that

$$e_{\mathfrak{n}}(B) = e_{\mathfrak{m}_1}(A)e_{\mathfrak{n}/\mathfrak{m}_1 B}(B/\mathfrak{m}_1 B) = e_{\mathfrak{m}_1}(A_1)e_{\mathfrak{m}_2}(A_2).$$

Suppose that $d = \dim A_1 > 0$. As in the proof of (1.3.5), we may assume that $\dim A_1/\mathfrak{p} = \dim A_1$, for any minimal prime $\mathfrak{p} \subset A_1$. Let $\mathfrak{q}_1 \subset \mathfrak{m}_1$ and $\mathfrak{q}_2 \subset \mathfrak{m}_2$ be reductions of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively, which are generated by systems of parameters: $\mathfrak{q}_1 = \langle x_1, \dots x_d\rangle$ and $\mathfrak{q}_2 = \langle y_1, \dots, y_e\rangle$. If $\mathfrak{m}_1^{r+1} = \mathfrak{q}_1\mathfrak{m}_1^r$ and $\mathfrak{m}_2^{s+1} = \mathfrak{q}_2\mathfrak{m}_2^s$ then

$$\begin{aligned}\mathfrak{n}^{r+s+1} &= (\mathfrak{m}_1 B + \mathfrak{m}_2 B)^{r+s+1}\\ &\subset \mathfrak{q}_1(\mathfrak{m}_1 B + \mathfrak{m}_2 B)^{r+s} + \mathfrak{q}_2(\mathfrak{m}_1 B + \mathfrak{m}_2 B)^{r+s} = (\mathfrak{q}_1 B + \mathfrak{q}_2 B)\mathfrak{n}^{r+s}\end{aligned}$$

so $\mathfrak{q} = \mathfrak{q}_1 B + \mathfrak{q}_2 B$ is a reduction of $\mathfrak{n}$. Now $x = x_1$ is a not a zero divisor in A_1, and hence it is not a zero divisor in $A_1 \otimes_\kappa A_2/\mathfrak{m}_2^j$ for any $j \geq 1$, or in B. Thus using [Ma, Thm 14.11] we find

$$e_{\mathfrak{n}}(B) = e_{\mathfrak{q}}(B) = e_{\mathfrak{q}/xB}(B/xB) = e_{\mathfrak{n}/xB}(B/xB).$$

Thus (1.3.8) is an equality in this context. Arguing exactly as in the last part of of the proof of (1.3.5) now proves that $e_{\mathfrak{n}}(B) = e_{\mathfrak{m}_1}(A_1)e_{\mathfrak{m}_2}(A_2)$ by induction on d. $\square$

(1.4) Deformation rings and pseudo-deformation rings: In this subsection we compare deformation rings of Galois representations with the corresponding pseudo-deformation ring.

(1.4.1) Let G be a group and R a commutative ring with 1. Recall [Ta 1, §1] that a pseudo-representation of G over R of dimension d is a function $T : G \to R$ such that T has the following properties of the trace of a representation of G on a finite free R-module.

(1) $T(1) = d$
(2) $T(g_1g_2) = T(g_2g_1)$ for $g_1, g_2 \in G$.
(3) $\sum_{\sigma\in S_{d+1}} \varepsilon(\sigma)T_\sigma(g_1, \dots, g_{d+1}) = 0$ for $g_1, \dots g_{d+1} \in G$, where S_{d+1} is the symmetric group on $d+1$ letters, $\varepsilon(\sigma)$ denotes the sign of σ, and if σ has the cycle decomposition

$$(i_1^1, i_1^2, \dots, i_1^{k_1})(i_2^1, \dots, i_2^{k_2})\dots(i_{m_\sigma}^1, \dots, i_{m_\sigma}^{k_{m_\sigma}})$$

then $T_\sigma : G^{d+1} \to R$ is the function

$$(g_1, \dots, g_{d+1}) \mapsto T(i_1^1 \dots i_1^{k_1})T(i_2^2 \dots i_2^{k_2}) \dots T(i_{m_\sigma}^1 \dots i_{m_\sigma}^{k_{m_\sigma}}).$$

If $A \to A'$ is a surjection of rings, and $T_{A'} : G \to A'$ is a pseudo-character, then by a deformation of $T_{A'}$ to A we mean a lifting of $T_{A'}$ to an A-valued pseudo-character. If T is a pseudo-representation of G over R, then we may regard T as map $R[G] \to R$ by linearity.

In the following we shall work with a profinite, finitely topologically generated group G. Let κ be a topological field. If κ is discrete and has characteristic $p > 0$, then we set W equal either to κ or to a Cohen ring for κ. In all other cases, we set $W = \kappa$.

Suppose that $T_\kappa : G \to \kappa$ is a continuous pseudo-representation of dimension d. For a local Artinian W-algebra A with residue field κ, denote by $D^{\text{ps}}_{T_\kappa}(A)$ the set of continuous deformations of T_κ to A.

Lemma (1.4.2). *$D^{\text{ps}}_{T_\kappa}$ is (pro-)represented by a Noetherian, complete local W-algebra $R^{\text{ps}}_{T_\kappa}$.*

Proof. By [Ta 1, Thm. 1] there is a finite subset $S \subset G$ such that a continuous pseudo-representation of G is determined by its values on S. This implies that the tangent space $D^{\text{ps}}_{T_\kappa}(\kappa[\epsilon])$ is finite dimensional over κ. The lemma now follows directly from Grothendieck's representability criterion [Maz, §18]. □

Lemma (1.4.3). *Let V_κ be a finite dimensional κ-vector space equipped with a continuous action of G such that $\operatorname{End}_{\kappa[G]} V_\kappa = \kappa$. Let R_{V_κ} denote the universal deformation ring of V_κ and T_κ the pseudo-deformation corresponding to V_κ. Let*

$$\theta : R^{\text{ps}}_{T_\kappa} \to R_{V_\kappa}$$

denote the map induced by sending a G-representation to its trace.

(1) *If V_κ is absolutely irreducible, the θ is an isomorphism.*
(2) *If $\operatorname{char} \kappa \neq 2$, V_κ is a non-trivial extension of ω_1 by ω_2 for two distinct character ω_1 and ω_2 of G, and $\operatorname{Ext}^1_{\kappa[G]}(\omega_1, \omega_2)$ is 1-dimensional over κ, then θ is a surjection.*

Proof. (1) follows from a result of Nyssen [Ny]. To prove (2) we shall adapt an argument of Carayol which applies when V_κ is absolutely irreducible [Ca, Thm. 1].

It suffices to show that θ induces a surjection on tangent spaces. For this, fix a basis of V_κ such that the resulting representation $\bar\rho : \kappa[G] \to \mathrm{GL}_2(\kappa)$ is upper triangular. Let $A = \kappa[\epsilon]/\epsilon^2$ denote the dual numbers over κ, and suppose that $\rho : A[G] \to \mathrm{GL}_2(A)$ is a deformation of ρ, which satisfies $\mathrm{tr}\rho(\sigma) = \mathrm{tr}\bar\rho(\sigma)$ for $\sigma \in A[G]$. Write $\rho(\sigma) = \bar\rho(\sigma) + \Delta(\sigma)$ where $\Delta(\sigma) \in \mathrm{M}_2(\epsilon\kappa)$. Since ρ is a ring map. one sees that

$$\mathrm{tr}(\bar\rho(\sigma_1)\Delta(\sigma_2) + \Delta(\sigma_1)\bar\rho(\sigma_2)) = \mathrm{tr}(\Delta(\sigma_1\sigma_2)) = 0$$

for $\sigma_1, \sigma_2 \in \kappa[G]$. Taking $\sigma_1 \in \ker \bar\rho$ we see that $\mathrm{tr}(\Delta(\sigma_1)\bar\rho(\sigma_2)) = 0$ for all $\sigma_2 \in \kappa[G]$. Our hypotheses imply that $\bar\rho(\kappa[G])$ consists of all upper triangular matrices in $\mathrm{M}_2(\kappa)$, so $\Delta(\sigma_1)$ has the form $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$.

Now for $\sigma \in A[G]$ write $\Delta(\sigma) = \begin{pmatrix} \alpha(\sigma) & \beta(\sigma) \\ \gamma(\sigma) & -\alpha(\sigma) \end{pmatrix} \cdot \epsilon$, and $\bar\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ 0 & d(\sigma) \end{pmatrix}$. If σ satisfies $\bar\rho(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ computing the trace of $\Delta(\sigma^2)$ shows that $\alpha(\sigma) = 0$. Choose $\sigma_0 \in A[G]$ so that $\bar\rho(\sigma_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then the calculations above show that for any $\sigma \in A[G]$ $\alpha(\sigma) = \alpha(\sigma_0)b(\sigma)$. Hence after replacing ρ by $U\rho U^{-1}$, where $U = \begin{pmatrix} 1 & 0 \\ \alpha(\sigma_0) & 1 \end{pmatrix}$ we may assume that $\alpha(\sigma) = 0$ for all $\sigma \in A[G]$.

But now $\sigma \mapsto b(\sigma) + \beta(\sigma)$ gives a $\mathbb{F}[\epsilon]^\times$-valued cocycle corresponding to an extension of ω_1 by ω_2. Since $\mathrm{Ext}^1_{\mathbb{F}[G]}(\omega_1, \omega_2)$ is 1-dimensional, this cocycle vanishes on $\ker\bar\rho$, and so Δ vanishes on the kernel of $\bar\rho$. We can now conclude as in Carayol's argument: Δ corresponds to a derivation of $\bar\rho(\kappa[G])$, which is necessarily inner, and this shows that ρ is equivalent to $\bar\rho$. $\square$

Corollary (1.4.4). *Suppose that $G = G_{\mathbb{Q}_p}$ and $\kappa \subset \bar{\mathbb{F}}_p$, and that V_κ is as in (1.4.3) and satisfies one of the conditions (1) or (2). Then the map*

$$(R^{\mathrm{ps}}_{T_\kappa}[1/p])^{\mathrm{red}} \to (R_{V_\kappa}[1/p])^{\mathrm{red}} \tag{1.4.5}$$

induced by θ is an isomorphism.

Proof. When V_κ satisfies (1.4.3)(1), there is nothing to show. Suppose that it satisfies (1.4.3)(2). Note that (1.4.5) is a surjection between reduced Jacobson rings, and so it suffices to check that induces a surjection on closed points. If $E/W[1/p]$ is a finite extension, and x an E-valued point of $(R^{\mathrm{ps}}_{T_\kappa}[1/p])^{\mathrm{red}}$, then after replacing E by a finite extension, we may assume that x corresponds to a G-representation V_x.

Let $\mathcal{O}_E$ denote the ring of integers of E, and π_E a uniformiser. If V_x is absolutely irreducible, then it contains a lattice whose reduction mod π_E is an extension of ω_1 by ω_2, and so x corresponds to a point of R_{V_κ}. If V_x is reducible, then it is an extension of two characters $\tilde\omega_1$ and $\tilde\omega_2$ lifting ω_1 and ω_2 respectively. Now any extension of $\tilde\omega_1$ by $\tilde\omega_2$ gives rise to the pseudo-representation corresponding to x. Thinking of $\tilde\omega_1$ and $\tilde\omega_2$ as $\mathcal{O}_E^\times$-valued characters, consider the map

$$\mathrm{Ext}^1_{\mathcal{O}_E[G_{\mathbb{Q}_p}]}(\tilde\omega_1, \tilde\omega_2) \to \mathrm{Ext}^1_{\kappa[G_{\mathbb{Q}_p}]}(\omega_1, \omega_2).$$

Since the right hand side is a finitely generated $\mathcal{O}_E$-module, the image of this map is non-zero, and hence it is surjective. It follows that we may assume that V_x has a lattice which gives rise to V_κ, which again shows that x is induced by a point of R_{V_κ}. $\square$

Corollary (1.4.6). *Let $T_{\mathbb{F}}$ be a two dimensional pseudo-representation of $G_{\mathbb{Q}_p}$ over $\mathbb{F}$ and assume that $T_{\mathbb{F}}$ is either irreducible, or a sum of two distinct pseudo-representations of dimension* 1*, given by $\mathbb{F}^\times$-valued characters ω_1 and ω_2 of $G_{\mathbb{Q}_p}$. If $p = 3$ assume also that $\omega_1\omega_2^{-1} \neq \omega$.*

Denote by $R^{\mathrm{ps},\circ}_{T_{\mathbb{F}}}$ the image of $R^{\mathrm{ps}}_{T_{\mathbb{F}}}$ in $(R^{\mathrm{ps}}_{T_{\mathbb{F}}}[1/p])^{\mathrm{red}}$. Then there is a finite free $R^{\mathrm{ps},\circ}_{T_{\mathbb{F}}}$-module M of rank 2*, equipped with a continuous action of $G_{\mathbb{Q}_p}$, such that for $\sigma \in G_{\mathbb{Q}_p}$ the trace of σ on M is given by $T(\sigma) \in R^{\mathrm{ps},\circ}_{T_{\mathbb{F}}}$.*

Proof. This follows from (1.4.4) once we remark that, ω_1 and ω_2 are distinct, then $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\omega_1, \omega_2)$ is one dimensional, provided that $\omega_2\omega_1^{-1}$ is not the mod p cyclotomic character. Since we can exchange the roles of ω_1 and ω_2, the only case in which (1.4.4) does not apply is when $\omega_2\omega_1^{-1} = \omega = \omega^{-1}$, which can happen only if $p = 3$. $\square$

Proposition (1.4.7). *Suppose $\omega_1, \omega_2 : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ are two $\mathbb{F}$-valued characters such that $\omega_1\omega_2^{-1} \notin \{\omega, \omega^{-1}, 1\}$, let $V_{\mathbb{F}} = \omega_1 \oplus \omega_2$ and denote by $T_{\mathbb{F}}$ the corresponding pseudo-representation. Write $R^{\square}_{V_{\mathbb{F}}}$ for the universal framed deformation ring of $V_{\mathbb{F}}$ as in (1.1), and $\mathfrak{m}_{T_{\mathbb{F}}}$ and $\mathfrak{m}_{V_{\mathbb{F}}}$ for the radicals of $R^{\mathrm{ps}}_{T_{\mathbb{F}}}$ and $R_{V_{\mathbb{F}}}$ respectively. Then*

(1) $\dim R^{\square}_{V_{\mathbb{F}}} = \dim R^{\mathrm{ps}}_{T_{\mathbb{F}}} + \dim R^{\square}_{V_{\mathbb{F}}}/\mathfrak{m}_{T_{\mathbb{F}}} R^{\square}_{V_{\mathbb{F}}}$.
(2) $\dim R^{\square}_{V_{\mathbb{F}}}/\mathfrak{m}_{T_{\mathbb{F}}} R^{\square}_{V_{\mathbb{F}}} = 3$.
(3) $e(R^{\square}_{V_{\mathbb{F}}}/\mathfrak{m}_{T_{\mathbb{F}}} R^{\square}_{V_{\mathbb{F}}}) = 2$.

