GEOMETRY OF, AND VIA, SYMMETRIES

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It is well known that Lie groups and homogeneous spaces provide a rich source of interesting examples for a variety of geometric aspects. Likewise it is often the case that topological and geometric restrictions yield the existence of isometries in a more or less direct way. The most obvious example of this is the group of deck transformations of the universal cover of a non-simply connected manifold. More subtle situations arise in the contexts of rigidity problems and of collapsing with bounded curvature.

Our main purpose here is to present the view point that the geometry of isometry groups provide a natural and useful link between theory and examples in Riemannian geometry. This fairly unexplored territory is fascinating and interesting in its own right. At the same time it enters naturally when such groups arise in settings as above. More importantly, perhaps, this study also provides a systematic search for geometrically interesting examples, where the group of isometries is short of acting transitively in contrast to the case of homogeneous spaces mentioned above.

Although the general philosophy presented here applies to many different situations, we will illustrate our point of view primarily within the context of manifolds with nonnegative or positive curvature.

We have divided our presentation into five sections. The first section is concerned with basic equivariant Riemannian geometry of smooth compact transformation groups, including a treatment of Alexandrov geometry of orbit spaces. Section two is the heart of the subject. It deals with the geometry and topology in the presence of symmetries. It is here we explain our guiding principle which provides a systematic search for new constructions and examples of manifolds of positive or nonnegative curvature. In the third section we exhibit all the known constructions and examples of such manifolds. The topic of section four is geometry via symmetries. We display three different types of problems in which symmetries are not immediately present from the outset, but where their emergence is crucial to their solutions. In the last section we discuss a number of open problems and conjectures related either directly, potentially or at least in spirit to the subject presented here.

Our exposition assumes basic knowledge of Riemannian geometry, and a rudimentary familiarity with Lie groups. Although we use Alexandrov geometry of spaces with a lower curvature bound our treatment does not require prior knowledge of this subject. Our intentions have been that anyone with these prerequisites will be able to get an impression of the subject, and guided by the references provided here will be able to go as far as their desires will take them.

1. Geometry of Isometry Groups

When we talk about geometry of individual, or groups of isometries we think of geometric entities associated with the isometry, or the group such as fixed point sets, invariant geodesics, displacement functions, orbits, strata of orbits, orbit types, orbit space, etc.

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Let us begin by reviewing the most basic geometric aspects of compact transformation groups. Throughout G will denote a compact Lie group which acts isometrically on a compact Riemannian manifold M. (We choose to view this as a left action). We denote by $G_x = \{g \in G | gx = x\}$ the *isotropy group* at $x \in M$, and by $Gx = \{gx | g \in G\} \simeq G/G_x$ the *orbit* of G through x. The ineffective kernel of the action is the subgroup $K = \cap_{x \in M} G_x$. Unless otherwise stated we assume that G acts *effectively* on M, i.e., $K = \{1\}$ is the trivial group. The action is called *almost effective* if K is finite. G is said to act *freely*, respectively *almost freely* if all isotropy groups are trivial, respectively finite. In these cases the orbits of G are the leaves of a smooth *foliation* of M. When the action is free, the space of orbits form a smooth manifold $B = M/G$ and the quotient map $M \to B$ is a *principal* bundle with group G.

The *fixed point set* of an element $g \in G$ is denoted by $M^g = \{x \in M | gx = x\}$. Similarly, $M^L = \cap_{g \in L} M^g$ will denote the fixed point set of a subgroup $L \subset G$. Note that $M^g = M^{<g>}$, where $< g >$ denotes the subgroup of G generated by g. It is important that each M^L is a finite disjoint union of closed totally geodesic submanifolds of M. This and other simple facts stated below are proved using that the exponential map $\exp : TM \to M$ is *equivariant* relative to the natural action of G on TM by the differentials.

Possibly the most crucial basic result in the theory of compact transformation groups is the following so-called *slice theorem*:

LEMMA 1.1 (Slice Theorem). *For any $x \in M$, a sufficiently small tubular neighborhood $D(Gx)$ of Gx is equivariantly diffeomorphic to $G \times_{G_x} D^{\perp}_x$.*

Here $D(Gx)$ is a suitable r-neighborhood of Gx and $D^{\perp}_x$ is the corresponding r-ball at the origin of the normal space $T^{\perp}_x$ to Gx at x. The usual *tubular neighborhood theorem* asserts that the normal exponential map of Gx, provides a diffeomorphism between the normal r-discbundle of Gx and $D(Gx)$, when r is sufficiently small. Since the G-equivariance is automatic, the essence of the slice theorem is the claim that the normal bundle of Gx is G-diffeomorphic to the bundle $G \times_{G_x} T^{\perp}_x = (G \times T^{\perp}_x)/G_x$ with fiber $T^{\perp}_x$ associated with the principal bundle $G \to G/G_x$ with group G_x. (In this construction, note that G_x acts on $T^{\perp}_x$ from the left, but on G from the right.)

Note that the isotropy group $G_{gx} = gG_xg^{-1}$ is conjugate to G_x. Two orbits Gx and Gy are said to be of the same *type* if G_x and G_y are conjugate subgroups in G. If a conjugate of G_x is a subgroup of G_y, Gx is said to have at least as large type as Gy. This is equivalent to saying that there a G-map from $Gx \simeq G/G_x$ onto $Gy \simeq G/G_y$. It is easy to see that each component of the collection of all orbits of the same type denoted (L), form a submanifold of M (typically not closed). The codimension of an orbit Gx in such a component is the dimension of the fixed point set of G_x in $T^{\perp}_x$. Since the *mean curvature* vector field of such a component is clearly fixed under the action of G, it follows that

PROPOSITION 1.2. *Each component of orbits of the same type form a minimal submanifold of M.*

This is just one simple illustration of how the slice representation can be used to yield geometric information.

For minimal geodesics between different orbits, the following simple but very useful fact was found in Kleiner's thesis [45].

