Dixmier's Problem on Amenability

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Plan

1. Dixmier's problem
2. Non-unitarizability of $\mathbb{F}_\infty$
3. Similarity Problems and Completely Bounded Maps
4. Length and amenability

DIXMIER'S PROBLEM (1950)

Definition

- We will consider **uniformly bounded** representations $\pi : \; G \to B(H)$ and we set

$$|\pi| = \sup_{g \in G} \|\pi(g)\|$$

- uniformly bounded means $|\pi| < \infty$
- π unitary IFF $|\pi| = 1$

Theorem

(Sz. Nagy, Day, Dixmier 1950) *Any amenable group G (for example* **Z***) is "unitarizable" i.e. every uniformly bounded representation*

$$\pi : \ G \to B(H)$$

is unitarizable, i.e. : $\exists \xi : \ H \to H$ *invertible such that*

$$g \to \xi \pi(g) \xi^{-1}$$

is a unitary representation.

Counterexample : Ehrenpreis-Mautner (1955)

$$\mathbf{SL_2}(\mathbb{R})$$

is NOT unitarizable

More generally : free groups with $N > 1$ generators

Day-Dixmier : G *amenable* $\Rightarrow$ G *unitarizable*

Open Problem (Dixmier) :

$$G \quad \textit{unitarizable} \overset{?}{\Rightarrow} G \quad \textit{amenable ?}$$

"YES'" known for all CONNECTED locally compact groups (Kunze-Stein 1962....Cowling)

Principal case still open : DISCRETE groups

"YES'" known if G contains a copy of $\mathbb{F}_2$ free group with 2 generators)

Mantero-Zappa 1983, Pytlik-Szwarc 1986

For Linear Groups : Recall Tits's alternative

$$G \quad \textit{unitarizable} \Rightarrow G \not\supset \mathbb{F}_2$$

Indeed unitarizable passes to **subgroups** (and quotients) because induction works for unif. bounded reps defined on a subgroup

Possible Counterexamples :

must be *non-amenable groups not containing* $\mathbb{F}_2$

The existence of such groups remained open for a long time, but they **exist** : A. Olshanskii 1980

Are Burnside Groups unitarizable ?
Adian-Novikov Adian 1982 : odd $n \geq 665$
$\exists G$ infinite , finitely generated and periodic i.e. such that

$$g^n = 1 \quad \forall x \in G$$

More recently (Publ IHES 2002) Sapir and Olshanskii constructed finitely generated **finitely presented** non amenable groups not containing $\mathbb{F}_2$

Are Golod-Shafarevitch groups unitarizable ?

cf. recent work by Ershov and Bartholdi (cf. also Elek, Gromov)

A simple construction for the free group $G = \mathbb{F}_\infty$

Reference : Fendler, cf. also Bożejko.
Let S be the set of free generators and their inverses and consider the associated Cayley graph : (V, E), where $V = G$ and $E = \{(s, t) \in G \times G \mid |st^{-1}| = 1\}$.

Note : If $|st^{-1}| = 1$ then either $|s| = |t| + 1$ or $|s| = |t| - 1$

Crucial Obs :

The set of edges E can be partitioned into two subsets E_+ and E_-

$$E_+ = \{(s,t) \in E \mid |s| > |t|\} \qquad E_- = \{(s,t) \in E \mid |s| < |t|\}$$

such that

(1) $$\sup_{s \in G} |\{t \in G \mid (s,t) \in E_+\}| \leq 1$$

(2) $$\sup_{t \in G} |\{s \in G \mid (s,t) \in E_-\}| \leq 1$$

Indeed, given s, to say $(s,t) \in E_+$ (resp. $(t,s) \in E_-$) means that t is obtained from s by erasing the first (resp. last) letter.