Proof. A standard cohomological calculation shows that $\dim R^{\square}_{V_{\mathbb{F}}} = 8$ (cf. [Ki 2, 2.3.4]), and using (1.4.3) and (1.4.6), one obtains in a similar way that $\dim R^{\mathrm{ps}}_{T_{\mathbb{F}}} = 5$. Hence we need to show that $R = R^{\square}_{V_{\mathbb{F}}}/\mathfrak{m}_{T_{\mathbb{F}}} R^{\square}_{V_{\mathbb{F}}}$ is 3-dimensional, and that if $\mathfrak{m}$ denotes its maximal ideal then $e_{\mathfrak{m}}(R) = 2$.

Now consider the functor which to a local $\mathbb{F}$-algebra, with residue field $\mathbb{F}$ assigns the set of framed deformations V_A of $V_{\mathbb{F}}$ to A, such that V_A is an extension of ω_2 by ω_1 (viewed as A-valued characters). Since $\omega_1 \neq \omega_2$, there is a unique finite free, rank 1 A-submodule $L_A \subset V_A$ on which $G_{\mathbb{Q}_p}$ acts via ω_1. Using this, one sees easily that this functor is representable by a complete local $\mathbb{F}$-algebra R_{ω_1}. We define in a similar way a complete local $\mathbb{F}$-algebra R_{ω_2}. Since

$$H^2(G_{\mathbb{Q}_p}, \omega_1\omega_2^{-1}) = H^2(G_{\mathbb{Q}_p}, \omega_2\omega_1^{-1}) = 0,$$

R_{ω_1} and R_{ω_2} are formally smooth, and since the space of extensions of ω_2 by ω_1 (resp. ω_1 by ω_2) is 1-dimensional one finds that

$$\dim R_{\omega_1} = \dim R_{\omega_2} = 3.$$

Now let V_R denote the tautological framed deformation of $V_{\mathbb{F}}$ to R. Let $\mathfrak{p}$ be a minimal prime of R, and $\kappa(\mathfrak{p})$ the residue field of $\mathfrak{p}$. Write $V_{R/\mathfrak{p}} = V_R \otimes_R R/\mathfrak{p}$. Then $(V_{R/\mathfrak{p}})_{\mathbb{F}(\mathfrak{p})}$ has semi-simplification $\omega_1 \oplus \omega_2$. Suppose that $(V_{R/\mathfrak{p}})_{\mathbb{F}(\mathfrak{p})}$ is an extension of ω_2 by ω_1. We claim that $\mathfrak{p}$ is induced by a prime of R_{ω_1}. To see this, for any $\mathbb{F}[G_{\mathbb{Q}_p}]$-module M and $i = 1, 2$ write $M[\omega_i] \subset M$ for the submodule on which $G_{\mathbb{Q}_p}$ acts by ω_i. Then $V_{R/\mathfrak{p}}[\omega_1]$ has $R/\mathfrak{p}$-rank 1, and $G_{\mathbb{Q}_p}$ acts on $V_{R/\mathfrak{p}}/(V_{R/\mathfrak{p}}[\omega_1])$ via ω_2. Since $V_{R/\mathfrak{p}}/(V_{R/\mathfrak{p}_1}[\omega_1]) \otimes_R R/\mathfrak{m}$ is a quotient of $V_R \otimes_R R/\mathfrak{m}$, it follows that it is one dimensional over $\mathbb{F}$, so that $V_{R/\mathfrak{p}}/(V_{R/\mathfrak{p}}[\omega_1])$ is a free $R/\mathfrak{p}$-module of rank 1, and the same argument now shows that $V_{R/\mathfrak{p}}$ is free of rank 1 over $R/\mathfrak{p}$.

This shows that $\mathfrak{p}$ contains the kernel of $R \to R_{\omega_1}$. Now since $\mathfrak{p}$ is a minimal prime of R, V_R is a non-trivial extension of ω_2 by ω_1. By (1.4.3), applied with $\kappa = \kappa(\mathfrak{p})$, any deformation of $(V_{R/\mathfrak{p}})_{\kappa(\mathfrak{p})}$ which induces a trivial deformation on pseudo-representations is trivial. Hence $R \to R_{\omega_1}$ is an isomorphism at $\mathfrak{p}$.

It follows that the map

$$R \to R_{\omega_1} \oplus R_{\omega_2}$$

is an isomorphism at the minimal primes of R, so R is 3-dimensional. Since we have already seen that R_{ω_1} and R_{ω_2} are formally smooth we find (cf. [Ma, Thm. 14.7])

$$e_{\mathfrak{m}}(R) = e_{\mathfrak{m}}(R_{\omega_1}, R) + e_{\mathfrak{m}}(R_{\omega_2}, R) = 2.$$

□

(1.5) $\mathrm{GL}_2(\mathbb{Q}_p)$**-representations mod** p**:** In this subsection we study certain (pro-)finite length, admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-representation built out of irreducible mod $\mathrm{GL}_2(\mathbb{Z}_p)$-representations, and the Galois representations obtained from them by applying the functor V_ψ introduced in (1.2).

(1.5.1) As in (1.2), fix an integer $r \in [0, p-1]$, and we consider the representation $\sigma = \mathrm{Sym}^r\mathbb{F}^2$ of KZ obtained by letting $p \in K$ act trivially. We also fix a character $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$, and an element $\lambda \in \mathbb{F}$.

The operator T introduced in (1.2.1) acts on $I(\sigma) = \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r\mathbb{F}^2$ and hence on $I_\chi(\sigma) = I(\sigma) \otimes \chi \circ \det$. We set

$$\Pi(r, \lambda, \chi) = \varprojlim I_\chi(\sigma)/(T - \lambda)^n I_\chi(\sigma)$$

The $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\Pi(r, \lambda, \chi)$ is naturally a module of $\mathbb{F}[\![S]\!]$, where S acts on $I_\chi(\sigma)/(T-\lambda)^n I_\chi(\sigma)$ by $T - \lambda$. We will sometimes write $T - \lambda$ for S. As mentioned in (1.2.5), inverse limits on the category of admissible representations are exact. In particular one sees that $\Pi(r, \lambda, \chi)/(T - \lambda)^n \xrightarrow{\sim} I_\chi(\sigma)/(T - \lambda)^n$. The list of possibilities for $\pi(r, \lambda\, \chi) = \Pi(r, \lambda, \chi)/(T - \lambda)$ is given in (1.2.2).

Denote by ψ the central character of $I_\chi(\sigma)$ (and hence also of $\Pi(r, \lambda, \chi)$).

Lemma (1.5.2). *Let*

$$V_\psi(\Pi(r, \lambda, \chi)) = \varprojlim V_\psi(I_\chi(\sigma)/(T-\lambda)^n I_\chi(\sigma)) = \varprojlim V_\psi(\Pi(r, \lambda, \chi)/(T-\lambda)^n \Pi(r, \lambda, \chi))$$

Then $V_\psi(\Pi(r, \lambda, \chi))$ *is a finite free* $\mathbb{F}[\![S]\!]$*-module which has rank* 1 *if* $\lambda \neq 0$ *and has rank* 2 *if* $\lambda = 0$*.*

Proof. Let $i = 1$ if $\lambda \neq 0$ and 2 if $\lambda = 0$. The exactness of V_ψ and (1.2.4) implies that

$$V_\psi(\Pi(r, \lambda, \chi))/SV_\psi(\Pi(r, \lambda, \chi)) \xrightarrow{\sim} V_\psi(\pi(r, \lambda, \chi))$$

has $\mathbb{F}$-dimension i, and hence one sees that there is a surjection

$$\mathbb{F}[\![S]\!]^i/S^n \to V_\psi(\Pi(r, \lambda, \chi)/(\pi - \lambda)^n \Pi(r, \lambda, \chi)).$$

Since $I_\chi(\sigma)$ has no $T-\lambda$ torsion (this is easily seen using the fact that the functions in $I_\chi(\sigma)$ are compactly supported), $\Pi(r, \lambda, \chi)/(\pi - \lambda)^n \Pi(r, \lambda, \chi)$ has a filtration of length n where the associated graded pieces are isomorphic to $\pi(r, \lambda, \chi)$. Hence $V_\psi(\Pi(r, \lambda, \chi)/(\pi - \lambda)^n \Pi(r, \lambda, \chi))$ has length n by (1.2.4), and this surjection is an isomorphism. The lemma follows by passing to the limit over n. $\square$

Lemma (1.5.3). $V_\psi(\Pi(r, 0, \chi))$ *is a deformation to* $\mathbb{F}[\![T]\!]$ *of the absolutely irreducible* 2*-dimensional* $\mathbb{F}$*-representation* $V_\psi(\pi(r, 0, \chi))$ *of* $G_{\mathbb{Q}_p}$*. If* R *denotes the universal deformation ring of this representation, then the map* $R \to \mathbb{F}[\![T]\!]$ *is surjective.*

Proof. It suffices to consider the case when χ is trivial. We first consider the case when $r \in [0, p-2]$. Let E be a finite extension of $W(\mathbb{F})[1/p]$, with ring of integers $\mathcal{O}$ and uniformizer π_E, and let $E(T) \in W(\mathbb{F})[T]$ be the Eisenstein polynomial of π_E. Write $e = [E : W(\mathbb{F})[1/p]]$.

Consider $\mathrm{Sym}^r W(\mathbb{F})^2$ viewed as a KZ-module, by letting $p \in KZ$ act trivially. The compact induction $\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r W(\mathbb{F})^2$ is a $W(\mathbb{F})[T]$-module, where T acts via the KZ-bivariant function on $\mathrm{Sym}^r W(\mathbb{F})^2$ which is supported on KZ and takes $\begin{bmatrix} 1 & 0 \\ 0 & p^{-1} \end{bmatrix}$ to $\mathrm{Sym}^r \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$ (cf. 1.2.1). Then [Br 2, Prop. 3.3.3] asserts that

$$(1.5.4) \qquad (\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r W(\mathbb{F})^2)/(E(T)) \xrightarrow{\sim} (\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r \mathcal{O}^2)/(T - \pi_E)$$

is p-torsion free, and that its reduction modulo π_E is isomorphic to $\pi(r, 0, 1)$. Let $\tilde{\psi}$ denote the central character of (1.5.4). (So $\tilde{\psi}$ corresponds via class field theory to $\chi_{\mathrm{cyc}}^{r+1}$.) By [BB 1, Thm. 4.3.1, 5.3.2] taking the p-adic completion of $\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r W(\mathbb{F})^2/(E(T))$ and applying $V_{\tilde{\psi}}$ yields a lattice in a two dimensional, crystalline E-representation V_{π_E} of $G_{\mathbb{Q}_p}$, having Hodge-Tate weights $0, r+1$. Moreover, if $D^*_{\mathrm{cris}}(V_{\pi_E})$ denotes the weakly admissible module contravariantly associated to V_{π_E}, then the trace of the Frobenius φ on $D^*_{\mathrm{cris}}(V^*_{\pi_E})$ is equal to π_E.

Let $R^{0,r+1}$ denote the quotient of R corresponding to crystalline deformations having Hodge-Tate weights $0, r+1$. Suppose that A is any finite local $W(\mathbb{F})$-algebra, and consider a map of $W(\mathbb{F})$-algebras $\theta : R^{0,r+1} \to A$. Denote by V_A the corresponding A-representation of $G_{\mathbb{Q}_p}$. The theory of Fontaine-Laffaille [FL] implies that there is an element $a_p \in R^{0,r+1}$ such that for any A and θ as above, the trace of φ on $D_{\mathrm{cris}}(V_A^*)$ is equal to $\theta(a_p)$. Here V_A^* denotes the A-dual of V_A.

Now the reduction of (1.5.4) modulo p is

$$\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r \mathbb{F}^2/T^e \xrightarrow{\sim} \Pi(r, 0, 1)/T^e \Pi(r, 0, 1).$$

It follows that $R \to \mathbb{F}[\![T^e]\!]$ factors through the quotient $R^{0,r+1}$ and sends a_p to T. Since this is true for any e, the lemma follows when $r \in [0, p-2]$. When $r = p-1$ it follows from the case $r = 0$ and (1.5.5) below. $\square$

Lemma (1.5.5). *There is a morphism of* $\mathbb{F}[T][\mathrm{GL}_2(\mathbb{Q}_p)]$*-modules*

$$\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1} \mathbb{F}^2 \to \mathrm{Ind}_{KZ}^G \mathbf{1}$$

which induces a continuous isomorphism of $\mathbb{F}[\![T]\!][\mathrm{GL}_2(\mathbb{Q}_p)]$*-modules*

$$\Pi(0, \lambda, \chi) \xrightarrow{\sim} \Pi(p-1, \lambda, \chi)$$

for $\lambda \in \mathbb{F} \backslash \{\pm 1\}$.

Proof. It suffices to consider the case $\chi = 1$.

We recall the notation of [Br 1, 2.3]. Suppose that σ, V_σ and $I(\sigma)$ are as in (1.2). If $g \in G$ and $v \in V_\sigma$ we denote by $[g, v] \in I(\sigma)$ the function which is supported KZg^{-1} and given by $[g, v](g') = \sigma(g'g)v$ for $g' \in KZg^{-1}$. If $\varphi : G \to \mathrm{End}_{\mathbb{F}} V_\sigma$ is a KZ-bivariant function, then the corresponding operator T_φ on $I(\sigma)$ is given by [Br 1, 2.4]

$$T_\varphi([g, v]) = \sum_{g'KZ \in G/KZ} [gg', \varphi(g'^{-1})(v)].$$

We identify $\mathrm{Sym}^{p-1}\mathbb{F}^2$ with the space of polynomials in $\mathbb{F}[x, y]$ which are homogeneous of degree $p-1$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^{p-1-j} y^j = (ax + cy)^{p-1-j} (bx + dy)^j.$$

Let $I \subset \mathrm{GL}_2(\mathbb{Z}_p)$ denote the Iwahori subgroup consisting of matrices whose reduction modulo p is upper triangular. Then we identify $I\backslash K$ with $\mathbb{P}^1(\mathbb{F}_p)$ via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c, d)$, and we may think of x, y as projective co-ordinates on $\mathbb{P}^1(\mathbb{F}_p)$, so that Sym^{p-1} becomes a subspace of $\mathrm{Ind}_I^K \mathbf{1}$, consisting of the functions with average value 0.