LEMMA 1.3 (Isotropy Lemma). *Let $c : [0, d] \to M$ be a minimal geodesic between the orbits $Gc(0)$ and $Gc(d)$. Then for any $t \in (0, d)$, $G_{c(t)} = G_c$ is a subgroup of $G_{c(0)}$ and of $G_{c(1)}$.*

Using this one easily proves the following important and well known

THEOREM 1.4 (Principal Orbit Theorem). *There is a unique maximal orbit type. These so-called principal orbits form an open and dense subset of M.*

The usual distance between compact subsets defines a natural metric on the space of G-orbits in M denoted by M/G. By definition, this metric is a so-called *length metric*, indeed there is even a shortest curve, a geodesic, between any two orbits. We now proceed to describe the geometry of the orbit space M/G in more detail.

Throughout we will consider the orbit space M/G equipped with the above mentioned so-called *orbital metric*, and denote the quotient map by $\pi : M \to M/G$. When we view an orbit $Gx \subset M$ as a point in M/G, we will also use the notation $\pi(x) = [x]$. The following is immediate

PROPOSITION 1.5. *The orbit map $\pi : M \to M/G$ is a* submetry*, i.e., $\pi(B_x(r)) = B_{[x]}(r)$ for all $x \in M$ and all $r \geq 0$.*

Here $B_x(r)$ denotes the open r-ball centered at x. This has a very important consequence:

THEOREM 1.6. *The orbit space M/G has the structure of an Alexandrov space with locally totally geodesic orbit strata.*

The image under π of a component of orbits of the same type is what we call an *orbit stratum*. That each of these are locally totally geodesic in M/G is an easy consequence of the slice theorem (1.1).

A finite dimensional length space (X, dist) is called and *Alexandrov space* if it has curvature bounded from below, say $\mathrm{curv}\, X \geq k$. For a Riemannian manifold M the property $\mathrm{curv}\, M \geq k$ is equivalent to saying that its *sectional curvature* is bounded below by k, or in short $\sec M \geq k$. For a general metric space (X, dist), the property $\mathrm{curv}\, X \geq k$ can be expressed by the requirement that any four tuple of points $x = (x_0, x_1, x_2, x_3) \in X^4$ can be isometrically embedded in the simply connected 3-manifold, $S^3_{k(x)}$ with constant curvature $k(x) \geq k$. This is equivalent to the statement

$$\angle_{x_1,x_2}(k) + \angle_{x_2,x_3}(k) + \angle_{x_3,x_1}(k) \leq 2\pi \tag{1.7}$$

where $\angle_{x_i,x_j}(k)$, the so-called *comparison angle*, is the angle at $x_0(k)$ in the geodesic triangle in S^2_k with vertices $(x_0(k), x_i(k), x_j(k))$ the isometric image of (x_0, x_i, x_j) (see [6]). Using that π is a submetry it is immediate to check that indeed M/G is an Alexandrov space with $\mathrm{curv}\, X \geq k$ if $\sec M \geq k$.

Even general Alexandrov spaces have an amazingly rich structure (see [12], [53] for the basic facts described here, and [49] for deeper developments). An important notion in this context is that of the *space of directions* S_x at $x \in X$. By definition this space is the completion of the space of *geodesic directions* at x, i.e., of germs of unit speed geodesics emanating from $x = x_0$. The curvature bound yields a natural notion of *angle* between geodesic directions represented by c_1 and c_2, namely $\angle(c_1, c_2) = \lim_{t\to 0} \angle_{c_1(t),c_2(t)}(k)$. In M/G all directions are geodesic directions and although not completely trivial one has the following natural

PROPOSITION 1.8. *The space of directions $S_{[x]}$ at $[x] = \pi(x) \in M/G$ is isometric to $S^{\perp}_x/G_x$.*

The euclidean cone $CS_x = T_xX$ on S_x is called the *tangent space* to X at x. The metric on $CS_x = (S_x \times [0, \infty))/\{(u, 0) = (v, 0)\}$ is defined so that the distance between (u, s) and (v, t) is the distance in the euclidean plane between the end points of a hinge with sides of lengths u and v and angle $\mathrm{dist}(u, v)$. One also has the characterization $T_xX = \lim_{\lambda\to\infty} \lambda(X, x)$, where this limit refers to the so-called Gromov-Hausdorff limit (cf. [12]). In our case, $T_{[x]}M/G \simeq T^{\perp}_x/G_x$.

In a general Alexandrov space X its *boundary* ∂X consists of those points x for which ∂S_x is non-empty. This inductive definition is based on the fact that $\dim S_x = \dim X - 1$, and that the only compact one-dimensional Alexandrov spaces are the circle without boundary and the interval with two boundary points.

Let us now analyze the structure of M/G as an Alexandrov space a little closer. For simplicity we will use the abbreviation $F_x = (T_x^\perp)^{G_x}$ and consider the orthogonal decomposition $T_x^\perp = F_x + F_x^\perp$. At the corresponding point $[x] \in M/G$ we have $T_{[x]} \simeq F_x \times (F_x^\perp)/G_x$. Here F_x is canonically isomorphic to the tangent space at $[x]$ of the orbit stratum containing $[x]$. The cone $(F_x^\perp)/G_x$ should be viewed as the "normal space" of this stratum in M/G.

First note that for the *regular part* M_0 of M consisting of all the principal orbits, i.e., orbits of maximal type, we have $F_x = T_x^\perp$. Hence this set projects to exactly those points $[x] \in M/G$ for which $T_{[x]} \simeq T_x^\perp$ is euclidean. From the isotropy lemma (1.3) we see that this manifold $\pi(M_0) = M_0/G$ is a convex subset of M/G.

Now suppose Gx is an orbit of next largest type, i.e., only principal orbits have larger type. In this case, the action of G_x on the unit sphere $SF_x^\perp$ of $F_x^\perp$ has only one orbit type. In particular $SF_x^\perp/G_x$ is a smooth manifold and $SF_x^\perp \to SF_x^\perp/G_x$ is a locally trivial fiber bundle with fiber G_x/H, where H is the principal isotropy group. It is well known that proper fibrations of spheres with connected fibers have fibers with the homotopy type of either S^1, S^3 or S^7 and that the latter only occurs when the total space is S^{15} [11]. Since homogeneous manifolds homotopy equivalent to spheres are spheres (cf. [10, p.195]), we know that in our case the connected components of each fiber is a 1-,3- or 7-sphere unless G_x/H is finite or G_x acts transitively on $SF_x^\perp$. In all but the last case $[x]$ is an interior point of M/G whose normal sphere of its orbit stratum is either a complex projective space or the locally symmetric $\mathbb{Z}_2$ quotient of it, a quaternion projective space, the Cayley plane, or a non-simply connected space form. In the last case, however, where G_x acts transitively on $SF_x^\perp$, the normal space of the stratum at $[x]$ is a halfline, and M/G near $[x]$ is a manifold with totally geodesic boundary, the stratum of $[x]$. We will view such an orbit stratum as a *boundary face* of M/G. It follows again from the slice theorem, that if $[x]$ is an arbitrary boundary point of M/G, then it is in the closure of a boundary face. Thus, if M/G has non empty boundary it consists of a finite union of closures of locally totally geodesic faces.