Consider (formally) the unbounded operator

$$\Delta = \sum_{|w|=1} \lambda(w) = \sum_{(s,t)\in E} e_{s,t} = \sum_{(s,t)\in E_+} e_{s,t} + \sum_{(s,t)\in E_-} e_{s,t} = \Delta_+ + \Delta_-$$

Operator interpretation of (1) (resp. (2)) : $\Delta_+ : \ell_\infty(G) \to \ell_\infty(G)$ (resp $\Delta_- : \ell_1(G) \to \ell_1(G)$) has norm at most 1.
Let $\rho : G \to B(\ell_2(G))$ be the right regular representation, we define (note $[\rho, \lambda] = 0$)

$$D(g) = [\rho(g), \Delta_+] = -[\rho(g), \Delta_-]$$

Then

$$\|D(g) : \ell_2(G) \to \ell_2(G)\| \leq 2$$

and hence the representation

$$\pi(g) = \begin{pmatrix} \rho(g) & D(g) \\ 0 & \rho(g) \end{pmatrix}$$

is uniformly bounded with norm ≤ 3.
But π is NOT unitarizable because the matrix coefficient $D(g)_{e,e}$ (which is a coefficient of π) is equal to indicator function of S :

$$D(g)_{e,e} = 1_{\{|g|=1\}}$$

and the latter is not in the space $B(G)$ of coefficients of unitary representations on G.
Indeed, any function in $B(G)$ that is supported in $\{|g| = 1\}$ must be in ℓ_2.

Note that this proves more :

Definition

We say $\Lambda \subset G$ is an L-set when, if we use Λ as set of generators, there is a partition of the edges into two sets E_+, E_- such that we have both

$$\sup_{s \in G} |\{t \in G \mid (s,t) \in E_+\}| < \infty$$

$$\sup_{t \in G} |\{s \in G \mid (s,t) \in E_-\}| < \infty$$

Note : As pointed out in the audience during the talk, if Λ is symmetric, we may without loss of generality assume that E_- is the image of E_+ under transposition, so that only the first condition is needed (the other one is then identical).

Theorem

If G contains an infinite L-set $\Lambda \subset G$ then G is not unitarizable.

Indeed, the above construction shows that not only 1_Λ but any function with the same modulus as 1_Λ is in $B(G)$ and (in any group) this can only be true for finite sets.

Question I do not know whether the presence of an infinite L-set implies the presence of $\mathbb{F}_\infty$ as a subgroup.

Day-Dixmier Theorem : If G is amenable
$|\pi| < \infty$ implies $\exists \xi : \ H \to H$ invertible such that $g \to \xi\pi(g)\xi^{-1}$ is unitary
Proof : Very simple averaging argument let $C = |\pi|$

$$|||h||| = (\int \|\pi(g)h\|^2 d\phi(g))^{1/2}$$

$$C^{-1}\|h\| \leq |||h||| \leq C\|h\|$$

Let $\xi : \ H \to H$ with $\xi > 0$ such that

$$|||h|||^2 =< \xi^2 h, h >= \int < \pi(g)^*\pi(g)h, h > \phi(dg)$$

Translation invariance of ϕ
implies that $g \to \xi\pi(g)\xi^{-1}$ is unitary ■

This proof yields two “supplements" :

ADD 1 We have

$$\inf \|\xi\| \cdot \|\xi^{-1}\| \le |\pi|^2$$

where $|\pi| \stackrel{\text{def}}{=} \sup_{g \in G} \|\pi(g)\|$.

ADD 2 The “similarity” ξ can be chosen so that $\xi U = U\xi \ \forall U$ unitary such that $\forall g \ U\pi(g) = \pi(g)U$.
In other words, $\xi \in VN(\pi(G))$, indeed (abusive notation !) :

$$\xi = \left(\int \pi(g)^* \pi(g) \phi(dg) \right)^{1/2}$$

Open Problem :
Dixmier's Question :

$$G \text{ unitarizable} \stackrel{?}{\Rightarrow} G \text{ amenable}$$

Partial Results :

$$\begin{pmatrix} G \text{ unitarizable} \\ + \text{ ADD 1} \end{pmatrix} \Rightarrow G \text{ amenable}$$

$$\begin{pmatrix} G \text{ unitarizable} \\ + \text{ ADD 2} \end{pmatrix} \Rightarrow G \text{ amenable}$$

More precisely :

Theorem (1)

$$\left\{\exists\alpha < 3, \exists K, \forall\pi \quad \inf \|\xi\|\|\xi^{-1}\| \leq K|\pi|^{\alpha}\right\} \Rightarrow G \text{ amenable}$$

Theorem (2)

If every $\pi : \ G \to B(H)$ assumed unitarizable admits a unitarizing similarity ξ in $VN(\pi(G))$, then G is amenable.