Set $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$, and denote by T the operator introduced in (1.2), which corresponds to the KZ-bi-invariant function φ_α supported on $KZ\alpha^{-1}KZ$ and sending α^{-1} to $\mathrm{Sym}^r \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. A simple calculation (cf. [Br 1, Prop. 4.1.2]) shows that the elements $[\alpha, 1]$ and $[1, 1]$ of $\mathrm{Ind}_{KZ}^G \mathbf{1}$ are I-invariant. Let $b \in \mathrm{Ind}_{KZ}^G \mathbf{1}$ be an element contained in the $\mathbb{F}$-span of $\{T^j[\alpha, 1], T^j[1, 1]\}_{j \geq 0}$. Since the stabilizer of $x^{p-1} \in \mathrm{Sym}^{p-1}\mathbb{F}^2 \subset \mathrm{Ind}_I^K \mathbf{1}$ is $I \subset K$, there is a unique map of $\mathbb{F}[\mathrm{GL}_2(\mathbb{Z}_p)]$-modules

$$\mathrm{Sym}^{p-1}\mathbb{F}^2 \to \mathrm{Ind}_{KZ}^G \mathbf{1}$$

taking x^{p-1} to b, and we denote by

$$h_b : \mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1}\mathbb{F}^2 \to \mathrm{Ind}_{KZ}^G \mathbf{1} = I(\mathbf{1})$$

the map obtained by Frobenius reciprocity. h_b is characterised by the property that $h_b([1, x^{p-1}]) = b$.

We now find a b such that the composite of h_b with the projection $I(\mathbf{1}) \to I(\mathbf{1})/T^n$ is compatible with the action of T. Let $C \subset K$ denote the set of matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ i & 1 \end{smallmatrix}\right)$ with $i = 0, 1, \dots p-1$ together with the matrix $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then C consists of a set of representatives for $KZ\alpha KZ/KZ$, and we compute

$$\begin{aligned} T([1, x^{p-1}]) &= \sum_{gKZ \in G/KZ} [g, \varphi_\alpha(g^{-1})(x^{p-1})] \\ &= \sum_{k \in C} (k\alpha) \cdot [1, \varphi_\alpha(\alpha^{-1}k^{-1})(x^{p-1})] = \sum_{k \in C} (k\alpha) \cdot [1, (\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} k^{-1})(x^{p-1})] \\ &= \sum_{k \in C\backslash\{1\}} (k\alpha) \cdot [1, y^{p-1}] = \sum_{k \in C\backslash\{1\}} (k\alpha w) \cdot [1, x^{p-1}] \end{aligned}$$

Hence we have

$$h_b(T[1, x^{p-1}]) = \sum_{k \in C\backslash\{1\}} (k\alpha w) \cdot b. \tag{1.5.6}$$

Suppose that $b = T^j[\alpha, 1]$ for some $j \geq 0$. Then

$$k\alpha w T^j[\alpha, 1] = T^j[k\alpha w\alpha, 1] = T^j[kpw, 1] = T^j[1, 1].$$

In particular the summands in the right hand side of (1.5.6) do not depend on k. Since there are p summands, we see that $h_b(T[1, x^{p-1}]) = 0$.

Suppose that $b = T^j[1, 1]$ for some $j \geq 0$. Then

$$h_b(T[1, x^{p-1}]) = \sum_{k \in C\backslash\{1\}} T^j[k\alpha w, 1] = T^j(\sum_{k \in C\backslash\{1\}} [k\alpha, 1]) = T^j(T[1, 1] - [\alpha, 1]).$$

Hence if we set $b = [\alpha, 1] - T[1,1]$, then we find $h_b(T[1, x^{p-1}]) = Th_b([1, x^{p-1}])$, so h_b is a map of $\mathbb{F}[T][\mathrm{GL}_2(\mathbb{Q}_p)]$-modules.

Now let $\lambda \in \mathbb{F}$. Then h_b is non-zero modulo $T - \lambda$, for if $(T-\lambda)c = b$ for some $c \in \mathrm{Ind}_{KZ}^G \mathbf{1}$ then by comparing supports one find that c must be in $\mathbb{F} \cdot [1,1]$ (see [BL, Lem. 20]). But then $[\alpha, 1]$ would be in the span of $[1,1]$ and $T[1,1]$ which is not the case.

Suppose that $\lambda \neq \pm 1$. Taking the reduction of h_b modulo $(T-\lambda)^n$ gives a map of $\mathbb{F}[T][\mathrm{GL}_2(\mathbb{Q}_p)]$-modules

$$I(\mathrm{Sym}^{p-1}\mathbb{F}^2)/(T-\lambda)^n \to I(\mathbf{1})/(T-\lambda)^n. \tag{1.5.7}$$

Since $I(\mathbf{1})/(T-\lambda)$ is irreducible, and (1.5.7) is non-zero modulo $T-\lambda$, it is surjective by Nakayama's lemma. Since both sides have the same length, (1.5.7) is an isomorphism. Passing to the limit over n, yields the isomorphism of the lemma. □

Lemma (1.5.8). *If $\lambda \in \mathbb{F}^\times$, then the action of $G_{\mathbb{Q}_p}$ on $V_\psi(\Pi(r, \lambda, \chi))$ is given by the $\mathbb{F}[\![S]\!]^\times$-valued character $\chi\mu_{T^{-1}}$, where $\mu_{T^{-1}}$ is the unramified character of $G_{\mathbb{Q}_p}$ sending the geometric Frobenius corresponding to p, Frob_p^{-1} to $T^{-1} = (S+\lambda)^{-1}$.*

Proof. We use the notation of the proof of (1.5.4). Again it suffices to consider the case when χ is trivial. Let $[\lambda] \in W(\mathbb{F})$ be the Teichmüller representative of λ and consider the quotient

$$(\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r W(\mathbb{F})^2)/(E(T - [\lambda])) \xrightarrow{\sim} (\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r \mathcal{O}^2)/(T - ([\lambda] + \pi_E)) \tag{1.5.9}$$

Let $\tilde{\psi}$ denote the central character of (1.5.9) (as in (1.5.4), $\tilde{\psi}$ is induced by $\chi_{\mathrm{cyc}}^{r+1}$). By [BB 2, §7.2] p-adically completing (1.5.9) and applying $V_{\tilde{\psi}}$ produces an unramified character $G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ sending Frob_p^{-1}, to the inverse of the unit root of the quadratic equation $X^2 - ([\lambda] + \pi_E)X + p^{r+1}$. Hence applying V_ψ to $(\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r \mathbb{F}^2)/(T-\lambda)^e$ produces the character

$$G_{\mathbb{Q}_p} \to (\mathcal{O}/\pi_E^e)^\times \xrightarrow{\sim} (\mathbb{F}[\![S]\!]/S^e)^\times$$

given by sending Frob_p^{-1} to $T^{-1} = (S+\lambda)^{-1}$. The lemma follows as in (1.5.4). □

Lemma (1.5.10). *Let $\bar{\mathfrak{r}} : G_{\mathbb{Q}_p} \to \mathbb{F}$ be a 2-dimensional pseudo-representation with determinant $\psi\chi_{\mathrm{cyc}}$, and denote by $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$ its universal deformation ring. Suppose that $V_\psi(\pi(r, \lambda, \chi))$ is a factor of the semi-simple $\mathbb{F}$-representation $V_{\mathbb{F}}$ of $G_{\mathbb{Q}_p}$ attached to $\bar{\mathfrak{r}}$.*

Then there is map $\theta : R^{\mathrm{ps}}(\mathfrak{r}) \to \mathbb{F}[\![S]\!]$ such that for $\sigma \in G_{\mathbb{Q}_p}$, the element $\theta(T(\sigma)) \in \mathbb{F}[\![S]\!]$ acts on $V_\psi(\Pi(r, \lambda, \chi))$ by $\sigma + \psi\chi_{\mathrm{cyc}}(\sigma)\sigma^{-1}$.

Moreover, the map θ is surjective unless $(\chi\mu_{\lambda^{-1}})^2 = \psi\chi_{\mathrm{cyc}}$ (that is $V_{\mathbb{F}}$ is scalar), in which case the image of θ has the form $\mathbb{F}[\![S']\!]$, where $S' \in \mathbb{F}[\![S]\!]$ is an element of S-adic valuation 2.

Proof. If $V_{\mathbb{F}}$ is irreducible, this follows from (1.5.3), since $V_\psi(\Pi(r, \chi, \lambda))$ is a deformation of $V_{\mathbb{F}}$.

If $V_{\mathbb{F}}$ is reducible, then $V_\psi(\Pi(r, \chi, \lambda))$ is a direct summand of the deformation $\chi\mu_{T^{-1}} \oplus \chi^{-1}\psi\chi_{\mathrm{cyc}}\mu_T$ of $V_{\mathbb{F}}$ by (1.5.8), and this gives a map $\theta : R^{\mathrm{ps}}(\mathfrak{r}) \to \mathbb{F}[\![S]\!]$, with $T(\sigma)$ acting as claimed.

Now if $\sigma \in G_{\mathbb{Q}_p}$ acts via the geometric Frobenius on the residue field of $\bar{\mathbb{Q}}_p$, then

$$\theta(T(\sigma)) = \chi(\sigma)(S+\lambda)^{-1} + \chi^{-1}\psi\chi_{\mathrm{cyc}}(\sigma)(S+\lambda). \tag{1.5.11}$$

The coefficient of S in the above expression is $-\chi(\sigma)\lambda^{-2} + \chi^{-1}\psi\chi_{\mathrm{cyc}}(\sigma)$, which is 0 if and only if for all such σ $(\chi\mu_{\lambda^{-1}})^2 = \psi\chi_{\mathrm{cyc}}$.

For $i \geq 2$, the coefficient of S^i in (1.5.11) is $(-1)^i\chi(\sigma)\lambda^{-i-1} \neq 0$, so if the coefficient of S in (1.5.11) is 0, then we may take $S' = \sum_{i=2}^{\infty} S^i\lambda^{-i-1}$. $\square$

(1.6) Local patching and multiplicities: In this subsection we give a construction of certain finite modules over deformation rings for certain 2-dimensional pseudo-representations of $G_{\mathbb{Q}_p}$.

(1.6.1) We now return to the notation of (1.1). In particular $k \geq 2$, $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ as in (1.1.1), $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ is of Galois type, and

$$L_{k,\tau} \subset \sigma(k,\tau) = \sigma(\tau) \otimes_E \mathrm{Sym}^{k-2} E^2$$

is a $\mathrm{GL}_2(\mathbb{Z}_p)$-stable $\mathcal{O}$-lattice. We will regard ψ as a character of $\mathbb{Q}_p^\times$, as before, and we assume that the central character of $\sigma(k,\tau)$ is $\psi|_{\mathbb{Z}_p^\times}$.

We also fix a 2-dimensional pseudo-representation $\bar{\mathfrak{t}}$ of $G_{\mathbb{Q}_p}$ over $\mathbb{F}$. Thus $\bar{\mathfrak{t}}$ is the trace of a unique semi-simple, 2-dimensional $\mathbb{F}$-representation of $G_{\mathbb{Q}_p}$, which we denote by $V_{\mathbb{F}}$ [Ta 1, Thm. 1]. We denote by $R^{\mathrm{ps}}(\bar{\mathfrak{t}})$ the universal deformation ring of $\bar{\mathfrak{t}}$.

Suppose that E' is a finite extension of E with ring of integers $\mathcal{O}_{E'}$, and that $\mathfrak{t}$ is a deformation of $\bar{\mathfrak{t}}$ to $\mathcal{O}_{E'}$. Regarding $\mathfrak{t}$ as an E'-valued pseudo-representation, there is a representation $V_{\mathfrak{t}}$ of $G_{\mathbb{Q}_p}$ on a 2-dimensional E'-vector space, so that $\mathfrak{t}$ is given by the trace of $V_{\mathfrak{t}}$. Moreover the semi-simplification of $V_{\mathfrak{t}}$ is uniquely determined.

Suppose that $V_{\mathfrak{t}}$ is potentially semi-stable of type τ with Hodge-Tate weights, 0, $k-1$, and has determinant $\psi\chi_{\mathrm{cyc}}$. We will say that $V_{\mathfrak{t}}$ has type (k,τ,ψ). By (1.2.6) there is an admissible $\mathcal{O}_{E'}$-lattice $\Pi_{\mathfrak{t}}$ such that $V_{\mathfrak{t}} \xrightarrow{\sim} V_\psi(\Pi_{\mathfrak{t}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $\Pi_{\mathfrak{t}}$ has central character ψ and there is a $K = \mathrm{GL}_2(\mathbb{Z}_p)$-equivariant embedding $\sigma(k,\tau) \to \Pi_{\mathfrak{t}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Since $\Pi(V_{\mathfrak{t}})$ has central character ψ, this embedding becomes KZ-equivariant, if we let Z act on $\sigma(k,\tau)$ via ψ, and hence we obtain a map

$$\mathrm{Ind}_{KZ}^G \sigma(k,\tau) \to \Pi_{\mathfrak{t}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Multiplying this map by a power of p, if necessary, we may assume that it induces a map

$$\mathrm{Ind}_{KZ}^G L_{k,\tau} \to \Pi_{\mathfrak{t}}. \tag{1.6.2}$$

Denote by $\Pi(\mathfrak{t})$ the closure of the image of (1.6.2). It is an admissible $\mathcal{O}$-lattice, whose E'-span is $\Pi_{\mathfrak{t}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ if $V_{\mathfrak{t}}$ is absolutely irreducible, and is a proper closed submodule otherwise. Let $V(\mathfrak{t}) = V_\psi(\Pi(\mathfrak{t}))$. The E'-span of the image of the composite

$$V(\mathfrak{t}) \to V_\psi(\Pi_{\mathfrak{t}}) \to V_{\mathfrak{t}}$$

is $V_{\mathfrak{t}}$ if $V_{\mathfrak{t}}$ is absolutely irreducible and is a 1-dimensional E'-subspace of $V_{\mathfrak{t}}$ otherwise.