We will end this section by fixing some more or less standard terminology. The non-regular part $M - M_0$ of M is divided into the *singular* part M_s and the *exceptional* part M_e. Here M_s consists of those orbits Gx with $\dim G_x > \dim H$, and M_e consists of the non-principal orbits with $\dim G_x = \dim H$.

2. Structure and Classification Program

There is a rich theory of smooth transformation groups in which the existence of actions by a compact group is related to differential topological properties of the manifold on which it acts. Similarly in Riemannnian geometry one of the basic issues is to understand relations between geometric properties of the manifold and the differential topological properties of it.

Except for the immense and beautiful work on geometry and topology of symmetric and homogeneous spaces, only modest efforts have been devoted to the general topic of relations between geometry, topology and symmetry.

Here we will primarily discuss manifolds of positive or nonnegative (sectional) curvature and analyze the geometry and topology of such manifolds in the presence of groups of isometries. Our *guiding principle* can be expressed as

> Classify or determine the structure of positively/nonnegatively curved manifolds with large isometry groups.

By itself this opens a vast and interesting area of which we have so far only seen the beginning. In addition, partial answers can help solve problems in which initially no isometries are present (see section 4). Even more importantly perhaps, when pushed to the limit, the above principle provides a systematic search for new and interesting examples (see section 3).

Let us mention some simple examples of what "large" isometry group G could mean:

- Big $\dim G \geq 0$, i.e., large *degree of symmetry* relative to M
- Big $\operatorname{rank} G \geq 0$, i.e., large *symmetry rank* relative to M
- Small $\dim M \geq 0$, relative to G
- Small $\dim M/G \geq 0$, i.e., low *cohomogeneity*
- Small $\dim M/G - \dim M^G \geq 1$, i.e., low *fixed point cohomogeneity*

For finite groups G, where all these notion make G small, other notions such as the order $|G|$ of G, or the minimal number of generators for G or similar invariants for say abelian subgroups of G can be used to express largeness.

Before we discuss the above program in relation to any of these notions of "large", we will describe some useful tools based on the Alexandrov geometry of the orbit space M/G in the context of positive or nonnegative curvature.

One has the following orbit space analog of the celebrated *soul theorem* by Cheeger and Gromoll [16]:

THEOREM 2.1 (Soul Theorem). *If* $\operatorname{curv} M/G \geq 0$ *and* $\partial M/G \neq \emptyset$*, then there exists a totally convex compact subset* $S \subset M/G$ *with* $\partial S = \emptyset$*, which is a strong deformation retract of* M/G*. If* $\operatorname{curv} M/G > 0$*, then* $S = [s]$ *is a point, and* $\partial M/G$ *is homeomorphic to* $S_{[s]}M/G \simeq S_s^{\perp}/G_s$*.*

Note that $\operatorname{curv} M/G \geq 0$ or $\operatorname{curv} M/G > 0$ are ensured if for example $\sec M \geq 0$ or $\sec M > 0$.

The exact same statement is true in the general context of Alexandrov spaces (cf. [48]). For orbit spaces, however, the proof is considerably simpler. Here one can essentially reduce the proof to that of the original soul theorem, by noticing that geodesics in M/G are limits of geodesics emanating from the open, dense, convex regular part $(M/G)_0$ of M/G.

Similarly, Synge type techniques, yielding versions of Synge's theorem and of Frankel's theorem exist for general Alexandrov spaces (see [51]). By such techniques we refer to notions of *parallel transport*, and the fact that exponentiating parallel fields of directions along geodesics in a space of positive curvature yields shorter curves locally. We will not use these tools here, nor will we use more subtle equivariant versions of them.

The key point in the utility of the orbit space is most frequently connected with the *singularities* of it, as in the above soul theorem. This is expressed even more directly in the *extent lemma* below which yields bounds for the number of singular points taking into account how singular they are.

Recall that the $q-$*extent* of a compact metric space X (cf. [35]) is defined as

$$xt_q X = \frac{1}{\binom{q}{2}} \max_{(x_1,\ldots,x_q)} \sum_{i<j} \operatorname{dist}(x_i, x_j) \tag{2.2}$$

In other words, $xt_q X$ is the maximal average distance between points in q-touples in X. The reason for taking the average rather than the sum is that then the sequence $\{xt_q X\}$ is decreasing and hence has a limit called the infinity extent, or simply the extent xtX of X. Note that $xt_2 X = \operatorname{diam} X$ and $xtX \geq \frac{1}{2}\operatorname{diam} X$.

We can now formulate the

Lemma 2.3 (Extent Lemma). *If* $\operatorname{curv} M/G \geq 0$, *then* $\frac{1}{q+1}\{xt_q(S_{[x_0]}), \ldots, xt_q(S_{[x_q]})\} \geq \pi/3$ *for all* $(q+1)$ *touples* $([x_0], \ldots, [x_q])$ *in* M/G. *If* $\operatorname{curv} M/G > 0$, *then this average* q*-extent inequality is strict as well.*

The exact same statement holds in a general Alexandrov space with the same simple proof (cf. [35, 37] and [44]): Pick arbitrary minimal geodesics between any two points in the $(q+1)$ touple. At each point there are $\binom{q}{2}$ angles, i.e., there is a total of $(q+1)\binom{q}{2}$ angles. The same configuration has $\binom{q+1}{3}$ triangles. Since the sum of angles in a triangle is at least π by curvature comparison, the lemma follows from the definition of the extents.

The key point in the applications of this lemma is that the more singular a point is the smaller its space of directions is, and hence the smaller its q-extents are. The lemma therefor limits the number of very singular points.