Theorem (1) : valid for all locally compact

Theorem (2) : only for discrete groups

Complements

• If $\pi(G) \subset M$ with M a **finite** von Neumann algebra (i.e. carrying a finite faithful normal trace), then π is unitarizable (if $|\pi| < \infty$) (Vasilescu-Zsido 1974)
• Assume $G \supset \mathbb{F}_2$, then for any $1 \leq c < \infty$, there is a unif. bounded π on G that is not similar to any rep π' with $|\pi'| \leq c$.

KADISON'S PROBLEM (1955)

Consider the C^*-algebra (“full" ou “maximal") of G

$$A = C^*(G)$$

Definition : A is the completion of $\mathbb{C}[G]$ for the norm

$$\forall x \in \mathbb{C}[G] \quad \left\|\sum x(g)\delta_g\right\|_A \stackrel{\text{def}}{=} \sup_{\sigma\in\widehat{G}} \|\sum x(g)\sigma(g)\|$$

$$\widehat{G} \stackrel{\text{def}}{=} \{ \text{ unitary representations on } G\}.$$

If π is unitarizable

$$\exists\xi \quad \text{such that} \quad g \to \sigma(g) = \xi\pi(g)\xi^{-1}$$

is unitary therefore

$$\left\|\sum x(g)\pi(g)\right\| \leq C\left\|\sum x(g)\sigma(g)\right\| \leq C\left\|\sum x(g)\delta_g\right\|_A$$

$$\text{with} \quad C = \|\xi\|\|\xi^{-1}\|.$$

Hence if

$$u_\pi(x) \stackrel{\text{def}}{=} \sum x(g)\pi(g)$$

then $u_\pi\colon\ A \to B(H)$ is a bounded (unital) homomorphism.

KADISON' s Similarity Problem (1955)

Consider A C^*-algebra with unit

$$1 \in A \subset B(\mathscr{H})$$

$$u\colon \ A \to B(H)$$

homomorphism (unital) i.e.

$$\forall a, b \in A \quad u(ab) = u(a)u(b) \quad u(1) = I$$

Problem : $\|u\| < \infty \overset{??}{\Rightarrow}$ u similar to a *-homomorphism ?

We say u is similar to a $*$-homomorphism if $\exists \xi : \ H \to H$ invertible such that

$$x \mapsto \xi^{-1} u(x) \xi$$

is a $*$-homomorphism

Explicitly : $\forall x \in A \quad \xi^{-1} u(x^*) \xi = (\xi^{-1} u(x) \xi)^*$

$\Leftrightarrow$ u restricted to $\mathscr{U}(A)$ is unitarizable

Remark : $*$-homomorphism = C^*-algebraic representation

$\Leftrightarrow \|u\| = 1$
We will then say (abusively) that u is unitarizable, and we say A is **unitarizable** if every u that is bounded on A is unitarizable.
Note that G unitarizable $\Rightarrow A = C^*(G)$ unitarizable (but the converse is false)

With this terminology, Kadison's problem can be reformulated :

Is every C^*-algebra unitarizable ?

For this problem : Numerous partial results due to
E. Christensen (1976-1986)
and
U. Haagerup (1983) (*cyclic case*)

Known cases : A commutative

$$A = \widetilde{K} = K + \mathbb{C}I \qquad (K = \{\text{compacts}\})$$

A amenable C^*-algebra (=nuclear)

$A = B(H)$ (non nuclear)

$A = \widetilde{K} \otimes B \qquad B$ C^*-algebra arbitrary

$A = M$: type II_1 factor with property Γ

Example : $\bigotimes_{\mathbb{N}} (M_2, \tau_2)$.

Possible counterexamples

Open :

$$A = \text{ reduced } C^*\text{-algebra of } \mathbb{F}_2?$$

$$A = \left(\sum_n \bigoplus M_n\right)_\infty ?$$

Kadison's Similarity Problem

$$\Updownarrow$$

Derivation Problem

$$\begin{pmatrix}\text{Kirchberg}\\ \text{JOT 1996}\end{pmatrix}$$

Unitarizability criterion

Theorem

(Haagerup 1983) *u is unitarizable (= similar to a $*$-homomorphism)*
IFF
u is completely bounded.