Next suppose that we are given a finite collection of distinct deformations $U = \{\mathfrak{r}_1, \dots \mathfrak{r}_n\}$ of $\bar{\mathfrak{r}}$, and for each $\mathfrak{r}_i$ a potentially semi-stable representation $V_{\mathfrak{r}_i}$ of type (k, τ, ψ) giving rise to $\mathfrak{r}_i$. Then we obtain a map as in (1.6.2) for each $\mathfrak{r}_i$, and we denote by $\Pi(U)$ the closure of the image of

$$\mathrm{Ind}_{KZ}^{G} L_{k,\tau} \to \oplus_{i=1}^{n} \Pi_{\mathfrak{r}_i}.$$

This is again an admissible $\mathcal{O}$-lattice

Finally if we are given a countable collection $U = \{\mathfrak{r}_i\}_{i \geq 1}$ of deformations, and a a potentially semi-stable representation $V_{\mathfrak{r}_i}$ of type (k, τ, ψ) giving rise to $\mathfrak{r}_i$, then we set

$$\Pi(U) = \varprojlim \Pi(U')$$

where U' runs over finite subsets of U. We set $V(U) = \varprojlim V(\Pi(U'))$.

Lemma (1.6.3). *Suppose $U = \{\mathfrak{r}_i\}_{i \geq 1}$ is as above. Then*

(1) *$V(U)$ is naturally a $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-module.*
(2) *If $U' \subset U$ is any subset than the natural map $V(U) \to V(U')$ is a map of $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-modules*
(3) *If $\mathfrak{r} \in U$ is a deformation of $\bar{\mathfrak{r}}$ to $\mathcal{O}_{E'}$, then $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$ acts on $V(\mathfrak{r})$ via the image of the corresponding map $x_{\mathfrak{r}} : R^{\mathrm{ps}}(\bar{\mathfrak{r}}) \to \mathcal{O}_{E'}$. In particular $V(\mathfrak{r})$ is an $x_{\mathfrak{r}}(R^{\mathrm{ps}}(\bar{\mathfrak{r}}))$-module.*

Proof. It suffices to prove the lemma when $U = \{\mathfrak{r}_1, \dots, \mathfrak{r}_n\}$ is finite, where the $\mathfrak{r}_i$ are distinct pseudo-representations.

Note that we have an inclusion $V(U) \hookrightarrow \oplus_{i=1}^{n} V_{\mathfrak{r}_i}$, and $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$ acts on each $V_{\mathfrak{r}_i}$ via the corresponding character $x_{\mathfrak{r}_i} : R^{\mathrm{ps}} \to \mathcal{O}_{E'}$. We saw in (1.4) that R^{ps} is topologically generated by the elements $T(\sigma)$ with $\sigma \in G_{\mathbb{Q}_p}$. Hence it suffices to check that the map $T(\sigma) : V(U) \to \oplus_{r=1}^{n} V_{\mathfrak{r}_i}$ induced by $T(\sigma)$ has image in $V(U)$.

The operator $\sigma^2 - T(\sigma)\sigma + \psi(\sigma)\chi_{\mathrm{cyc}}(\sigma)$ acts on each $V_{\mathfrak{r}_i}$ by 0, so that $T(\sigma)$ on $\oplus_{i=1}^{n} V_{\mathfrak{r}_i}$ is given by $\sigma + \psi\chi_{\mathrm{cyc}}(\sigma)\sigma^{-1}$, which preserves $V(U)$ since $V(U) \subset \oplus_{i=1}^{n} V_{\mathfrak{r}_i}$ is a $G_{\mathbb{Q}_p}$-stable subspace. $\square$

(1.6.4) Suppose that Q is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on an $\mathbb{F}$-vector space, and that we given a finite collection P of representations of the form $\pi(r, \lambda, \chi)$, all with some fixed central character ψ. We set $Q_{\widehat{P}} = \varprojlim Q'$ where Q' runs over finite length quotients of $\mathrm{GL}_2(\mathbb{Q}_p)$ all of whose Jordan-Hölder factors are isomorphic to a subquotient of a representation $\pi(r, \lambda, \chi) \in P$.

It is clear that the functor $Q \mapsto Q_{\widehat{P}}$ is right exact. We write $V_{\psi}(Q_{\widehat{P}}) = \varprojlim V_{\psi}(Q')$.

Lemma (1.6.5). *Let $Q = \mathrm{Ind}_{KZ}^{G} L$ where $L = \mathrm{Sym}^r \mathbb{F}^2 \otimes \chi \circ \det$ is an irreducible representation of KZ on a finite dimensional $\mathbb{F}$-vector space (so $r \in [0, p-1]$). Then $Q_{\widehat{P}}$ is a successive extension of representations of the form $\Pi(r, \lambda, \chi)$ introduced in (1.5).*

Proof. By [BL, Prop. 32] any irreducible quotient of $\mathrm{Ind}_{KZ}^{G} L$ is a quotient of $\mathrm{Ind}_{KZ}^{G} L/(T - \lambda)\mathrm{Ind}_{KZ}^{G} L$. The lemma follows easily from this. $\square$

Lemma (1.6.6). *$V(U)$ is a finite $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-module of dimension $\leqslant 2$. In particular, if $R^{\mathrm{ps}}_U(\bar{\mathfrak{r}})$ denotes the image of $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$ in $\operatorname{End} V(U)$, then $R^{\mathrm{ps}}_U(\bar{\mathfrak{r}})$ is a flat $\mathcal{O}$-algebra of relative dimension at most 1.*

Proof. From the construction, one sees that $V(U)$ is p-adically separated (and even $\mathfrak{m}_{R^{\mathrm{ps}}(\mathfrak{r})}$-adically separated, where $\mathfrak{m}_{R^{\mathrm{ps}}(\mathfrak{r})}$ is the maximal ideal of R^{ps}). Hence it suffices to show that $V(U)/\pi V(U)$ is a finitely generated $R^{\mathrm{ps}}(\mathfrak{r})$-module of dimension at most 1. The claim regarding the dimension of $R^{\mathrm{ps}}_U(\mathfrak{r})$ follows from this.

Let P be the set of $\pi(r,\lambda,\chi)$ with central character ψ, such that $V_\psi(\pi(r,\lambda,\chi))$ is a factor in $V_{\mathbb{F}}$. Then P is a finite set, and $V(U)$ is a quotient of $(\mathrm{Ind}^G_{KZ}\bar{L}_{k,\tau})_{\widehat{P}}$ where $\bar{L}_{k,\tau} = L_{k,\tau}/pL_{k,\tau}$.

Let $\{0\} = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_m = \bar{L}_{k,\tau}$ be a filtration by KZ-stable subspaces, such L_{i+1}/L_i is an irreducible KZ-module for $i = 1,\dots m-1$. For $i = 1,\dots m$, let $V(U)_i$ denote the image of the composite

$$V_\psi((\mathrm{Ind}^G_{KZ}L_i)_{\widehat{P}}) \to V_\psi((\mathrm{Ind}^G_{KZ}\bar{L}_{k,\tau})_{\widehat{P}}) \to V(U)/\pi V(U).$$

Then $V(U)_i$ is a $G_{\mathbb{Q}_p}$-stable subspace of $V(U)$, and is hence $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-stable, since the elements $T(\sigma)$ of $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$ act on $V(U)/\pi V(U)$ via $\sigma + \psi\chi_{\mathrm{cyc}}\sigma^{-1}$. Hence it suffices to show that $V(U)_{i+1}/V(U)_i$ is a finitely generated $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-module of dimension at most 1.

Now for $i = 1,\dots r-1$, $(\mathrm{Ind}^G_{KZ}(L_{i+1}/L_i))_{\widehat{P}}$ is isomorphic to a successive extension of representations of the form $\Pi(r,\lambda,\chi)$ by (1.6.5), so the lemma follows from (1.5.10). (In fact it is not hard to check that $(\mathrm{Ind}^G_{KZ}(L_{i+1}/L_i))_{\widehat{P}}$ is actually isomorphic to a $\Pi(r,\lambda,\chi)$ unless $L_{i+1}/L_i \xrightarrow{\sim} \mathrm{Sym}^{p-2}\mathbb{F}^2 \otimes \chi\circ\det$, in which case it is an extension of two such spaces). $\square$

(1.6.7) Let $I_{\mathfrak{r}}$ be the kernel of the map $x_{\mathfrak{r}}$ of (1.6.3)(3), corresponding to a deformation $\mathfrak{r}$ of $\bar{\mathfrak{r}}$ which corresponds to a potentially semi-stable representation of type τ, Hodge-Tate weights $0, k-1$, and determinant $\psi\chi_{\mathrm{cyc}}$. Let I be the intersection of all the ideals $I_{\mathfrak{r}}$ with $\mathfrak{r}$ of this kind. The set of such $\mathfrak{r}$ has a countable subset U_0 such that $I = \cap_{\mathfrak{r}\in U_0} I_{\mathfrak{r}}$. (This holds for any set of ideals corresponding to closed points of $\operatorname{Spec} R^{\mathrm{ps}}(\bar{\mathfrak{r}})$.) In particular, any $R^{\mathrm{ps}}_U(\bar{\mathfrak{r}})$ is a quotient of $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$. We shall study specific components of $\operatorname{Spec} R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$.

Suppose now that $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ is indecomposable with trace given by $\bar{\mathfrak{r}}$. Set

$$\mu'_{\mathrm{Aut}} = \mu'_{\mathrm{Aut}}(k,\tau,\bar{\rho}) = \sum_{n,m} a(n,m)\mu'_{n,m}(\bar{\rho})$$

where $a(n,m)$ is as in (1.1.2), $\mu'_{n,m}(\bar{\rho}) = 0$ if $\mu_{n,m}(\bar{\rho}) = 0$ and $\mu'_{n,m}(\bar{\rho}) = 1$ otherwise.

We will use the notion of (1.3.2)

Lemma (1.6.8). *Let $\alpha = \{\bar{\rho}\}$ if $\bar{\rho}$ is absolutely irreducible, and $\alpha = \{\omega^{n+1+m}\mu_{\lambda\lambda'}\}$ if $\bar{\rho} \sim \begin{pmatrix} \omega^{n+1}\mu_\lambda & * \\ 0 & \mu_{\lambda^{-1}} \end{pmatrix} \otimes \omega^m\mu_{\lambda'}$ with $n,m \in [0,p-2]$ and $\lambda,\lambda' \in \mathbb{F}^\times$. If $\bar{\rho}$ is reducible and $n = 0$, then we suppose that $\lambda \neq \pm 1$. Then*

$$e_\alpha(V(U)/\pi V(U), R^{\mathrm{ps}}(\bar{\mathfrak{r}})) \leqslant \mu'_{\mathrm{Aut}}$$

unless $\bar{\rho}$ has scalar semi-simplification, in which case

$$e_\alpha(V(U)/\pi V(U), R^{\mathrm{ps}}(\bar{\mathfrak{r}})) \leqslant 2\mu'_{\mathrm{Aut}}.$$

Proof. We use the notation of the proof of (1.6.6). Let $i \in [1, p-1]$ and suppose that $L_{i+1}/L_i = \mathrm{Sym}^r \mathbb{F}^2 \otimes (\det)^s$. It suffices to show that

$$e_\alpha = e_\alpha(V_\psi((\mathrm{Ind}_{KZ}^G(L_{i+1}/L_i))_{\widehat{P}}), R^{\mathrm{ps}}(\bar{\mathfrak{r}})) = \mu'_{r,s}(\bar{\rho}) \tag{1.6.9}$$

unless $\bar{\rho}$ has scalar semi-simplification in which case it is equal to $2\mu'_{r,s}(\bar{\rho})$.

Comparing the definition of $\mu'_{r,s}(\bar{\rho})$ with the formulas of (1.2.5) one sees that $\mu'_{r,s}(\bar{\rho}) \neq 0$ if and only if there exists an extension χ of $(\det)^s$ to $\mathbb{Q}_p^\times$ such that $\psi = \omega^r \chi^2$, and a $\lambda \in \mathbb{F}$ such that $V_\psi(\pi(r, \lambda, \chi))$ is the unique element of α. (This is where we use the hypothesis that $\lambda \neq \pm 1$ if $\bar{\rho}$ is reducible and $n = 0$, since in this exceptional case whether $\mu'_{0,s}(\bar{\rho}) \neq 0$ also depends on the extension class $*$.) By (1.6.5) this is equivalent to asking that the left hand side of (1.6.9) is non-zero, in which case it is equal to $e_\alpha(V_\psi(\Pi(r, \lambda, \chi), R^{\mathrm{ps}}(\bar{\mathfrak{r}})))$, where the $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-module structure on $\Pi(r, \lambda, \chi)$ is given by (1.5.3) and (1.5.10). These lemmas also show that $e_\alpha = 1$ unless $\bar{\rho}$ has scalar semi-simplification, in which case $e_\alpha = 2$. □

Proposition (1.6.10). *Suppose that $\bar{\rho}$ is absolutely irreducible. Then*

$$e(R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})) \leqslant \mu_{\mathrm{Aut}}(k, \tau, \bar{\rho}).$$

Proof. By (1.4.6), $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$ is a quotient of the universal deformation ring of $V_{\mathbb{F}}$ and hence carries a finite free $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$-module of rank 2 equipped with a continuous action of $G_{\mathbb{Q}_p}$. Denote this module by $M(U_0)$.

Let $\sigma \in R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})[G_{\mathbb{Q}_p}]$. By definition

$$P_\sigma(X) = X^2 - T(\sigma)X + \frac{1}{2}(T(\sigma)^2 - T(\sigma^2))$$

is the characteristic polynomial of σ acting on $M(U_0)$, and by construction $P_\sigma(\sigma)$ annihilates $V(U_0)$. Hence, by (1.6.11) below, (and because $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$ is reduced by construction) $M(U_0)$ is a $R^{\mathrm{ps}}_{(U_0)}(\bar{\mathfrak{r}})[G_{\mathbb{Q}_p}]$-submodule of $V(U_0)$ at each generic point of $\mathrm{Spec}\, R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$. Hence there exists an inclusion of $R^{\mathrm{ps}}_{(U_0)}(\bar{\mathfrak{r}})[G_{\mathbb{Q}_p}]$-modules $M(U_0) \hookrightarrow V(U_0)$.