We will now illustrate how these techniques can be used in the context of the classification program. We begin by analyzing torus actions. These play a particularly important role for various reasons including phenomena related to collapse (see section 4).

Without reference to curvature one has the following simple identity for the *Euler characteristic* $\chi(M)$ of M due to Kobayashi (cf. [46])

Theorem 2.4. *If a torus* T *acts effectively on* M *then* $\chi(M) = \chi(M^T)$.

Note that from simple representation theory the codimension of any component of M^T in M is even. When M has positive curvature, Berger observed that any Killing vector field on M has a zero [7]. This has the following extension to isometric torus actions, which in essence may be proved via Synge type techniques (cf. also [67]).

Lemma 2.5 (Isotropy rank Lemma). *Suppose* $\sec M > 0$ *and* T *acts isometrically on* M. *Then* M^T *is non-empty when* $\dim M$ *is even, and if* $\dim M$ *is odd either* M^T *is non-empty, or* T *has a circle orbit.*

It is an immediate consequence of this lemma that if G acts isometrically on a manifold M with positive curvature then there is a point $x \in M$ such that $\operatorname{rank} G_x = \operatorname{rank} G$ if $\dim M$ is even, or else $\operatorname{rank} G - \operatorname{rank} G_x \leq 1$. In particular, if G acts freely, M must be odd dimensional and $\operatorname{rank} G \leq 1$. Let us now see how $\operatorname{rank} G$ itself is restricted in general when $\sec M > 0$.

Theorem 2.6 (Maximal rank Theorem). *If a torus* T *acts isometrically on* M *and* $\sec M > 0$ *then* $\dim T \leq [(\dim M + 1)/2]$, *and equality holds if and only if* M *is a sphere, a lens space, or a complex projective space.*

This result from [36] illustrates the classification program when the notion of large is big or rather maximal rank. The proof is carried out via induction on $\dim M$ for even and odd dimensions separately. The key points are, that if $\dim T > [(\dim M + 1)/2]$ then T acts ineffectively, and if $\dim T = [(\dim M + 1)/2]$ then there is a circle subgroup $S^1 \subset T$ with $\operatorname{codim} M^{S^1} = 2$. In the latter case it is clear that a component of M^{S^1} with codimension 2 project to boundary faces of M/S^1. It then follows from the soul theorem combined with critical point theory for distance functions that there is exactly one such component, exactly one S^1-orbit $S^1 s$ at maximal distance to this fixed point component, and all other orbits are principal. The three cases in the theorem then correspond to S^1_s being trivial, cyclic, or all of S^1 respectively.

Recently Rong [58] has obtained some partial results when $\dim T$ is almost maximal, i.e., when $\dim T = [(\dim M + 1)/2] - 1$. In particular, if M is a simply connected 5-manifold and $\dim T = 2$, then M is diffeomorphic to S^5. When $\dim M = 4$, almost maximal means $T = S^1$.

Here the beautiful work of Hsiang and Kleiner [44] (done before the development of Alexandrov geometry) was one of the key inspirations to the material discussed in this section.

Theorem 2.7 (Hsiang-Kleiner). *If S^1 acts isometrically on a simply connected 4-manifold M with $\sec M > 0$ then M is homeomorphic to either S^4, or $\mathbb{C}P^2$.*

The proof of this result is based on Freedman's topological classification of simply connected 4-manifolds (see [23]). It follows from this classification that it suffices to prove that under the assumptions of the theorem $\chi(M) \leq 3$. By (2.4) this means that $\chi(M^{S^1}) \leq 3$. But M^{S^1} consists of isolated points and 2-dimensional components. If there are 2-dimensional components the above discussion yields the conclusion even up to diffeomorphism. When the fixed point set consists of isolated points only, the extent lemma (2.3) implies that there are at most three isolated fixed points. The crucial point here is that $xt_3(S^3/S^1) \leq \pi/3$ whenever $(S^3)^{S^1}$ is empty, and this is indeed the case for the unit tangent sphere S^3 at an isolated fixed point.

Note that this provides significant information concerning the classical *Hopf conjecture* which asserts that there is no metric of positive curvature on $S^2 \times S^2$. Indeed if there is such a metric, then the isometry group of this metric can at most be finite, i.e., its degree of symmetry would be zero. When $\sec M \geq 0$ the only additional manifolds one has to add in the above theorem are $S^2 \times S^2$ and $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$, as was proved in the unpublished part of Kleiner's thesis and in [63].

There is no doubt that one of the most natural measurements for the group G being large is that the orbit space is small, in particular in the sense that $\dim M/G$ is small. We will discuss this in more detail in the next section. Here we will discuss a related idea which in turn is useful also in other measurements for largeness.

If the fixed point set M^G is non-empty, this already puts a constraint on M/G since we can view M^G it as a proper subset via the projection map π. In particular, the codimension $\dim M/G - \dim M^G$ is at least one. When G has no fixed points, this codimension is just $\dim M/G + 1$. Thus to analyze manifolds with minimal *fixed point cohomogeneity* has two parts: (a) Homogeneous manifolds G/H, and (b) G-manifolds M with $M^G \neq \emptyset$ of codimension one in M/G. We will refer to manifolds in class (b) as *fixed point homogeneous*, since they are as homogeneous as they can be given that G has fixed points. A circle action with fixed point set of codimsion two as in the discussion above is an example of a fixed point homogeneous manifold. If M has positive curvature, the arguments for circle actions extend to general fixed point homogeneous manifolds via the soul theorem (2.1), and yields a complete classification (see [37]). When M is simply connected, this classification is formulated in

Theorem 2.8. *Any simply connected fixed point homogeneous manifold with positive curvature is diffeomorphic to a compact rank one symmetric space.*

This class of spaces, i.e., the spheres, complex or quaternionic projective spaces , or the Cayley plane is simetimes denoted by CROSS. Since the classification of simply connected homogeneous manifolds with positive curvature was completed already in 1976 (cf. section 3), this result provides a complete classification of simply connected positively curved manifolds with minimal fixed point cohomogeneity one. It also turns out to be useful (as would extensions to higher fixed point homogeneity) in another classification problem, where large G means small M.

