

On Amenability of Group Algebras

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Amenable groups

Definition

A group G is *amenable* if it admits *almost invariant subsets*; i.e. for every $\epsilon > 1$ and every finite $S \subset G$, a finite subset $F \subset G$ such that

$$\#(F \cup FS) \leq \epsilon \#F.$$

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A G -set X is *amenable* if it admits *almost invariant subsets*; i.e. for every $\epsilon > 1$ and every finite $S \subset G$, a finite subset $F \subset X$ such that

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Theorem

A G -set of subexponential growth is amenable.

Let R be an associative algebra, and let M be a right R -module.

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M is *amenable* if it admits *almost invariant subspaces*; i.e. for every $\epsilon > 1$ and every finite-dimensional $S \leq R$, a finite-dimensional $F \leq M$ such that

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Theorem

An R -module of subexponential growth is amenable.

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*Let \mathbb{K} be a field, and let X be a right G -set.
Then X is amenable if and only if $\mathbb{K}X$ is amenable.*

Ascending filtrations

Let $G = \langle S \rangle$ be a finitely generated group. Filter

$$0 \leq \mathbb{K}S \leq \mathbb{K}S^2 \leq \dots,$$

and grade

$$(\mathbb{K}G)_0 = \bigoplus_{n \geq 0} \mathbb{K}S^{n+1} / \mathbb{K}S^n.$$

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Theorem

If G is amenable, then $(\mathbb{K}G)_0$ is amenable.

If G has dead ends, then $(\mathbb{K}G)_0$ is amenable.

Ascending filtrations on modules

More generally, let $M = TR$ be a finitely-generated R -module, where $R = \langle S \rangle$. Filter

$$T \leq T + TS \leq T + TS + TS^2 \leq \dots,$$

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$$M_0 = \bigoplus_{n \geq 0} (T + \dots + TS^{n+1}) / (T + \dots + TS^n).$$

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Question

If M is an amenable, is then M_0 also amenable?

Descending filtrations – main theorem

Assume now that R has an augmentation $\varepsilon : R \rightarrow \mathbb{K}$, with kernel ϖ . Let M be an R -module. Filter

$$M \supseteq M\varpi \supseteq M\varpi^2 \dots,$$

and grade

$$\overline{M} = \bigoplus_{n \geq 0} M\varpi^n / M\varpi^{n+1}.$$

Theorem

If G is an amenable, finitely generated group, then $\overline{\mathbb{K}G}$ has subexponential growth (so is amenable).

Descending filtrations – open questions

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Does \overline{R} necessarily have subexponential growth?*

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*Let M be an amenable finitely generated R -module.
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Golod-Shafarevich groups

For a group G and a prime p , let (G_n) denote the *Zassenhaus series*:

$$G_1 = G, \quad G_{n+1} = [G_n, G](G_{\lceil n/p \rceil})^p.$$

The *degree* of $g \in G$ is

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Definition

A *Golod-Shafarevich group* is a group $G = F_d / \langle \mathcal{R} \rangle^{F_d}$, with for some $t \in (0, 1)$

$$1 - dt + \sum_{r \in \mathcal{R}} t^{\deg r} < 0.$$

Proposition

All Golod-Shafarevich groups are infinite; they have (uniformly) exponential growth.

More precisely, $\dim \varpi^n / \varpi^{n+1}$ grows exponentially, at rate $\geq 1/t$.

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Corollary

Golod-Shafarevich groups are not amenable.

In particular, there exist non-amenable residually- p torsion groups.

Theorem

*Let R be an amenable i -algebra;
let $K \leq R$ be a finite-dimensional i -subspace;
let $\epsilon > 0$ be given;
let I_0, I_1, \dots be a sequence of finite-codimension ideals in R
such that $\bigcap_{n \in \mathbb{N}} I_n = \{0\}$.
Then for all $n \gg 0$ there exists a (K, ϵ) -invariant subspace
 $T_n \leq R$ that is a vector-space complement for I_n in R .*

Let $\zeta \geq 1$ and $\delta \in (0, 1)$ be constants to be fixed later. For $\mu \geq \delta$, consider the transformation $\Theta_\mu : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ given by

$$\Theta_\mu(\nu, \alpha) = \left(\nu + \mu(1 - \alpha), \alpha + \frac{\mu(1 - \alpha)}{1 - \delta} \zeta \right).$$