Set $\alpha = \{V_{\mathbb{F}}\}$. Since $M(U_0)$ is a finite free $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$-module of rank 2, and $V_{\mathbb{F}}$ is absolutely irreducible we see that the Hilbert-Samuel multiplicity of $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$ is equal to $e_\alpha(M(U_0)/\pi M(U_0), R^{\mathrm{ps}} U_0(\bar{\mathfrak{r}}))$. Using the above inclusion, together with (1.6.8) and (1.3.3), we find that

$$e_\alpha(M(U_0)/\pi M(U_0), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})) \leqslant e_\alpha(V(U_0)/\pi V(U_0), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})) \leqslant \mu'_{\mathrm{Aut}} = \mu_{\mathrm{Aut}}$$

□

Lemma (1.6.11). *Let κ be a field, and V and W representations of a group G on finite dimensional κ-vector spaces. Suppose that V is absolutely irreducible, and for $\sigma \in \kappa[G]$ let $P_\sigma(X) = \det(X - \sigma|V)$. If $P_\sigma(\sigma)|_W = 0$ for all $\sigma \in \kappa[G]$, then W is V-isotypic.*

Proof. It suffices to consider the case when W is absolutely irreducible. Let $I \subset \kappa[G]$ be the two-sided ideal generated by the elements $P_\sigma(\sigma)$ for $\sigma \in \kappa[G]$, and J (resp. J') the kernel of $\kappa[G]$ acting on V (resp. W). By Burnside's theorem $\kappa[G]$

surjects onto $\mathrm{End}_\kappa W$ and $\mathrm{End}_\kappa V$, so in particular $\kappa[G]/J$ and $\kappa[G]/J'$ are simple κ-algebras and $(J+J')/J'$ is either 0 of $\kappa[G]/J'$. If $\sigma \in J$, then $P_\sigma(X) = X^d$ where $d = \dim V$, and so $\sigma^d \in I$. Hence J is contained in the radical of $I \subset J'$ Hence $(J+J')/J' \neq \kappa[G]$, and so $J \subset J'$, and hence $J = J'$ as $\kappa[G]/J$ is simple.

It follows that V and W both have dimension d, and that if we consider $\kappa[G]$ as a $\kappa[G]$ module via multiplication on the left, then we find that

$$V^d \sim \kappa[G]/J = \kappa[G]/J' \sim W^d$$

hence $V \sim W$ as required. $\square$

(1.6.12) Suppose that $Z \subset \mathrm{Spec}\, R^{\mathrm{ps}}_{U_0}(\mathfrak{r})[1/p]$ is an irreducible component. We say that Z is of irreducible type if the pseudo-representation of $G_{\mathbb{Q}_p}$ at the generic point of Z corresponds to an absolutely irreducible representation. Otherwise we say that Z is of reducible type. Note that although the representation at the generic point of Z is a priori defined over some finite extension of the residue field at that point, (1.4.6) guarantees that it is actually defined over the residue field itself in almost all cases. Of course all components are of irreducible type if $V_{\mathbb{F}}$ is irreducible. In fact one can show that a component of irreducible type cannot meet a component of reducible type, but we shall not need this hear.

Suppose that $V_{\mathbb{F}} \sim \omega_1 \oplus \omega_2$ is reducible. Let Z be a component of reducible type, and $x \in Z$ a closed point, which corresponds to an absolutely reducible representation of $G_{\mathbb{Q}_p}$, V_x. Since V_x has distinct Hodge-Tate weights, V_x is in fact reducible, and its semi-simplification V_x^{ss} is uniquely determined by x. Suppose $V_x^{\mathrm{ss}} \sim \tilde{\omega}_1 \oplus \tilde{\omega}_2$ with $\tilde{\omega}_i$ reducing to ω_i, for $i = 1, 2$. If we insist that V_x be potentially semi-stable, and indecomposable then this determines which of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ appears as a subspace of V_x. We say that the point x is of type ω_i if $\tilde{\omega}_i$ appears as a subspace. Explicitly this means that the image of inertia in $\tilde{\omega}_i(1-k)$ is finite. It is not hard to see that either all points on Z are of type ω_1 or all are of type ω_2, and we say that Z is of type ω_1 or ω_2 respectively.

Proposition (1.6.13). *Suppose that $V_{\mathbb{F}} \sim \omega_1 \oplus \omega_2$, with $\omega_1 \neq \omega_2, \omega\omega_2$. Choose $U = U_{\omega_1}$ so that $\mathrm{Spec}\, R^{\mathrm{ps}}_U(\mathfrak{r}) \subset \mathrm{Spec}\, R^{\mathrm{ps}}_{U_0}(\mathfrak{r})$ is the closure of the union of the components of irreducible type and of type ω_1. Then*

$$e(R^{\mathrm{ps}}_U(\mathfrak{r})/\pi R^{\mathrm{ps}}_U(\mathfrak{r})) \leqslant \mu_{\mathrm{Aut}}(k, \tau, \bar{\rho}).$$

Proof. Let $I^{\mathrm{irr}} \subset R^{\mathrm{ps}}_U(\mathfrak{r})$ be the ideal corresponding to the components of irreducible type, and $I^{\omega_1} \subset R^{\mathrm{ps}}_U(\mathfrak{r})$ the ideal corresponding to components of type ω_1. Write $V(U_\omega)^{\mathrm{irr}}$ and $V(U)^{\omega_1}$ for $V(U)/I^{\mathrm{irr}}$ and $V(U)/I^{\omega_1}$ respectively.

Since $\mathrm{Ext}^1_{\mathbb{F}[G_{\mathbb{Q}_p}]}(\omega_2, \omega_1)$ is one dimensional, $R^{\mathrm{ps}}_U(\mathfrak{r})$ carries a finite free module of rank 2, $M(U)$ equipped with a continuous action of $G_{\mathbb{Q}_p}$, by (1.4.6), and $M(U)/I^{\omega_1}M(U)$ has a finite free rank 1-submodule L^{ω_1} on which $G_{\mathbb{Q}_p}$ acts via a character $\tilde{\omega}_1 : G_{\mathbb{Q}_p} \to (R^{\mathrm{ps}}_U(\mathfrak{r})/I^{\mathrm{irr}})^\times$.

The same argument as in (1.6.10), using (1.4.6) as well as (1.3.3) and (1.6.8) shows that

$$e_{\{\omega_1\}}(R^{\mathrm{ps}}_U(\mathfrak{r})/(I^{\mathrm{irr}}, \pi), R^{\mathrm{ps}}_U(\mathfrak{r})) \leqslant e_{\{\omega_1\}}(V(U)^{\mathrm{irr}}/\pi V(U)^{\mathrm{irr}}, R^{\mathrm{ps}}_U(\mathfrak{r})). \tag{1.6.14}$$

Similarly, using (1.6.11), one sees that there is a $G_{\mathbb{Q}_p}$-equivariant inclusion $L^{\omega_1} \hookrightarrow V(U)^{\omega_1}$, so that (1.3.3) gives

$$\begin{aligned} (1.6.15) \quad e(R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})/(I^{\omega_1},\pi), R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})) &= e_{\{\omega_1\}}(L^{\omega_1}/\pi L^{\omega_1}, R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})) \\ &\leqslant e_{\{\omega_1\}}(V(U)^{\omega_1}/\pi V(U)^{\omega_1}, R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})). \end{aligned}$$

Now the map

$$V(U) \to V(U)^{\mathrm{irr}} \oplus V(U)^{\omega_1}$$

is an isomorphism at all the generic points of $R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})$. Hence combining ((1.6.14) and (1.6.15), and using (1.3.3), and (.6.8) one finds that

$$\begin{aligned} e(R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{t}}), R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})) &= e(R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})/(I^{\mathrm{irr}},p), R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})) + e(R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})/(I^{\omega_1},\pi), R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})) \\ &\leqslant e_{\{\omega_1\}}(V(U)^{\mathrm{irr}}/\pi V(U)^{\mathrm{irr}}, R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})) + e_{\{\omega_1\}}(V(U)^{\omega_1}/\pi V(U)^{\omega_1}, R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})) \\ &= e_{\{\omega_1\}}(V(U)/\pi V(U), R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})) \leqslant \mu'_{\mathrm{Aut}} = \mu_{\mathrm{Aut}}. \end{aligned}$$

□

Corollary (1.6.16). *Suppose that $\bar{\rho}$ is either absolutely irreducible, or a non-trivial extension of ω_2 by ω_1, where $\omega_1, \omega_2 : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ are characters satisfying $\omega_1 \neq \omega\omega_2$. Then*

$$e(R^\psi(k,\tau,\bar{\rho})/\pi R^\psi(k,\tau,\bar{\rho})) \leqslant \mu_{\mathrm{Aut}}(k,\tau,\bar{\rho}).$$

Proof. Let $U = U_0$ if $\bar{\rho}$ is irreducible, and $U = U_{\omega_1}$ if not. By (1.4.6) there is a surjection $R^{\mathrm{ps}}(\bar{\mathfrak{t}}) \to R^\psi(k,\tau,\bar{\rho})$. By (16.10) and (1.6.13) it suffices to show that this surjection factors through $R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})$.

Since $R^\psi(k,\tau,\bar{\rho})$ is p-torsion free, and formally smooth after inverting p, by (1.1.1), it suffices to check that for any finite extension E'/E, an E'-valued point x of $R^\psi(k,\tau,\bar{\rho})$ gives rise to an E'-valued point of $R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})$. Now x corresponds to a 2-dimensional E'-representation which is a potentially semi-stable of type τ with Hodge-Tate weights $0, k-1$. Moreover V_x admits a lattice which is a deformation of $\bar{\rho}$ to the ring of integers $\mathcal{O}_{E'}$ of E'. Hence the trace of V_x is a deformation $\mathfrak{t}$ of $\bar{\mathfrak{t}}$. It follows from the maximality of $R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{t}})$ that x factors through $R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{t}})$, which completes the proof if $\bar{\rho}$ is absolutely irreducible. If $\bar{\rho}$ is reducible, then x lies either on a component of irreducible type or of type ω_1, and hence also factors through $R_U^{\mathrm{ps}}(\bar{\mathfrak{t}})$. □

Proposition (1.6.17). *Suppose that $\bar{\rho} \sim \omega_1 \oplus \omega_2$ where $\omega_1, \omega_2 \to \mathbb{F}^\times$, and $\omega_1\omega_2^{-1} \notin \{1, \omega, \omega^{-1}\}$. Then*

$$e(R^{\square,\psi}(k,\tau,\bar{\rho})/\pi R^{\square,\psi}(k,\tau,\bar{\rho})) \leqslant \mu_{\mathrm{Aut}}(k,\tau,\bar{\rho}).$$

Proof. Choose U^{irr} so that $\operatorname{Spec} R_{U^{\mathrm{irr}}}^{\mathrm{ps}}(\bar{\mathfrak{t}}) \subset \operatorname{Spec} R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{t}})$ is the union of the components of irreducible type. For $i = 1, 2$ choose $U_{\omega_i}^{\mathrm{red}}$ so that $\operatorname{Spec} R_{U_{\omega_i}^{\mathrm{red}}}^{\mathrm{ps}}(\bar{\mathfrak{t}}) \subset \operatorname{Spec} R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{t}})$ is the union of components of type ω_i. We set

$$R^{\mathrm{irr}} = \mathrm{Im}\,(R^{\square,\psi}(k,\tau,\bar{\rho}) \to R^{\square,\psi}(k,\tau,\bar{\rho}) \otimes_{R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{t}})} R_{U^{\mathrm{irr}}}^{\mathrm{ps}}(\bar{\mathfrak{t}})[1/p])$$

and similarly for $R^{\mathrm{red}}_{\omega_1}$ and $R^{\mathrm{red}}_{\omega_2}$. Note that for $i = 1, 2$, $R^{\mathrm{red}}_{\omega_i}$ is a quotient of the maximal quotient of $R^{\square,\psi}(k,\tau,\bar\rho)$ over which the universal representation of $G_{\mathbb{Q}_p}$ is an extension of $\tilde\omega'$ by $\tilde\omega$ for two characters $\tilde\omega', \tilde\omega$ with $\tilde\omega$ lifting ω_i.

Let $\mathfrak{m}_{U_0}$ denote the radical of $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$, and $\mathfrak{m}^{\mathrm{irr}}$, $\mathfrak{m}^{\mathrm{red}}_{\omega_1}$ and $\mathfrak{m}^{\mathrm{red}}_{\omega_2}$ the radicals of R^{irr}, $R^{\mathrm{red}}_{\omega_1}$ and $R^{\mathrm{red}}_{\omega_2}$ respectively. By (1.1.1) $\dim R^{\mathrm{irr}} = 5$, while $\dim R^{\mathrm{irr}}/\mathfrak{m}_{U_0}R^{\mathrm{irr}} \leqslant 3$, by (1.4.7) and $\dim R^{\mathrm{ps}}_{U^{\mathrm{irr}}}(\bar{\mathfrak{r}}) \leqslant 2$ by (1.6.6). Since

$$\dim R^{\mathrm{irr}} \leqslant \dim R^{\mathrm{irr}}/\mathfrak{m}_{U_0}R^{\mathrm{irr}} + \dim R^{\mathrm{ps}}_{U^{\mathrm{irr}}}(\bar{\mathfrak{r}})$$

by [Ma,Thm. 15.1], each of these inequalities is an equality. Hence, applying (1.3.5), we have

$$\text{(1.6.18)} \quad e(R^{\mathrm{irr}}/\pi R^{\mathrm{irr}}) \leqslant e(R^{\mathrm{ps}}_{U^{\mathrm{irr}}}(\bar{\mathfrak{r}})/\pi) e(R^{\mathrm{irr}}/\mathfrak{m}_{U_0}R^{\mathrm{irr}}) \leqslant 2e(R^{\mathrm{ps}}_{U^{\mathrm{irr}}}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U^{\mathrm{irr}}}(\bar{\mathfrak{r}})).$$

where the second inequality follows from (1.4.7). The same argument shows that for $i = 1, 2$

$$e(R^{\mathrm{red}}_{\omega_i}/\pi) \leqslant e(R^{\mathrm{ps}}_{U^{\mathrm{red}}_{\omega_i}/pi}(\bar{\mathfrak{r}})) e(R^{\mathrm{red}}_{\omega_i}/\mathfrak{m}_{U_0}R^{\mathrm{red}}_{\omega_i}).$$

and that $\dim R^{\mathrm{red}}_{\omega_i}/\mathfrak{m}_{U_0}R^{\mathrm{red}}_{\omega_i} = 3$. By construction $R^{\mathrm{red}}_{\omega_i}/\mathfrak{m}_{U_0}R^{\mathrm{red}}_{\omega_i}$ carries a reducible representation of $G_{\mathbb{Q}_p}$, and since $\omega_1 \neq \omega_2$ one sees easily that this representation is an extension of ω_2 by ω_1 if $i = 1$, and of ω_1 by ω_2 if $i = 2$. Hence $R^{\mathrm{red}}_{\omega_i}/\mathfrak{m}_{U_0}R^{\mathrm{red}}_{\omega_i}$ is a quotient of the ring R_{ω_1} introduced in the proof of (1.4.7), and since R_{ω_i} is formally smooth of dimension 3, $R^{\mathrm{red}}_{\omega_i}/\mathfrak{m}_{U_0}R^{\mathrm{red}}_{\omega_i} = R_{\omega_i}$. In particular we see that $e(R^{\mathrm{red}}_{\omega_i}/\mathfrak{m}_{U_0}R^{\mathrm{red}}_{\omega_i}) = 1$, so that

$$\text{(1.6.19)} \qquad e(R^{\mathrm{red}}_{\omega_i}/\pi) \leqslant e(R^{\mathrm{ps}}_{U^{\mathrm{red}}_{\omega_i}}(\bar{\mathfrak{r}})/\pi).$$

Now let $\bar\rho_{\omega_1}$ be a non-trivial extension of ω_2 by ω_1 and $\bar\rho_{\omega_2}$ be a non-trivial extension of ω_1 by ω_2.