The idea here is to fix the group G and ask for a classification of all sufficiently low dimensional manifolds of positive curvature on which G acts isometrically, and at least almost effectively. When $G = T$ is a torus, the maximal rank theorem (2.6) provides an answer: The lowest dimensional manifolds of positive curvature on which T^k can act isometrically are S^{2k-1} and $S^{2k-1}/\mathbb{Z}_q$. Moreover, if $\dim M = 2k$, then M is either S^{2k}, $\mathbb{R}P^{2k}$ or $\mathbb{C}P^k$. For manifolds with $\dim M > 2k$, the problem is wide open. For $\dim M = 2k+1$ see [58] though. If G is one of the simply connected classical groups $\mathrm{Spin}(n)$, $\mathrm{SU}(n)$ or $\mathrm{Sp}(n)$ classification through a much larger dimension interval is known. Similar statements hold for G_2 and for F_4. The fact is that through these ranges of dimensions (roughly up to twice the minimal dimension) only homogeneous manifolds of positive curvature occur (cf. [37]). This kind of statement illustrates very well the strategy in finding new examples: Suppose $\dim M$ exceeds this range by one. What can be said about the structure of M if it exists, and is it then a new example? For the other exceptional groups E_6, E_7 and E_8 we do not even know the minimal dimension of a positively curved manifold on which it acts isometrically. If this does not arise from the lowest dimensional linear representation, it is indeed a new example!

This kind of problem has not been analyzed for products of groups other than for products of circles as discussed above.

3. Constructions and Examples

In the previous section we presented various classification problems, any one of which potentially constitutes a systematic search for new examples of manifolds with positive or nonnegative curvature. In this section we will exhibit the known types of such manifolds and describe their constructions.

The simplest of all construction, which however only works within the class of manifolds of nonnegative curvature, is to take products $V \times W$, where both V and W have nonnegative curvature. For Alexandrov spaces there is an analogous construction of joins between positively curved spaces, but in Riemannian geometry this is valid only between unit spheres. What is important about these constructions is that they are dimension increasing.

The only construction so far by means of which manifolds of positive curvature are constructed, is based on the fact already touched upon in connection with Alexandrov spaces that submetries are curvature nondecreasing. In the context of manifolds, the equations expressing this phenomenon for *Riemannian or semi-Riemannian submersions* are referred to as the *Gray-O'Neill* formulas (see [47] and [24]). A special consequence of these equations is that if $\pi : M \to N$ is a Riemannian submersion and $\sec M \geq 0$ then $\sec N \geq 0$, and sometimes even $\sec N > 0$. On the other hand, to go in the opposite direction, i.e., to construct metrics on the total space of a bundle with positive or nonnegative sectional curvature is exceedingly difficult (for partial results see [74] and below).

Except for special *gluing methods* to be discussed below, these and combinations of them are the only known methods for constructing manifolds with nonnegative or positive curvature.

The single basic source for manifolds with nonnegative/positive curvature is the class of semisimple Lie groups. This is tied to the non-commutativity of the group. Indeed, if G is equipped with a biinvariant metric then $\sec(\mathrm{span}\{X, Y\}) = \frac{1}{4}\,\|[X, Y]\|^2$, where $[X, Y]$ is the Lie bracket between $X, Y \in \mathfrak{g} \simeq T_e G$.

A simple example of a Riemannian submersion is the quotient map $\pi : M \to M/G = N$, when there is only one orbit type. In particular, any homogeneous space G/H arises in this way and hence has a G-invariant metric with $\sec G/H \geq 0$. The question whether G/H has a G-invariant metric with positive curvature is much more involved and the complete answer (in the simply connected case) is contained in the work of Berger [8], Wallach [68], Aloff-Wallach [1], and Berard-Bergery [5].

Theorem 3.1 (Homogeneous Classification). *Aside from the rank one symmetric spaces only the following simply connected manifolds:*

- $W^7_{1,1} = (\mathrm{SU}(3) \times \mathrm{SO}(3))/\,\mathrm{U}(2)$, $\quad B^7 = \mathrm{SO}(5)/\,\mathrm{SO}(3)$, $\quad B^{13} = \mathrm{SU}(5)/(\mathrm{Sp}(2) \times S^1)$
- $W^6 = \mathrm{SU}(3)/T^2$, $\quad W^{12} = \mathrm{Sp}(3)/\,\mathrm{Sp}(1)^3$, $\quad W^{24} = \mathrm{F}_4\,/\,\mathrm{Spin}(8)$
- $W^7_{k,l} = \mathrm{SU}(3)/S^1_{k,l}$, $\gcd(k,l) = 1, kl(k+l) \neq 0$

have a homogeneous metric with positive curvature.

We will not describe the precise embedding $H \subset G$ in these examples other than saying that $\mathrm{SO}(3) \subset \mathrm{SO}(5)$ is maximal, $T^2 \subset \mathrm{SU}(3)$ is maximal and $S^1_{k,l} \subset T^2$ is the obvious circle winding k times around one S^1-factor of $T^2 = S^1 \times S^1$ and l times around the other S^1-factor. The first three spaces in this theorem are normal homogeneous, i.e., the metric on G/H is the orbital metric induced by a biinvariant metric of G. The first of these, however, was not on Berger's list [8], but was recognized in Wilking [72].

Note that the classification of homogeneous non-simply connected manifolds with homogeneous metrics of positive curvature is not to be found in the literature. Also, the difficult problem of determining the space of all homogeneous metrics with positive curvature on the above examples is open. This is particularly interesting for determining optimal pinching constants for these manifolds (cf. [56]).