Moreover :

$$\inf\{\|\xi^{-1}\| \cdot \|\xi\| \mid \xi \text{ unitarizing } u\} = \|u\|_{cb}$$

In particular for **group representations** :

$$\inf\{\|\xi^{-1}\| \cdot \|\xi\| \mid \xi \text{ unitarizing } \pi\} = \|u_\pi\|_{cb}$$

Completely bounded linear maps

$$\boxed{\|u\|_{cb} \stackrel{\text{def}}{=} \sup_{n\geq 1} \|u_n\|}$$

$$\begin{array}{ccc} B(\mathscr{H}) & & u \text{ linear} \\ \cup & & \\ A & \xrightarrow{u} & B(H) \end{array}$$

$$\begin{array}{rccc} u_n : & M_n(A) & \to & M_n(B(H)) \\ & [a_{ij}] & \mapsto & [u(a_{ij})] \end{array}$$

Note : $\|u\|_{cb} \geq \|u\| = \|u_1\|$!

$$a_{ij} \in B(H) \quad [a_{ij}] \in M_n(B(H)) = B(H \oplus \cdots \oplus H)$$

$$\|[a_{ij}]\| = \sup \left\{ \left(\sum_i \left\| \sum_j a_{ij} h_j \right\|^2 \right)^{1/2} \quad \Big| \quad \sum \|h_j\|^2 \leq 1 \quad h_j \in H \right\}$$

$$u_n([a_{ij}]) = [u(a_{ij})]$$

References : Stinespring 1950's
Arveson 1960's
Wittstock, Haagerup, Paulsen 1980's
$\mapsto$OPERATOR SPACE THEORY VERY ACTIVE since 1990
(Effros, Ruan, Blecher, Paulsen, Junge, Le Merdy, Xu, Ozawa, Oikhberg,...)

Unitarizability criterions

Theorem

G is unitarizable IFF

$$\exists d > 0 \quad \exists C \; \forall \pi \text{ unif. bounded rep. } \|u_\pi\|_{cb} \leq C|\pi|^d.$$

Moreover,

$$d_{\min}(G) \in \mathbb{N}$$

Theorem

A is unitarizable IFF

$$\exists d > 0 \quad \exists C \; \forall u \text{ bounded hom. } \|u\|_{cb} \leq C\|u\|^d.$$

Moreover,

$$d_{\min}(A) \in \mathbb{N}$$

Amenability criterions

(assuming G infinite, $\dim(A) = \infty$) :

Theorem (St Petersburg Math. J. 1999)

G is amenable IFF

$$d_{\text{min}}(G) = 2$$

Theorem (Pub. RIMS Kyoto, 2006)

A is amenable (=nuclear), IFF

$$d_{\text{min}}(A) = 2$$

Notion of LENGTH

Analogous to the following simple example :
Example. Γ semi-group with unit e

$$e \in S \subset \Gamma \quad \text{generating subset}$$

We say S *generates* Γ *with length* $\leq d$ if every x in Γ can be written as a product

$$x = s_1 s_2 \ldots s_d \quad \text{with} \quad s_i \in S.$$

We will study analogues of this notion for operator algebras with close ties to unitarizability.

$A \subset B(\mathscr{H})$ closed subalgebra (" operator algebra")

Definition. ("length of A")

$$K(A) = \overline{\bigcup_n M_n(A)}$$

$$M_n(A) \subset M_{n+1}(A)$$

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

Notation :

$$\widetilde{u}\colon\ [a_{ij}] \mapsto [u(a_{ij})]$$

Easy :

$$\|u\|_{cb} = \|\widetilde{u}\colon\ K(A) \to K(B(H))\|$$

LENGTH of A

$$\boxed{\ell(A) \leq d}$$

$$\Updownarrow$$

$$\forall x \in K(A)$$

$$\exists \alpha_0, \alpha_1, \ldots, \alpha_d \in K(\mathbb{C})$$

$$\exists D_1, \ldots, D_d \in K(A) \cap \{\text{diagonal}\}$$

$$\boxed{x = \alpha_0 D_1 \alpha_1 \ldots D_d \alpha_d}$$

with (actually automatic)

$$\prod_0^d \|\alpha_i\| \prod_1^d \|D_i\| \leq C\|x\|$$

Equivalent Definition : We say that $\ell(A) \le d$ if

$$\exists C \ \ \forall n \ \ \forall x \in M_n(A) \quad \exists N = N(n, x)$$

$$\exists \alpha_0 \in M_{n,N}(\mathbb{C}) \ \exists \alpha_1, \ldots, \alpha_{d-1} \in M_N(\mathbb{C}) \exists \alpha_d \in M_{N,n}(\mathbb{C}) \quad (\underline{\text{scalar}})$$

$$\exists D_i \in M_N(A) \quad (\underline{\text{diagonal}})$$

such that

$$x = \alpha_0 D_1 \alpha_1 \ldots D_d \alpha_d$$

and

$$\prod_0^d \|\alpha_i\| \prod_1^d \|D_i\| \le C\|x\|$$

REF : Notion inspired by infinite factorizations appeared in (Blecher–Paulsen, Proc. AMS 1991).