Using (1.6.18) and (1.6.19), together with (1.6.13) we compute

$$\begin{aligned} e(R^{\square,\psi}(k,\tau,\bar\rho)/\pi) &= e(R^{\mathrm{irr}}/\pi) + e(R^{\mathrm{red}}_{\omega_1}/\pi) + e(R^{\mathrm{red}}_{\omega_2}/\pi) \\ &\leqslant 2e(R^{\mathrm{ps}}_{U^{\mathrm{irr}}}(\bar{\mathfrak{r}})/\pi) + e(R^{\mathrm{ps}}_{U^{\mathrm{red}}_{\omega_1}}/\pi) + e(R^{\mathrm{ps}}_{U^{\mathrm{red}}_{\omega_2}}/\pi) \\ &= e(R^{\mathrm{ps}}_{U_{\omega_1}}(\bar{\mathfrak{r}})/\pi) + e(R^{\mathrm{ps}}_{U_{\omega_2}}/\pi) \leqslant \mu_{\mathrm{Aut}}(k,\tau,\bar\rho_{\omega_1}) + \mu_{\mathrm{Aut}}(k,\tau,\bar\rho_{\omega_2}) \\ &= \mu_{\mathrm{Aut}}(k,\tau,\bar\rho) \end{aligned}$$

where the first two equalities follows from (1.3.3)(2), and the final equality follows from the definition of μ_{Aut}. $\square$

§2 Modularity via the Breuil-Mézard conjecture

(2.1) Quaternionic forms: We recall some standard facts and notation from the theory of quaternionic forms. Further details may be found in [Tay 2, §1] or [Ki 2, §3].

(2.1.1) Let F be a totally real field, and D a quaternion algebra with center F which is ramified at all the infinite places of F and at a set of finite places Σ, which does not contain any primes dividing p. We fix a maximal order $\mathcal{O}_D$ of D, and for each finite place $v \notin \Sigma$, an isomorphism $(\mathcal{O}_D)_v \xrightarrow{\sim} M_2(\mathcal{O}_{F_v})$. For each finite place

v of F we will denote by $\mathbf{N}(v)$ the order of the residue field at v, and by $\pi_v \in F_v$ a uniformizer.

Let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact open subgroup contained in $\prod_v (\mathcal{O}_D)_v^\times$. We assume that if $v \in \Sigma$, then $U_v = (\mathcal{O}_D)_v^\times$, and that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ for $v|p$.

Let A be a topological $\mathbb{Z}_p$-algebra. For each $v|p$, we fix a continuous representation $\sigma_v : U_v \to \mathrm{Aut}(W_{\sigma_v})$ on a finite free A-module. We write $W_\sigma = \otimes_{v|p,A} W_{\sigma_v}$ and we denote by $\sigma : \prod_{v|p} U_v \to \mathrm{Aut}(W_\sigma)$ the corresponding representation. We regard σ as being a representation of U by letting U_v act trivially if $v \nmid p$.

Finally, we fix a continuous character $\psi : (\mathbb{A}_F^f)^\times / F^\times \to A^\times$ such that for any place v of F, σ on $U_v \cap \mathcal{O}_{F_v}^\times$ is given by multiplication by ψ^{-1}. We think of $(\mathbb{A}_F^f)^\times$ as acting on W_σ via ψ^{-1}, so that W_σ becomes a $U(\mathbb{A}_F^f)^\times$-module.

Let $S_{\sigma,\psi}(U, A)$ denote the set of continuous functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \to W_\sigma$$

such that for $g \in (D \otimes_F \mathbb{A}_F^f)^\times$ we have $f(gu) = \sigma(u)^{-1} f(g)$ for $u \in U$, and $f(gz) = \psi(z) f(g)$ for $z \in (\mathbb{A}_F^f)^\times$. If we write $(D \otimes_F \mathbb{A}_F^f)^\times = \coprod_{i \in I} D^\times t_i U (\mathbb{A}_F^f)^\times$ for some $t_i \in (D \otimes_F \mathbb{A}_F^f)^\times$ and some finite index set I, then we have

$$S_{\sigma,\psi}(U, A) \underset{f \mapsto \{f(t_i)\}}{\xrightarrow{\sim}} \oplus_{i \in I} W_\sigma^{(U(\mathbb{A}_F^f)^\times \cap t_i^{-1} D^\times t_i)/F^\times}.$$

We will make the following assumption:
(2.1.2)
For all $t \in (D \otimes_F \mathbb{A}_F^f)^\times$ the group $(U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t)/F^\times$ has prime to p-order.

The calculations of [Tay 1, 1.1] show that $(U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t)/F^\times$ is automatically finite, and a 2-group if U is sufficiently small. Thus (2.1.2) holds for U sufficiently small.

Assuming (2.1.2), $S_{\sigma,\psi}(U, A)$ is a finite projective A-module, and the functor $W_\sigma \mapsto S_{\sigma,\psi}(U, A)$ is exact in W_σ.

(2.1.3) Let Q be a finite set of finite primes of F, such that for $v \in Q$, D is unramified at v and $v \nmid p$. Suppose that for each $v \in Q$,

$$U_v = \{g \in \mathrm{GL}_2(\mathcal{O}_{F_v}) : g = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) (\pi_v)\}.$$

For $v \in Q$ fix a quotient Δ_v of $(\mathcal{O}_{F_v}/\pi_v \mathcal{O}_{F_v})^\times$ of p-power order, and write $\Delta = \prod_{v \in Q} \Delta_v$. Define a compact open subgroup $U_\Delta = \prod_v (U_\Delta)_v \subset U$ by setting $(U_\Delta)_v = U_v$ if $v \notin Q$, and $(U_\Delta)_v$ the set of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_v$ such that ad^{-1} maps to 1 in Δ_v. Then $\Delta \xrightarrow{\sim} U/U_\Delta$ acts naturally $S_{\sigma,\psi}(U_\Delta, A)$ via the right multiplication of U on $D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times$. For $h \in \Delta$ we denote by $\langle h \rangle$ the corresponding operator on $S_{\sigma,\psi}(U_\Delta, A)$.

Lemma (2.1.4). *We have*

(1) *The operator* $\sum_{h \in \Delta} \langle h \rangle$ *on* $S_{\sigma,\psi}(U_\Delta, A)$ *induces an isomorphism*

$$\sum_{h \in \Delta} \langle h \rangle : S_{\sigma,\psi}(U_\Delta, A)_\Delta \xrightarrow{\sim} S_{\sigma,\psi}(U, A)$$

(2) $S_{\sigma,\psi}(U, A)$ *is a finite projective* $A[\Delta]$*-module*

Proof. The argument in [Tay 2, 2.3] uses duality on the space $S_{\sigma,\psi}(U_\Delta, A)$, which is not available in our level of generality. However we have the following more direct argument: It suffices to show that Δ acts freely on $D^\times\backslash(D\otimes_F\mathbb{A}_F^f)^\times/U_\Delta(\mathbb{A}_F^f)^\times$. If $u\in U$ fixes one of these double cosets, then there exists $t\in(D\otimes_F\mathbb{A}_F)^\times$ and $v\in U_\Delta(\mathbb{A}_F^f)^\times$ such that $uv^{-1}\in t^{-1}D^\times t\cap U_\Delta(\mathbb{A}_F)^\times$. Hence $(uv^{-1})^{2^r}\in F^\times$ for some $r>0$, and in particular $u^{2^r}\in U\cap U_\Delta(\mathbb{A}_F^f)^\times=U_\Delta$. Since U/U_Δ is a p-group, we are done. $\square$

(2.1.5) Let S be a set of primes containing Σ, the primes dividing p, and the primes v of F such that $U_v\subset D_v^\times$ is not maximal compact. Let $\mathbb{T}^{\mathrm{univ}}_{S,A}=A[T_v,S_v]_{v\notin S}$, be a commutative polynomial ring in the indicated formal variables. For each finite prime v of F we fix a uniformiser π_v of F_v. We consider the left action of $(D\otimes_F\mathbb{A}_F^f)^\times$ on W_σ-valued functions on $(D\otimes_F\mathbb{A}_F^f)^\times$ given by the formula $(gf)(z)=f(zg)$. Then $S_{\sigma,\psi}(U,A)$ becomes a $\mathbb{T}^{\mathrm{univ}}_{S,A}$-module with S_v acting via the double coset $U\begin{pmatrix}\pi_v & 0\\ 0 & \pi_v\end{pmatrix}U$ and T_v via $U\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}U$. These operators do not depend on the choice of π_v. We write $\mathbb{T}_{\sigma,\psi}(U,A)$ or simply $\mathbb{T}_{\sigma,\psi}(U)$ for the image of $\mathbb{T}^{\mathrm{univ}}_{S,A}$ acting on $S_{\sigma,\psi}(U,A)$.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{\mathrm{univ}}_{S,A}$. We say that $\mathfrak{m}$ is in the support of (σ,ψ) if $S_{\sigma,\psi}(U,A)_\mathfrak{m}$ is non-zero. We say that $\mathfrak{m}$ is *Eisenstein* if $T_v-2\in\mathfrak{m}$ for all but finitely many primes which split in some fixed abelian extension of F.

(2.1.6) Let Q be a finite set of primes of F which is disjoint from S, and for each $v\in Q$ fix a quotient Δ_v of $(\mathcal{O}_{F_v}/\pi_v)^\times$ of p-power order. Define compact open subgroups U_Q and U_Q^- of $\prod_v(\mathcal{O}_D)_v^\times$, by setting $(U_Q)_v=(U_Q^-)_v=U_v$ if $v\notin Q$, and defining

$$(U_Q^-)_v=\{g\in\mathrm{GL}_2(\mathcal{O}_{F_v}):\ g=\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\ (\pi_v)\}$$

and

$$(U_Q)_v=\{g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in(U_Q^-)_v:\ ad^{-1}\mapsto 1\in\Delta_v\}.$$

Suppose that $\mathcal{O}$ and $\mathbb{F}$ are as in (1.1.1). We will assume here that $\mathbb{F}$ is a finite field. We fix a maximal ideal $\mathfrak{m}\subset\mathbb{T}^{\mathrm{univ}}_{S,\mathcal{O}}$ such that $\mathfrak{m}$ is induced by a maximal ideal of $\mathbb{T}_{\sigma,\psi}(U)=\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$, and for $v\in Q$ the Hecke polynomial $X^2-T_vX+\mathbf{N}(v)S_v$ has distinct roots in $\mathbb{T}^{\mathrm{univ}}_{S,\mathcal{O}}/\mathfrak{m}$. After increasing $\mathbb{F}$, we may assume each of these polynomials has two distinct roots $\alpha_v,\beta_v\in\mathbb{F}$.

Write $S_Q=S\cup Q$. Consider the polynomial ring $\mathbb{T}^{\mathrm{univ}}_{S_Q,\mathcal{O}}[U_{\pi_v}]$ over $\mathbb{T}^{\mathrm{univ}}_{S_Q,\mathcal{O}}$ in the formal symbols U_{π_v} for $v\in Q$. Let $\mathfrak{m}_Q$ denote the ideal $\mathbb{T}^{\mathrm{univ}}_{S_Q,\mathcal{O}}[U_{\pi_v}]$ generated by $\mathfrak{m}\cap\mathbb{T}^{\mathrm{univ}}_{S_Q,\mathcal{O}}$ and the elements $U_{\pi_v}-\tilde{\alpha}_v$, where $\tilde{\alpha}_v\in\mathcal{O}$ is any lifting of α_v.

Denote by $\tilde{\mathbb{T}}_{\sigma,\psi}(U_Q)$ (resp. $\tilde{\mathbb{T}}_{\sigma,\psi}(U_Q^-)$) the rings of endomorphisms of $S_{\sigma,\psi}(U_Q,\mathcal{O})$ (resp. $S_{\sigma,\psi}(U_Q^-,\mathcal{O})$) generated by the elements of $T^{\mathrm{univ}}_{S_Q,\mathcal{O}}[U_{\pi_v}]$ where U_{π_v} acts by the endomorphisms corresponding to the double cosets $U_v\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}U_v$.

Lemma (2.1.7). *The ideal $\mathfrak{m}_Q$ induces proper, maximal ideals in $\tilde{\mathbb{T}}_{\sigma,\psi}(U_Q)$ and $\tilde{\mathbb{T}}_{\sigma,\psi}(U_Q^-)$. If $\alpha_v\beta_v^{-1}\neq\mathbf{N}(v)^{\pm1}$ for all $v\in Q$, then the natural map*

$$S_{\sigma,\psi}(U,\mathcal{O})_\mathfrak{m}\to S_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}_Q}\qquad(2.1.8)$$

is an isomorphism of $\mathbb{T}^{\mathrm{univ}}_{S_Q,\mathcal{O}}$-modules.

Proof. The first claim follows from the fact that the Hecke polynomial $X^2 - T_vX + \mathbf{N}(v)S_v$ vanishes at $X = U_{\pi_v}$. To see the second claim, it suffices to consider the case when Q consists of a single element. Since $\alpha_v\beta_v^{-1} \neq \mathbf{N}(v)^{\pm 1}$, the map

$$S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}}^{\oplus 2} \to S_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}}; \quad (f_1,f_2) \mapsto f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} f_2$$

is an isomorphism after inverting p, and a calculation using the fact that $\alpha_v \neq \beta_v$ shows that it is an isomorphism (see for example [Ki 3, 7.5]). Here the subscript $\mathfrak{m}$ on the right hand side means localisation with respect to the ideal $\mathfrak{m} \cap T^{\mathrm{univ}}_{S_Q,\mathcal{O}}$.

Since $X^2 - T_vX + \mathbf{N}(v)S_v$ has distinct roots in $\mathbb{F}$, by Hensel's lemma it has two distinct roots $A_v, B_v \in \mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}}$, lifting α_v and β_v respectively. Then

$$(U_{\pi_v} - B_v)(f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} f_2) = (U_{\pi_v} - B_v)(f_1 + B_vf_2),$$

and since $\alpha_v \neq \beta_v$, $U_{\pi_v} - B_v$ induces an automorphism of $S_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}_Q}$. This shows that (2.1.8) is a surjection between finite free $\mathcal{O}$-modules of the same rank, and hence an isomorphism. $\square$

(2.2) Global patching and multiplicities: We now carry out the Taylor-Wiles style patching argument (as modified in [Di] and [Ki 2]), which allows to relate the local deformation rings studied in §1, with patched Hecke algebras.