More generally, a so-called *biquotient* denoted $G//H$ is the orbit space of a Lie group G by a subgroup $H \subset G \times G$ acting from left and right on G. Here we consider only those H which act freely so that $G//H$ is a manifold. The interest in these spaces began with the discovery by Gromoll and Meyer [27] that one of the exotic Milnor spheres can be presented in this way. Since obviously any biquotient admits a metric with nonnegative curvature, this exhibited for the first time a metric of nonnegative curvature on an exotic sphere. Moreover, the particular metric constructed by them had points with positive curvature. Several attempts to deform the metric to have positive curvature have failed. To some extent these attempts were supported by a general *deformation conjecture* proposing that a manifold with nonnegative curvature and positive curvature at a point can be deformed to have positive curvature, in analogy to the case of Ricci curvature (cf. [2, 18]). The first inhomogeneous biquotients with positive curvature were found by Eschenburg [20, 19]. He found one example in dimension 6 and an infinite family in dimension 7. More recently Bazaikin [3] found another similar infinite family in dimension 13. It is a striking fact, that so far all known simply connected manifolds with positive curvature (including the homogeneous ones above) are constructed in this way. It is also curious that except for the rank one symmetric spaces they occur only in the dimensions 6, 7, 12, 13 and 24. Just very recently, Wilking [71] made the remarkable discovery that there are biquotients with positive curvature on an open and dense set in infinitely many dimensions. In particular all the projective tangent bundles of the projective spaces admit such metrics. One of the striking examples from this very interesting list is $\mathbb{R}P^2 \times \mathbb{R}P^3$. By Synge's theorem this manifold does not carry a metric with positive curvature since it is odd dimensional but not orientable. Therefor this provides a *counterexample* to the *deformation conjecture* mentioned above. It is also worth noticing that this most likely also will provide a simply connected counterexample to the deformation conjecture. If not it will yield a counterexample to the generalized *Hopf conjecture* asserting that higher rank symmetric spaces do not admit metrics with positive curvature. The idea to investigate manifolds with *quasipositive curvature*, i.e., nonnegatively curved manifolds with positive curvature at a point, and manifolds with *almost positive curvature*, i.e., with positive curvature on an open and dense set, as natural classes between nonnegatively and positively curved manifolds was initiated and promoted by Wilhelm and Petersen (cf. e.g. [50]).

The first gluing construction of manifolds with nonnegative curvature was done by Cheeger. He showed that the connected sum of any two rank one symmetric spaces admits a metric with nonnegative curvature. The key point in his construction [14] is to show that the complement of a disc in a rank one symmetric space admits a metric of nonnegative curvature which near the boundary sphere is a product metric. The specific description of this complement as a disc bundle is crucial for the construction.

In this context it was shown by Guijarro [41] that the metric on any complete noncompact manifold M with nonnegative curvature can be deformed near its soul S in such a way that in the new metric with nonnegative curvature the metric is a product outside a tubular neighborhood of the soul. In particular, if the boundaries of two such tubular neighborhoods are isometric they can be glued together to form a compact manifold of nonnegative curvature. It is this condition on the metrics of the boundaries which is difficult to achieve in general. Below we will describe a different somewhat general situation in which this can be done.

Above we saw that any homogeneous space, i.e., space of cohomogeneity 0, admits a metric of nonnegative curvature, and that those with positive curvature have been classified in the simply connected case at least. Guided by the classification program of the previous section it is natural to attempt to take the next step for this notion of large, i.e., to attempt to classify those manifolds of cohomogeneity one admitting metrics of nonnegative or positive curvature.

When M/G is one dimensional, it is either a circle S^1, or an interval, say $I = [-1, 1]$. In the first case all G orbits are principal and hence $\pi : M \to M/G$ is a bundle map. In this case it is easy to see that M admits a G-invariant metric with nonnegative curvature. Since, however, the fundamental group $\pi_1(M)$ of M is infinite it cannot carry a metric with positive curvature by the classical Bonnet-Myers theorem. In the second case there are precisely two non-principal G-orbits corresponding to the endpoints ± 1 of I, and M is decomposed as the union of two tubular neighborhoods of the non-principal orbits, with common boundary a principal orbit. Specifically, if $x_\pm \in M$ realize the distance between the non-principal orbits $B_\pm = \pi^{-1}(\pm 1)$ relative to a G-invariant Riemannian metric on M,then

$$M = D(B_-) \cup_E D(B_+) \,, \tag{3.2}$$

and by the slice theorem $D(B_\pm) = G \times_{K_\pm} D^{\ell_\pm+1}$, where $K_\pm = G_{x_\pm}$. Here $E = \pi^{-1}(0)$, the orbit $Gx_0 = G/H$ through the midpoint of a minimal geodesic c from x_- to x_+ is canonically identified with the boundaries $\partial D(B_\pm) = G \times_{K_\pm} S^{\ell_\pm}$, via the maps $G \to G \times S^{\ell_\pm}$, $g \to (g, \mp\dot{c}(\pm 1))$. Note also that $\partial D^{\ell_\pm+1} = S^{\ell_\pm} = K_\pm/H$. All in all we see that we can *reconstruct* M from G and the isotropy subgroups H and $K_\pm$.

In general, suppose G is a compact Lie group and $H \subset K_\pm \subset G$ are closed subgroups such that $K_\pm/H = S^{\ell_\pm}$ are spheres. It is well known (cf. [10, p.195]) that a transitive action of a compact Lie group K on a sphere S^ℓ is linear and is determined by its isotropy group $H \subset K$. Thus the diagram of inclusions

$$\begin{tikzcd}[column sep=small]
B_- = G/K_- & & G & & B_+ = G/K_+ \\
& K_- \arrow[ur, dash, "j_-"] & & K_+ \arrow[ul, dash, "j_+"'] & \\
S^{\ell_-} = K_-/H & & H \arrow[ul, dash, "i_-"] \arrow[ur, dash, "i_+"'] & & S^{\ell_+} = K_+/H
\end{tikzcd} \tag{3.3}$$

determines a manifold

$$M = G \times_{K_-} D^{\ell_-+1} \cup_{G/H} G \times_{K_+} D^{\ell_++1} \tag{3.4}$$

on which G acts by cohomogeneity one via the standard G action on $G \times_{K_\pm} D^{\ell_\pm+1}$ in the first coordinate. Thus the diagram (3.3) defines a cohomogeneity one manifold, and we will refer to it as a cohomogeneity one group diagram, which we sometimes denote by $H \subset \{K_-, K_+\} \subset G$. We also denote the common homomorphism $j_+ \circ i_+ = j_- \circ i_-$ by $j_0 \colon H \to G$.