LENGTH of G

The length of G is defined similarly, with $A = C^*(G)$, we factorize elements of unit ball of $K(A)$) as before but the diagonal matrices $D_1, \cdots, D_d$ must be taken with diagonal coefficients in $\mathbb{C} \times G$, so that we still have for such a D

$$\|u_\pi(D)\| \leq \|D\||\pi|$$

Main Result

$$d_{\min}(A) = \ell(A)$$

$$d_{\min}(G) = \ell(G)$$

Proof of $d_{\min}(A) \le \ell(A)$ or $d_{\min}(G) \le \ell(G)$

Suppose $\ell(A) \le d$

Let $x \in M_n(A)$.
$x = \alpha_0 D_1 \dots D_d \alpha_d$
$\Pi\|\alpha_i\|\Pi\|D_i\| \le K\|x\|$
Recall :

$$\|u\|_{cb} = \sup_n \{\|u_n(x)\| \mid \ \|x\| \le 1\}$$

$$u_n(x) = \alpha_0 u_N(D_1)\alpha_1 \dots u_N(D_d)\alpha_d$$

$$\|u_N(D_i)\| \le \|u\|\|D_i\|$$

$$\begin{aligned}\|u_n(x)\| &\le \Pi\|\alpha_i\|\Pi\|u_N(D_i)\| \\ &\le \Pi\|\alpha_i\|\Pi\|D_i\|\|u\|^d \\ &\le K\|u\|^d\|x\|.\end{aligned}$$

Main Result $1 \in A \subset B(\mathscr{H})$
closed subalgebra

A unitarizable $\Leftrightarrow \ell(A) < \infty$. Moreover, let :

$$d_{\min}(A) = \inf\{d \geq 0 \mid \exists C\ \forall u\colon A \to B(H) \text{ hom. } \|u\|_{cb} \leq C\|u\|^d\}$$

Then :

$$d_{\min}(A) = \ell(A)$$

(and the inf is attained).

Connection with the derivation problem in C^*-alg. case

$$\|\delta\|_{cb} \leq a\|\delta\| \quad \forall \delta \text{ derivation}$$

$$\Downarrow$$

$$\|u\|_{cb} \leq \|u\|^a \quad \forall u \text{ homomorphism}$$

Conjecture : smallest $a_{\min} \in \mathbb{N}$ in C^*-case ? ?

Examples

$$1 < \dim(A) < \infty \qquad \underline{\ell(A) = 1}$$

Suppose $\dim(A) = \infty$

$$\left\{\begin{array}{l} A \ \ C^*\text{-algebra commutative} \\ A = K + \mathbb{C}I = \widetilde{K} \\ A \quad \text{nuclear} \end{array}\right\} \qquad \underline{\ell(A) = 2}$$

$$\left\{\begin{array}{ll} A & = B(H) \\ A & = \bigotimes_{\mathbb{N}} M_2 \ (\text{hyperfinite factor}) \end{array}\right\} \qquad \underline{\ell(A) = 3}$$

$$A = \widetilde{K} \otimes B \qquad \underline{\ell(A) \leq 3}$$

B arbitrary

$A = \mathrm{II}_1$ factor with property Γ

$d_{\min}(A) \leq 44$ Christensen JOT (1986),

$3 \leq \ell(A) \leq 5$ (2001)

and finally

$\underline{d_{\min}(A)(= \ell(A)) = 3}$ Christensen JFA 2002

Theorem

(Publ. RIMS Kyoto, 2006)

$$A \quad C^*\text{-algebra} \quad \dim(A) = \infty$$

$$\ell(A) = 2 \Leftrightarrow A \ \text{ nuclear}$$

A C^*-algebra A is called nuclear if for all C^*-algebra B, there is a unique C^*-norm on $A \otimes B$.