Keeping the notation above, we denote by $G_{F,S}$ the Galois group of the maximal extension of F, which is unramified outside S. For each finite prime prime v we denote by G_{F_v} the absolute Galois group of F_v, and we fix a map $G_{F_v} \to G_{F,S}$ induced by the inclusion of an algebraic closure of F into an algebraic closure of F_v. We also fix a continuous absolutely irreducible representation

$$\bar\rho : G_{F,S} \to \mathrm{GL}_2(\mathbb{F}).$$

Write $V_{\mathbb{F}}$ for the underlying $\mathbb{F}$-vector space of $\bar\rho$ and fix a basis for $V_{\mathbb{F}}$.

Let $\Sigma_p = \Sigma \cup \{v\}_{v|p}$. For $v \in \Sigma_p$, we denote by $R_v^{\square}$ the universal framed deformation $\mathcal{O}$-algebra of $\bar\rho|_{G_{F_v}}$ (considered with the chosen basis for $V_{\mathbb{F}}$), and by $R_v^{\square,\psi}$ the quotient of $R_v^{\square}$ corresponding to deformations with determinant ψ. We denote by $R^{\psi}_{F,S}$ the universal framed deformation $\mathcal{O}$-algebra of $\bar\rho$, and by $R^{\square,\psi}_{F,S}$ the complete local $\mathcal{O}$-algebra representing the functor which assigns to a local Artinian $\mathcal{O}$-algebra A, the set of isomorphism classes of tuples $\{V_A, \beta_v\}_{v\in\Sigma_p}$, where V_A is a deformation of $V_{\mathbb{F}}$ to A having determinant ψ, and β_v is a lifting of the chosen basis of $V_{\mathbb{F}}$ to an A-basis of V_A. For $v \in \Sigma_p$, the functor $\{V_A, \beta_w\}_{w\in\Sigma_p} \mapsto \{V_A, \beta_v\}$ induces the structure of an $R_v^{\square,\psi}$-algebra on $R^{\square,\psi}_{F,S}$. Finally we set $R^{\square,\psi}_{\Sigma_p} = \widehat{\otimes}_{\mathcal{O}} R_v^{\square,\psi}$, where in the tensor product v runs over the elements of Σ_p.

We now assume the following conditions hold.

(1) $\bar\rho$ is unramified outside the primes of F dividing p, and has odd determinant.
(2) The restriction of $\bar\rho$ to $G_{F(\zeta_p)}$ is absolutely irreducible.
(3) If $p > 3$, then $[F(\zeta_p) : F] > 2$.
(4) If $v \in S \setminus \Sigma_p$, then

$$(1 - \mathbf{N}(v))((1 + \mathbf{N}(v))^2 \det\bar\rho(\mathrm{Frob}_v) - (\mathbf{N}(v))(\mathrm{tr}\bar\rho(\mathrm{Frob}_v))^2) \in \mathbb{F}^{\times}.$$

Here, Frob_v denotes an arithmetic Frobenius at v.

Then as in [Ki 2, 2.3.5], we have

Proposition (2.2.1). *Set $g = \dim_{\mathbb{F}} H^1(G_{F,S}, \mathrm{ad}^0\bar{\rho}(1)) - [F:\mathbb{Q}] + |\Sigma_p| - 1$. For each positive integer n, there exists a finite set of primes Q_n of F, which is disjoint from S, and such that*

(1) *If $v \in Q_n$, then $\mathbf{N}(v) = 1(p^n)$ and $\bar{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues.*
(2) *$|Q_n| = \dim_{\mathbb{F}} H^1(G_{F,S}, \mathrm{ad}^0\bar{\rho}(1))$. If $S_{Q_n} = S \cup Q_n$, then as an $R^{\square,\psi}_{\Sigma,p}$-algebra $R^{\square,\psi}_{F,S_{Q_n}}$ is topologically generated by g elements. In particular $g \geq 0$.*

(2.2.2) Suppose now that $\mathfrak{m} \subset \mathbb{T}^{\mathrm{univ}}_{S,\mathcal{O}}$ is as in (2.1.6), and that $\mathfrak{m}$ is non-Eisenstein, with associated representation $\bar{\rho}$. That is, if $v \notin S$, and $\mathrm{Frob}_v \in G_{F,S}$ is an arithmetic Frobenius, then $\bar{\rho}(\mathrm{Frob}_v)$ has trace equal to the image of T_v in $\mathbb{F}$.

For $n \geq 1$ fix a set Q_n as in (2.2.1). Let Δ_v be the maximal p-quotient of $(\mathcal{O}_{F,v}/\pi_v)^\times$, and $\Delta_{Q_n} = \prod_{v\in Q_n} \Delta_v$. For each $v \in Q_n$ we fix a choice of zero in $\mathbb{F}$ of the polynomial $X^2 - T_v X + \mathbf{N}(v)S_v$ (increasing $\mathbb{F}$ if necessary), and we denote by $\mathfrak{m}_{Q_n} \in \mathbb{T}^{\mathrm{univ}}_{S_{Q_n},\mathcal{O}}$ the corresponding maximal ideal. We apply the discussion of (2.1.6) and (2.1.7) to each of these Q_n.

There is a map of $\mathcal{O}$-algebras $R^{\psi}_{F,S_{Q_n}} \to \mathbb{T}_{\sigma,\psi}(U_{Q_n})_{\mathfrak{m}_{Q_n}}$ such that for $v \notin S_{Q_n}$, the trace of Frob_v on the tautological $R^{\psi}_{F,S_{Q_n}}$-representation of $G_{F,S_{Q_n}}$ maps to T_v. Thus, we regard $S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}$ as an $R^{\psi}_{F,S}$-module via this map. Moreover $R^{\psi}_{F,S_{Q_n}}$ has a natural structure of $\mathcal{O}[\Delta_{Q_n}]$-algebra so that the induced $\mathcal{O}[\Delta_{Q_n}]$-structure on $S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}$ is the one given by (2.1.4) [Ta 2, 1.3, 2.1]. By (2.1.4) this is a finite free $\mathcal{O}[\Delta_{Q_n}]$-module, whose rank does not depend on n. Denote this rank by r. We now set

$$M_n = R^{\square,\psi}_{F,S_{Q_n}} \otimes_{R^{\psi}_{F,S_{Q_n}}} S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}.$$

for $n \geq 0$, where $S_{Q_0} = S$.

Fix a filtration by $\mathbb{F}$-subspaces

$$0 = L_0 \subset L_1 \subset \cdots \subset L_s = W_\sigma \otimes_{\mathcal{O}} \mathbb{F} = W_{\bar{\sigma}}$$

on $W_{\bar{\sigma}}$ such that L_i is $\mathrm{GL}_2(\mathbb{Z}_p)$-stable, and $\sigma_i = L_{i+1}/L_i$ is absolutely irreducible. This induces a filtration on $S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}} \otimes_{\mathcal{O}} \mathbb{F}$ whose associated graded pieces are the finite free $\mathbb{F}[\Delta_{Q_n}]$-modules $S_{\sigma_i,\psi}(U_{Q_n},\mathbb{F})_{\mathfrak{m}_{Q_n}}$. We denote by

$$0 = M_n^0 \subset M_n^1 \subset \ldots M_n^s = M_n \otimes_{\mathcal{O}} \mathbb{F},$$

the induced filtration in M_n, obtained by extension of scalars.

Following [Ki 2], set $j = 4|\Sigma_p| - 1$, $h = |Q_n|$, and $d = [F:\mathbb{Q}] + 3|\Sigma_p|$. Then $g = h + j - d$. We fix surjections

$$\mathcal{O}[\![y_1, \ldots, y_h]\!] \to \mathcal{O}[\Delta_{Q_n}] \tag{2.2.3}$$

Then $M_0 \xrightarrow{\sim} M_n/(y_1, \ldots, y_h)$ by (2.1.4) and (2.1.7).

The map $R^{\psi}_{F,S_{Q_n}} \to R^{\square,\psi}_{F,S_{Q_n}}$ is formally smooth of relative dimension j. We extend the maps (2.2.3) to maps

$$\mathcal{O}[\![y_1 \ldots, y_{h+j}]\!] \to R^{\square,\psi}_{F,S_{Q_n}} \tag{2.2.4}$$

in such a way that $R^{\square,\psi}_{F,S_{Q_n}}$ is identified with $R^{\psi}_{F,S_{Q_n}}[\![y_{h+1},\dots y_{h+j}]\!]$. We also fix surjections of $R^{\square,\psi}_{\Sigma_p}$-algebras

$$R^{\square,\psi}_{\Sigma_p}[\![x_1,\dots,x_g]\!] \to R^{\square,\psi}_{F,S_{Q_n}} \tag{2.2.5}$$

and a lifting of the maps in (2.2.4) to maps

$$\mathcal{O}[\![y_1.\dots,y_{h+j}]\!] \to R^{\square,\psi}_{\Sigma_p}[\![x_1,\dots,x_g]\!].$$

We regard each M_n as a $R^{\square,\psi}_{\Sigma_p}[\![x_1,\dots,x_g]\!]$-module via (2.2.5) and the map $R^{\psi}_{F,S_{Q_n}} \to \mathbb{T}_{\sigma,\psi}(U_{Q_n})_{\mathfrak{m}_{Q_n}}$ introduced above.

For $n \geq 1$ let

$$\mathfrak{c}_n = (\pi^n, (y_1+1)^{p^n}-1,\dots,(y_h+1)^{p^n}-1, y_{h+1}^{p^n},\dots,y_{h+j}^{p^n}) \subset \mathcal{O}[\![y_1,\dots,y_{h+j}]\!].$$

The proof of [Ki 2, 3.3.1] (which is of course based on the argument of Taylor-Wiles) shows that, after replacing the sequence $\{Q_n\}_{n\geq 1}$ by a subsequence, we may assume that there exist maps of $R^{\square,\psi}_{\Sigma_p}[\![x_1,\dots,x_g]\!]$-modules $f_n : M_{n+1}/\mathfrak{c}_{n+1}M_{n+1} \to M_n/\mathfrak{c}_nM_n$ which reduce modulo $(y_1,\dots,y_h)+\mathfrak{c}_n$ to the identity on $M_0/\mathfrak{c}_n$. Moreover, the same finiteness argument as in *loc. cit* implies that we may assume that this map is compatible with the filtration on $M_n\otimes_{\mathcal{O}}\mathbb{F}$ defined above.

Passing to the limit over n, we obtain a map of $R^{\square,\psi}_{\Sigma_p}[\![x_1,\dots,x_g]\!]$-modules

$$M_\infty \to M_\infty/(y_1,\dots y_h)M_\infty \xrightarrow{\sim} M_0.$$

Since M_n is a finite free $\mathcal{O}[\Delta_{Q_n}][\![y_{h+1},\dots,y_{h+j}]\!]$-module $M_n/\mathfrak{c}_nM_n$ is a finite free $\mathcal{O}[\![y_1,\dots y_{h+j}]\!]/\mathfrak{c}_n$-module, and M_∞ is a finite free $\mathcal{O}[\![y_1,\dots y_{h+j}]\!]$-module. Moreover, $M_\infty\otimes_{\mathcal{O}}\mathbb{F}$ has a filtration

$$0 = M^0_\infty \subset M^1_\infty \subset \dots M^s_\infty = M_\infty\otimes_{\mathcal{O}}\mathbb{F}$$

and since M^i_n/M^{i-1}_n is a finite free $\mathcal{O}[\Delta_{Q_n}][\![y_{h+1},\dots,y_{h+j}]\!]$-module, $M^i_\infty/M^{i-1}_\infty$ is a finite free $\mathcal{O}[\![y_1,\dots,y_{h+j}]\!]$-module for $i=1,\dots s$.

(2.2.6) We now assume that p splits in F, so that $F_v = \mathbb{Q}_p$ for $v|p$. For each such v, let $I_v \subset G_{F_v}$ denote the inertia subgroup, and fix a representation $\tau_v : I_v \to \mathrm{GL}_2(E)$ of Galois type. Suppose that for $v|p$ the representation W_{σ_v} of $U_v \xrightarrow{\sim} \mathrm{GL}_2(\mathcal{O}_{F_v})$ has the form $\sigma(k_v,\tau_v) = \mathrm{Sym}^{k_v-2}\mathcal{O}^2_{F_v}\otimes\sigma(\tau_v)$ where $k_v\geq 2$, and $\sigma(\tau_v)$ is a representation with open kernel which is associated to τ_v by the local Langlands correspondence in the sense explained in (1.1.3).

For each $v\in\Sigma_p$ we now define a quotient $\bar{R}^{\square,\psi}_v$ of $R^{\square,\psi}_v$ such that the action of $R^{\square,\psi}_v$ on each M_n factors through $\bar{R}^{\square,\psi}_v$. If $v|p$ $\bar{R}^{\square,\psi}_v$ is the ring denoted $R^{\square,\psi}(k_v,\tau_v,\bar{\rho})$ in (1.1.1). That the action of $R^{\square,\psi}_v$ on M_n factors through $\bar{R}^{\square,\psi}_v$ follows from the fact that the Galois representations attached to Hilbert modular eigenforms are compatible with the local Langlands correspondence [Ki 1], as well as the compatibility of the local and global Jacquet-Langlands correspondences.

For $v\nmid p$ we fix an unramified character $\gamma : G_{F_v}\to\mathcal{O}^\times$ such that $\gamma^2 = \psi|_{G_{F_v}}$, and $\bar{\rho}|_{G_{F_v}}$ is an extension of γ by $\gamma(1)$. Again, the fact that the action of $R^{\square,\psi}_v$ on

M_n factors through $\bar{R}_v^{\square,\psi}$ is a consequence of the compatibility between the local and global Langlands and Jacquet-Langlands correspondences.

We set $\bar{R}_{\Sigma_p}^{\square,\psi} = \widehat{\otimes}_{\mathcal{O}} \bar{R}_v^{\square,\psi}$ where v runs over Σ_p. The relative dimension, over $\mathcal{O}$ of $\bar{R}_v^{\square,\psi}$ is $3 + [F_v : \mathbb{Q}_p] = 4$ if $v|p$, and 3 if $v \nmid p$. In particular $\bar{R}_{\Sigma_p}^{\square,\psi}$ has relative dimension $[F : \mathbb{Q}_p] + 3|\Sigma_p|$.