The above description of cohomogeneity one manifolds lends itself to the following simple but crucial construction of *principal bundles* over cohomogeneity one manifolds in [40]:
Let L be any compact Lie group, and M any cohomogeneity one manifold with group diagram $H \subset \{K_-, K_+\} \subset G$. For any Lie group homomorphisms $\phi_\pm : K_\pm \to L$, $\phi_0 : H \to L$ with $\phi_+ \circ i_+ = \phi_- \circ i_- = \phi_0$, let P be the cohomogeneity one $L \times G$-manifold with diagram

$$\begin{tikzcd} & L \times G & \\ K_- \arrow[ur, dash, "(\phi_-{,}j_-)"] & & K_+ \arrow[ul, dash, "(\phi_+{,}j_+)"'] \\ & H \arrow[ul, dash, "i_-"] \arrow[ur, dash, "i_+"'] & \end{tikzcd} \tag{3.5}$$

Clearly the subaction of $L \times G$ by $L = L \times \{e\}$ on P is free, and $P/L = M$ since it has a cohomogeneity one description $H \subset \{K_-, K_+\} \subset G$. It is also apparent that the non-principal orbits in P have the same codimension as the non-principal orbits in M. In summary:

Proposition 3.6. *For every cohomogeneity one manifold M as in* (3.3) *and every choice of homomorphisms $\phi_\pm : K_\pm \to L$ with $\phi_+ \circ i_+ = \phi_- \circ i_-$, the diagram* (3.5) *defines a principal L bundle over M.*

In view of the special structure of cohomogeneity one manifolds described above, it is tempting to make the following

Conjecture. *Any cohomogeneity one manifold supports an invariant metric of nonnegative sectional curvature.*

To prove this in general is impossible with gluing techniques as described above. However, it is seems likely that combining such techniques with perturbations will yield metrics with *almost nonnegative curvature*. In [40] where the above conjecture was made the following partial case was settled.

Theorem 3.7. *Any cohomogeneity one manifold with codimension two singular orbits admits a nonnegatively curved invariant metric.*

This class of manifolds is much larger than one might first anticipate. The most striking immediate application of this is illustrated in

Theorem 3.8. *Each of the four (oriented) diffeomorphism types homotopy equivalent to $\mathbb{R}P^5$ support metrics with non-negative sectional curvature.*

In fact, each of these manifolds (cf. [43, 40]) support infinitely many cohomogeneity one actions with codimension two singular orbits descending from similar actions on S^5 discovered by Calabi.
More importantly, however, the above principal bundle construction yields many such examples including all principal SO(3) and SO(4) bundles over S^4 with its cohomogeneity one action given by the maximal subgroup SO(3) of SO(5). When this is combined with the basic submersion-construction, many associated bundles support such metrics as well. In particular this yields

Theorem 3.9. *The total space of every vector bundle and every sphere bundle over S^4 admits a complete metric of non-negative sectional curvature.*

In Cheeger and Gromoll [16] it was asked whether every vector bundle over S^n admits a metric of nonnegative curvature. This remains open for $n \geq 6$. The statement about sphere bundles is particularly interesting when the fiber is S^3, since many exotic 7-spheres admit such a description. In fact

Theorem 3.10. *Ten of the 14 exotic spheres in dimension 7 admit metrics of non-negative sectional curvature.*

In this formulation we have used the fact that in the Kervaire-Milnor group, $\mathbb{Z}_{28} = \mathrm{Diff}^+(S^6)/\mathrm{Diff}^+(D^7)$, of oriented diffeomorphism types of homotopy 7-spheres, a change of orientation corresponds to the inverse and hence the numbers 1 to 14 correspond to the distinct diffeomorphism types of exotic 7-spheres.

4. Emergence of Isometries

The simplest case where isometries appear naturally is in the context of deck transformations. For manifolds of negative or nonpositive curvature this is in fact the "bread and butter" of the subject. As in the previous sections we will confine our attention to the "opposite classical" subject of nonnegative or positive curvature.

We begin by illustrating *size restrictions* again. The following is due to Gromov:

Theorem 4.1. *The fundamental group of any nonnegatively curved n-manifold can be generated by a set of at most $c(n)$ elements.*

By the soul theorem it suffices to consider closed manifolds M. Now fix x in the universal cover of M and minimize over all sets of generators $\{g_1, \ldots, g_k\}$ of $\pi_1(M)$ the sum of displacements $\sum_{i=1,\ldots,k} \mathrm{dist}(x, g_i(x))$. If we join x to each $g_i(x)$ by a minimal geodesic c_i the claim is that $\angle(c_i, c_j) \geq \pi/3$ for $i \neq j$. If not the angle comparison theorem would imply that the sum would be smaller when replacing say g_i by $g_j g_i^{-1}$. In other words the open $\pi/6$-balls centered at $\dot{c}_i(0)$ in the unit tangent sphere S^{n-1} at x are disjoint. A simple volume arguments now completes the proof.

Another size restriction for the fundamental group $\pi_1(M)$ of a positively curved manifold M was proposed by Chern, who in [17] asked if any abelian subgroup of $\pi_1(M)$ is cyclic. A detailed analysis of the full isometry group of the examples in (3.1) (see [65]) reveals that there are counterexamples to this so-called Chern conjecture. In particular [64, 38],

Theorem 4.2. *$W^7_{1,1}$ admits a free isometric action by $\mathbb{Z}_2 \times \mathbb{Z}_2$, and when $3 \nmid kl(k+l)$ then $\mathbb{Z}_3 \times \mathbb{Z}_3$ acts freely and isometrically on $W^7_{k,l}$.*

The first example follows directly from Wilking's description $(\mathrm{SU}(3) \times \mathrm{SO}(3))/\,\mathrm{U}(2)$ of the Aloff-Wallach example $W^7_{1,1}$ as a normal homogeneous space (see [64]). Note that by (2.5) only cyclic subgroups of tori can act freely on a positively curved manifold supporting the conjecture. In retrospect, both cases stem from the fact that the isometry groups contain non-toral rank two abelian subgroups, although this by itself of course not is sufficient (see [38] and also [4]).