Lemma

Let $\Omega = \mathbb{k}\mathcal{C}$ be a finite-dimensional R -module; let B be a subspace of Ω and let K, L be finite-dimensional i -subspaces of R ; assume that $xK \cong K$ for all $x \in \mathcal{C}$. Let $\alpha, \nu \in [0, 1)$ and $\zeta \geq 1$ be such that

$$\begin{aligned} \dim(KL^*) &\leq \zeta \dim K, & \dim(BK^*) &\leq \alpha \dim \Omega, \\ \dim B &= \nu \dim \Omega, & \dim(BL^*) &\leq \alpha \dim \Omega; \end{aligned}$$

Let furthermore $\delta \in (0, 1)$ be given.

Then there exist $s \geq 1$, elements $x_1, \dots, x_s \in \mathcal{C}$, and $\mu \geq \delta$ such that, setting $B_0 = B$ and $B_i = B_{i-1} + x_i K$ for $1 \leq i \leq s$ and $\Theta_\mu(\nu, \alpha) = (\nu', \alpha')$, we have

$$\dim(x_i K \cap B_{i-1}) \leq \delta \dim K \text{ for all } i \in \{1, \dots, s\}; \quad (1)$$

$$\dim B_s = \nu' \dim \Omega; \quad (2)$$

$$\dim(B_s L^*) \leq \alpha' \dim \Omega. \quad (3)$$

Constants

- 1 Choose $\delta > 0$, the **allowed overlap** as in (1), such that $\delta \dim K < \frac{\epsilon}{2}$ and $(1 + \frac{\epsilon}{2})(1 - \delta) > 1$;
- 2 Choose $\zeta > 1$ such that $(1 - \delta)/\zeta > 1 - \epsilon/(2 \dim K)$, the **relative Følner constant**.
- 3 Let $t \in \mathbb{N}$, the **tower height**, be such that $\overline{\nu}_t > 1 - \epsilon/(2 \dim K)$.
- 4 Construct finite-dimensional i -subspaces $K_0 = K, K_1, \dots, K_t$ of R , the **Rokhlin tower**, such that for all $j < i$ in $\{1, \dots, t\}$ one has $\dim(K_j K_j^*) < \zeta \dim K_i$, and such that

$$\dim(K_i K) \leq (1 + \frac{\epsilon}{2})(1 - \delta) \dim K_i \text{ for all } i \in \{1, \dots, t\}. \quad (4)$$

- 5 Choose $n \in \mathbb{N}$, the **quotient index**, large enough so that $K_i K_j^* \cap I_n = (0)$ for all $i \in \{1, \dots, t\}$.

Lemma

Let R be a unital \mathbb{K} -algebra generated by a finite-rank subspace S ; let $I \triangleleft R$ be a right ideal in R ; and let $F \leq R$ be a complement of I , so we have $R = I \oplus_{\mathbb{K}} F$. Let $x \mapsto \bar{x}$ be the projection $R \rightarrow F$; assume that $1 \in F$ and $\bar{1} = 1$.

Then

$$I = \langle fs - \overline{fs} : f \in F, s \in S \rangle$$

as a right ideal.

Reidemeister-Schreier for algebras

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as a right ideal.

Corollary

Let $R = \langle S \rangle$ be an augmented algebra, and consider $I \triangleleft R$ with $R = I \oplus_{\mathbb{K}} F$.

Then $I/I\varpi$ is spanned by the image of $(F + FS) \cap I$ in $I/I\varpi$.

Proof of the main theorem

Let $S \subset G$ and $\epsilon > 1$ be given. To show $\lim \sqrt[n]{\dim(\mathbb{K}G/\varpi^n)} < \epsilon$.

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$$\dim(F_n + F_n S) < \epsilon \dim F_n.$$

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$$\dim(\mathbb{K}G/\varpi^{n+1}) =$$

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On amenability of group algebras, I
to appear in *Isr. J. Math.*

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On amenability of group algebras, II
submitted

[arXiv.org/math.GR/0611709](https://arxiv.org/math.GR/0611709)