$$\Leftrightarrow \quad \forall B \quad A \otimes_{\min} B = A \otimes_{\max} B \ \text{ isometrically}$$

Note :

$$A \text{ nuclear} \Leftrightarrow A \text{ amenable (B.E. Johnson's sense)}$$

Haagerup (Invent. 1983)

Proof that $\ell(B(H)) \leq 3$

Proposition

Fix $n \geq 1$.
Let W_1, W_2 be unitary matrices $n \times n$ such that $|W_1(i,j)| = |W_2(i,j)| = \sqrt{n}^{-1}$ (ex : Hadamard matrices) Then $\forall x \in M_n(B(H))$ with $\|x\| \leq 1$

$$\exists D_1, D_2, D_3 \in M_n(B(H))$$

diagonal with norm $\leq$ 1 such that

$$\boxed{x = D_1 W_1 D_2 W_2 D_3.}$$

Proof : very simple !
Let S_i be ("Cuntz isometries") such that

$$S_i^* S_j = \delta_{ij} I$$

set

$$D_1(i,i) = S_i^* \qquad D_3(j,j) = S_j$$

and

$$D_2(k,k) = n \sum_{ij} \overline{W_1(i,k)} S_i x_{ij} S_j^* \overline{W_2(k,j)}$$

we have then

$$\boxed{x = D_1 W_1 D_2 W_2 D_3.}$$

∎

Back to ADD2

Theorem (Tohoku Math. J, 2007)

If every $\pi: \ G \to B(H)$ assumed unitarizable admits a unitarizing similarity ξ in $VN(\pi(G))$ then G is amenable.

Amenability criterion (Bożejko-Wysoczanski) :

$$[\ell_\infty(\ell_2) + {}^t\ell_\infty(\ell_2)]_G \subset \ell_\infty(\ell_2)_G$$

where the index G means that we restrict to the subspace of elements $k(s,t)$ commuting with right translations by G., i.e. such that $k(s,t) = k(st^{-1}, 1)$

Sketch of proof :

We pass to derivations :
our assumption becomes :
$\exists C$ For any representation of G in $\mathscr{U}(H)$ (unitary group of H) every derivation

$$D: \mathbb{C}[G] \to B(H)$$

that is of norm ≤ 1 on G is automatically of the form

$$D(x) = [T, x]$$

with T in $VN(D(G), G)$ of norm $\leq C$.

Consider then a kernel $k(s,t) = K(st^{-1})$ in

$$[\ell_\infty(\ell_2) + {}^t\ell_\infty(\ell_2)]_G$$

$$K(st^{-1}) = a(s,t) + b(s,t)$$

with

$$\sup_s(\sum_t |a(s,t)|^2)^{1/2} + \sup_s(\sum_t |b(t,s)|^2)^{1/2} \leq 1$$

Therefore $k = \sum K(\theta)\lambda(\theta)$

$$K(\theta)\lambda(\theta) = A_\theta + B_\theta$$

with $A_\theta = \sum_{st^{-1}=\theta} a(s,t) \otimes e_{st}$
We will apply our assumption to the derivation

$$D: \ f \mapsto [\rho(f)\otimes 1, \sum A_\theta \otimes \lambda(g_\theta)] = [\rho(f)\otimes 1, T] \in B(H \otimes \ell_2(\mathbb{F}_G))$$

where g_θ is family of free generators of $\mathbb{F}_G$.

where

$$T = \sum A_\theta \otimes \lambda(g_\theta)$$

One can show that

$$\|T\| \leq 1$$

but one can remplace T by a T' in $VN(D(G), G)$ with $\|T'\| \leq C$ ONLY if K is in ℓ_2.
Therefore our assumption that this remplacement is always possible implies G amenable ■

CONCLUSION

By the preceding results :
"Every C^*-algebra is unitarizable"
$\Updownarrow$
$\exists d_0$ such that $\ell(A) \leq d_0 \ \forall A \ C^*$-algebra . Still open !

Nevertheless we have

Theorem

(Math. Zeit. 2000) *For any integer $d \geq 1$, there exist an operator algebra (not self-adjoint) A_d of length d ; i.e.*

$$\ell(A_d) = d.$$

But, we still have **no example** of group G such that

$$2 < \ell(G) < \infty \quad !!$$