The following lemma shows that to prove a modularity lifting theorem we are reduced to showing that M_∞ is a faithful $\bar{R}_\infty = \bar{R}_{\Sigma_p}^{\square,\psi}[\![x_1, \dots x_g]\!]$-module, or to a question on Hilbert-Samuel multiplicities.

Lemma (2.2.7). *The following conditions are equivalent.*

1. M_∞ *is a faithful* $\bar{R}_\infty$*-module*
2. M_∞ *is a faithful* $\bar{R}_\infty$*-module which has rank* 1 *at all generic points of* R_∞.
3. $e(\bar{R}_\infty/\pi\bar{R}_\infty) = e(M_\infty/\pi M_\infty, \bar{R}_\infty/\pi\bar{R}_\infty)$.
4. $e(\bar{R}_\infty/\pi\bar{R}_\infty) \leqslant e(M_\infty/\pi M_\infty, \bar{R}_\infty/\pi\bar{R}_\infty)$.

Moreover, if these conditions hold, and $\rho : G_{F,S} \to \mathrm{GL}_2(\mathcal{O})$ *is a deformation of* $\bar{\rho}$ *such that for* $v \in \Sigma_p$, $\rho|_{I_v}$ *is unipotent if* $v \nmid p$, *and* $\rho|_{G_{F_v}}$ *is potentially semi-stable of type* τ_v *and with Hodge-Tate weights* $0, k-1$ *if* $v|p$, *then* ρ *is modular, and arises from an eigenform in* $S_{\sigma,\psi}(U, \mathcal{O}) \otimes_{\mathcal{O}} E$.

Proof. Write $\mathcal{O}[\![\Delta_\infty]\!] = \mathcal{O}[\![y_1, \dots y_{h+j}]\!]$, and denote by $\mathbb{T}_\infty$ the image of $\bar{R}_\infty$ in $\mathrm{End}_{\mathcal{O}[\![\Delta_\infty]\!]}(M_\infty)$. Then $\mathbb{T}_\infty$ is a finite, torsion free $\mathcal{O}[\![\Delta_\infty]\!]$-module, and hence all its components have relative dimension $h + j$ over $\operatorname{Spec} \mathcal{O}$. Hence, if Z is such a component, then Z surjects onto $\operatorname{Spec} \mathcal{O}[\![\Delta_\infty]\!]$. This implies that the rank of $M_\infty|_Z$ is at most one, since otherwise $M_0 = M_\infty \otimes_{\mathcal{O}[\![\Delta_\infty]\!]} \mathcal{O}$ would have a fibre of dimension > 1 over some point of $\operatorname{Spec} R_{F,S}^\psi[1/p]$, and $S_{\sigma,\psi}(U, \mathcal{O})_{\mathfrak{m}}$ would have rank > 1 over some generic point of $\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}}$, which is impossible by the condition (4) in (2.2). Since M_∞ is a faithful $\mathbb{T}_\infty$-module its rank is exactly one on each irreducible component of $\operatorname{Spec} \mathbb{T}_\infty$.

This shows the equivalence of (1) and (2). Moreover, by (1.3.3)(2) we have

$$e(M_\infty/\pi M_\infty, \bar{R}_\infty/\pi\bar{R}_\infty) = e(M_\infty/\pi M_\infty, \mathbb{T}_\infty/\pi\mathbb{T}_\infty) = e(\mathbb{T}_\infty/\pi\mathbb{T}_\infty)$$

Since $\bar{R}_\infty$ is pure of relative dimension $d+g = h+j$ over $\mathcal{O}$, the inclusion $\operatorname{Spec} \mathbb{T}_\infty \hookrightarrow \operatorname{Spec} \bar{R}_\infty$ identifies $\operatorname{Spec} \mathbb{T}_\infty$ with a union of irreducible components of $\operatorname{Spec} \bar{R}_\infty$, and we have $e(\mathbb{T}_\infty/\pi\mathbb{T}_\infty) \leqslant e(\bar{R}_\infty/\pi\bar{R}_\infty)$ with equality if and only if the above inclusion is an isomorphism. This shows that (1), (3) and (4) are equivalent.

Suppose that the conditions (1)-(4) hold. Then ρ induces a map $\mathbb{T}_\infty = \bar{R}_\infty \to \mathcal{O}$, which kills the ideal $(y_1, \dots y_{h+j})$, and hence a map $\xi : \mathbb{T}_\infty/(y_1, \dots y_{h+j})[1/p] \to E$. Since M_∞ has positive rank on all components of $\mathbb{T}_\infty$, the fibre of M_0 over the closed point of $\mathbb{T}_\infty/(y_1, \dots y_{h+j})[1/p]$ corresponding to ξ is non-empty, and ξ induces a map $\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}} \to E$, which corresponds to the required eigenform in $S_{\sigma,\psi}(U, \mathcal{O}) \otimes_{\mathcal{O}} E$. $\square$

(2.2.8) Our next task is to compute $e(M_\infty/\pi M_\infty, \bar{R}_\infty/\pi\bar{R}_\infty)$. For $i = 1, \dots s$, write σ_i for the representation L_{i+1}/L_i. Thus σ_i has the form $\sigma_i = \otimes_{v|p}\sigma_{n_{i,v},m_{i,v}}$ where $(n_{i,v}, m_{i,v}) \in \{0, 1, \dots, p-1\} \times \{0, 1, \dots, p-2\}$, and $\sigma_{n_{i,v},m_{i,v}}$ is an irreducible constituent of $W_{\sigma_v}/\pi W_{\sigma_v}$.

For $v \in \Sigma$, we set $e_\Sigma = \prod_{v\in\Sigma} e(\bar{R}_v^{\square,\psi}/\pi\bar{R}_v^{\square,\psi})$.

Proposition (2.2.9). *The $\bar{R}_\infty$-module M^i_∞ is non-zero if and only if for each $v|p$ we have $\mu_{n_{i,v},m_{i,v}}(\bar\rho|_{G_{F_v}}) \neq 0$. If this condition hold for all $v|p$, and for each $v|p$ $\bar\rho|_{G_{F_v}} \nsim \begin{pmatrix} \chi & * \\ 0 & \chi \end{pmatrix}, \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ for any character $\chi : G_{F_v} \to \mathbb{F}^\times$ then*

$$e(M^i_\infty, \bar{R}_\infty/\pi\bar{R}_\infty) = e_\Sigma \prod_{v|p} \mu_{n_{i,v},m_{i,v}}(\bar\rho|_{G_{F_v}}) = e_\Sigma \tag{2.2.10}$$

Proof. The first statement follow from results of Gee [Ge 1], [Ge 2]. (When $F = \mathbb{Q}$ or $\bar\rho$ arises from a modular representation of $G_{\mathbb{Q}}$ this can be deduced from results asserting that the weights of modular forms giving rise to a given modular $\bar\rho$ are predicted by Serre's conjecture).

For $n, m \in \{0, 1, \dots, p-1\} \times \{0, 1, \dots, p-2\}$, let $\tilde\sigma_{n,m} = \mathrm{Sym}^n\mathcal{O}^2 \otimes \det^m$, and set $\tilde\sigma_i = \otimes_{v|p}\tilde\sigma_{n_{i,v},m_{i,v}}$. There is a surjection

$$\tilde{M}^i_n := R^{\square,\psi}_{F,S_{Q_n}} \otimes_{R^\psi_{F,S_{Q_n}}} S_{\tilde\sigma_i,\psi}(U_{Q_n}, \mathcal{O})_{\mathfrak{m}_Q} \to R^{\square,\psi}_{F,S_{Q_n}} \otimes_{R^\psi_{F,S_{Q_n}}} S_{\sigma_i,\psi}(U_{Q_n}, \mathbb{F})_{\mathfrak{m}_Q} = M^n_i$$

so for $v|p$, the action of $R^{\square,\psi}_v$ on M^i_∞ factors through $R^{\square,\psi}(n_{i,v}+1, \tau_{i,v}, \bar\rho)$ where $\tau_{i,v} = \chi^m_{\mathrm{cyc}} : I_{F_v} \to \mathcal{O}^\times$.

Now under our restrictions on $\bar\rho|_{G_{F_v}}$, $R^{\square,\psi}(n_{i,v}+1, \tau_{i,v}, \bar\rho)$ is a power series ring over $\mathcal{O}$. This follows, for example, from (1.6.10). (Of course in most cases, one does not need the ellaborate arguments of (1.6) to see this; when $\bar\rho$ is absolutely reducible it follows from a standard calculation using Galois cohomology, while when $\bar\rho$ is absolutely irreducible, this is a consequence of Fontaine-Laffaille theory when $n \leqslant p-2$.) Moreover in this situation all the terms in the product in (2.2.10) are 1, so the claim is that $e(M^i_\infty, \bar{R}_\infty/\pi\bar{R}_\infty) = e_\Sigma$.

To prove this, we will need to augment the situation considered in the patching argument of (2.2.2). Set

$$\bar{R}^{\square,\psi,i}_{\Sigma_p} = \widehat{\otimes}_{v\in\Sigma}\bar{R}^{\square,\psi}_v \widehat{\otimes}_{v|p} R^{\square,\psi}(n_{i,v}+1, \tau_{i,v}, \bar\rho).$$

and $\bar{R}^i_\infty = \bar{R}^{\square,\psi,i}_{\Sigma_p}[\![x_1, \dots, x_g]\!]$. The same finiteness argument used in (2.2.2) shows that after replacing the Q_n by a subsequence, we may assume that we have maps of $\bar{R}^i_\infty$-modules $f^i_n : \tilde{M}^i_{n+1}/\mathfrak{c}_n \to \tilde{M}^i_n/\mathfrak{c}_n$, lifting the maps $M^i_{n+1} \to M^i_n$ induced by the maps f_n of (2.2.2). We set $\tilde{M}^i_\infty = \varprojlim \tilde{M}^i_n/\mathfrak{c}_n$. The same argument as (2.2.2) shows that this is a finite flat $\mathcal{O}[\![\Delta_\infty]\!]$-module, and we have $\tilde{M}^i_\infty/\pi\tilde{M}^i_\infty \xrightarrow{\sim} M^i_\infty$. Now the image of $\bar{R}^i_\infty$ in $\mathrm{End}_{\mathcal{O}[\![\Delta_\infty]\!]}\tilde{M}^i_\infty$ has relative dimension $h+j$ over $\mathcal{O}$, and since $\bar{R}^i_\infty$ is a domain we find that $\tilde{M}^i_\infty$ is a faithful $\bar{R}^i_\infty$-module Thus, as in the proof of (2.2.7) we find

$$e(\tilde{M}^i_\infty/\pi\tilde{M}^i_\infty, \bar{R}^i_\infty/\pi\bar{R}^i_\infty) = e(\bar{R}^i_\infty/\pi\bar{R}^i_\infty) = e(\bar{R}^{\square,\psi,i}_{\Sigma_p}/\pi\bar{R}^{\square,\psi,i}_{\Sigma_p}) = e_\Sigma$$

where in the final equality we have used (1.3.9). $\square$

Corollary (2.2.11). *Suppose that for each* $v|p$ $\bar{\rho}|_{G_{F_v}} \nsim \begin{pmatrix} \chi & * \\ 0 & \chi \end{pmatrix}, \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ *for any character* $\chi : G_{F_v} \to \mathbb{F}^\times$. *Then* M_∞ *is a faithful* $\bar{R}_\infty$*-module, and any* $\rho : G_{F,S} \to \mathrm{GL}_2(\mathcal{O})$ *as in (2.2.7) is modular.*

Proof. Using (1.3.9) together with (1.6.16) and (1.6.17), we have

$$e(\bar{R}_\infty/\pi\bar{R}_\infty) = e_\Sigma \prod_{v|p} e(\bar{R}_v^{\square,\psi}) \leqslant e_\Sigma \prod_{v|p} \mu_{\mathrm{Aut}}(k_v, \tau_v, \bar{\rho}|_{G_{F_v}}).$$

Now under our assumptions $\mu_{\mathrm{Aut}}(k_v, \tau_v, \bar{\rho}|_{G_{F_v}})$ is equal to the number of irreducible constituents of $W_{\sigma_v}/\pi W_{\sigma_v}$ which have the form $\sigma_{n,m}$ with $\mu_{n,m}(\bar{\rho}) \neq 0$. (That is under our assumptions, $\mu_{n,m}(\bar{\rho}) \neq 0$ implies $\mu_{n,m}(\bar{\rho}) = 1$.) Hence (2.2.9) yields

$$e_\Sigma \prod_{v|p} \mu_{\mathrm{Aut}}(k_v, \tau_v, \bar{\rho}|_{G_{F_v}}) = \sum_{i=1}^{s} e(M_\infty^i, \bar{R}_\infty/\pi\bar{R}_\infty) = e(M_\infty/\pi M_\infty, \bar{R}_\infty/\pi\bar{R}_\infty).$$

Hence

$$e(\bar{R}_\infty/\pi\bar{R}_\infty) \leqslant e(M_\infty/\pi M_\infty, \bar{R}_\infty/\pi\bar{R}_\infty),$$

and the corollary follows from (2.2.7). □

Theorem (2.2.12). *Let* F *be a totally real field where* p *is totally split and*

$$\rho : G_{F,S} \to \mathrm{GL}_2(\mathcal{O})$$

a continuous representation. Assume (1.2.6) and suppose that

(1) *For* $v|p$ $\rho|_{G_{F_v}}$ *is potentially semi-stable with distinct Hodge-Tate weights.*
(2) $\bar{\rho} : G_{F,S} \xrightarrow{\rho} \mathrm{GL}_2(\mathcal{O}) \to \mathrm{GL}_2(\mathbb{F})$ *is modular and* $\bar{\rho}|_{F(\zeta_p)}$ *is absolutely irreducible.*
(3) *For* $v|p$ $\bar{\rho}|_{G_{F_v}} \nsim \begin{pmatrix} \chi & * \\ 0 & \chi \end{pmatrix}, \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ *for any character* $\chi : G_{F_v} \to \mathbb{F}^\times$.

Then ρ *is modular.*

Proof. This follows from (2.2.11) using the same base change arguments as in [Ki 2, 3.5]. Note that the relevant results on raising and lowering the level at $v \nmid p$ can be deduced from the case where all the W_{σ_v} are of the form $\mathrm{Sym}^{k_v-2} \otimes \det^{m_v}$ where $2 \leqslant k_v \leqslant p+1$, and in this case one has the relevant version of Ihara's lemma, (see [Ki 2, 3.1.8, 3.1.10]) □

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