A more subtle situation where isometries emerge from the geometric setting is often exhibited in connection with *rigidity problems*. For positively curved manifolds this is illustrated very well in the

THEOREM 4.3 (Diameter Rigidity Theorem). *A manifold M with* $\sec M \geq 1$ *and* $\operatorname{diam} M = \pi/2$ *is either homeomorphic to a sphere or else isometric to a locally rank one symmetric space.*

When M in the above theorem is not simply connected it is isometric either to a unique $\mathbb{Z}_2$-quotient of an odd dimensional complex projective space, or else isometric to a spherical space form whose fundamental group representation is reducible (see [25]). In the simply connected case, the rigidity cases correspond to the following geometric description of M, which is derived via comparison geometry methods and critical point theory [25]: There is a point $x \in M$, such that the set of all points A at maximal distance $\pi/2$ from x is a totally geodesic smooth submanifold of M. Moreover, any geodesic of length $\pi/2$ emanating from x ends at A, and any geodesic of length $\pi/2$ starting perpendicularly to A ends at x. The map defined in this way from the unit tangent sphere at x onto A is a Riemannian submersion. The rigidity claim in the above theorem then follows from the following metric classification of Riemannian submersions from the euclidean sphere.

THEOREM 4.4 (Characterization of Hopf fibrations). *Any Riemannian submersion with domain a sphere of constant curvature one and with connected fibers is metrically equivalent to a Hopf fibration.*

Topologically it is well known that fibrations of spheres with connected fibers must have fiber dimension 1,3 or 7 [11]. Moreover the latter occurs only for fibrations of S^{15}. This case was settled very recently in Wilking [70] by means of an ingenious novel approach to Morse theory for the space of closed curves on the base. The final conclusion obtained in this way is that the fibers are all totally geodesic, in which case it is easy to see that it is a Hopf fibration. When the fibers have dimension 1 or 3 a different approach was used in [26]. It is in this approach where group actions emerge from the geometric constraints in the situation. Specifically, one proves more generally that any Riemannian foliation of a unit sphere with leaf dimension $k \leq 3$ is in fact the orbit foliation of an isometric almost free action of a k-dimensional Lie group. The group is constructed via so-called local *integrability vector fields* of the foliation, and these in turn are shown to form a Lie algebra of action (Killing) fields. This then leads to a complete classification via representation theory.

A similar situation arises for Riemannian fibrations of euclidean spaces. Also here groups of isometries emerge and a complete classification is achieved in [28].

In the remaining part of this section we will discuss problems for manifolds with bounded curvature, in particular with pinched positive curvature.

Recall the *differentiable sphere theorem* which asserts that a simply connected n-manifold M with $\delta \leq \sec M \leq 1$ is diffeomorphic to S^n if δ is sufficiently close to 1. The non-simply connected case is covered via deck transformations in [34] by

THEOREM 4.5 (Equivariant Sphere Theorem). *There is a* $0 < \delta < 1$ *such that for any simply connected n-manifold M with* $\delta \leq \sec M \leq 1$ *there is a representation* $\phi : \operatorname{Isom}(M) \to \mathrm{O}(n+1)$ *and a corresponding equivariant diffeomorphism* $F : M \to S^n$.

The "Gauss map" proof of the sphere theorem due to Ruh [60] yields in a natural way a map $\phi_0 : G \to \mathrm{O}(n)$, where $G = \operatorname{Isom}(M)$ is the isometry group of M. For each $g \in G$, the isometry $\phi_0(g) \in \mathrm{O}(n+1)$ is C^1-close to the induced diffeomorphism FgF^{-1} of S^n, but the map ϕ_0 is not a homomorphism, only "almost". The non-linear notion of *center of mass* developed in this context (cf.[33] and below) can now be used in two different ways to provide a proof of the above result: In the first step one constructs an actual Lie homomorphism $\phi : G \to \mathrm{O}(n+1)$ close to ϕ_0, and in the second step one shows that this isometric G-action on S^n is smoothly conjugate to the induced G-action via F. Both of these steps are general in the sense that a sufficiently *almost homomorphism* between any compact Lie groups can be perturbed to a homomorphism, and any two sufficiently C^1-close actions by a compact Lie group G on a compact manifold M are smoothly conjugate.

The idea behind the center of mass is that a continuous *almost constant* map $f : X \to M$ with domain a probability space (X, μ) is canonically close to a constant map $\mathcal{C}(f)$. In fact, the collection of almost constant maps may be viewed as a tubular neighborhood of the submanifold M interpreted as the set of constant maps in the Banach manifold $C^0(X, M)$. Specifically, if $f(X)$ is contained in a sufficiently small convex ball $B \subset M$, then the function $c : B \to \mathbb{R}$ defined by

$$c(p) = \int_X dist^2(p, f(x)) d\mu$$

is strictly convex on B and hence has a unique minimum denoted $\mathcal{C}(f)$. This construction has two vital properties: It is invariant under measure preserving maps of X, and equivariant with respect to the actions of $\mathrm{Isom}(M)$. In the special case where X consists of two points, the center of mass of $\{p_0, p_1\}$ is the midpoint of the unique minimal geodesic between p_0 and p_1 in M.

The two applications mentioned above come about as follows. That $\phi_0 : G \to H$ is an almost homomorphism means that for each $g \in G$ the map $x \mapsto \phi_0(gx)\phi_0(x)^{-1}$ is an almost constant map from G to H. If we define $\phi_1(g)$ as the center of mass of this map it turns out that $\phi_1 : G \to H$ is a "better" almost homomorphism, and with care iteration leads to the desired homomorphism ϕ. Similarly, the fact that two G-actions $\star_0$ and $\star_1$ on M are C^0-close implies that for each $p \in M$ the map $g \mapsto g \star_0 g^{-1} \star_1 p$ is close to the constant map p. If $S(p)$ is the center of mass of this map, and $\star_1$ acts by isometries, one proves that $S : M \to M$ is diffeomorphism which conjugates the two actions when they are sufficiently C^1-close.

Let us conclude our illustrations of how groups of isometries arise naturally with the phenomenon of *collapse*. First recall that according to Cheeger's finiteness theorem [13] the class of manifolds with say $|\sec M| \le 1, \operatorname{diam} M \le D$ and $\operatorname{vol} M \ge v$ contains at most finitely many diffeomorphism types. This is no longer true without the last condition on the volume. Geometrically, manifolds with bounded curvature and diameter, but with very small volume, will on a definite scale appear to be of lower dimension. In fact, a sequence of manifolds with bounded curvature and diameter, but with volume tending to zero, will have a subsequence which converges to a lower dimensional space relative to the so-called *Gromov-Hausdorff metric*. It is this phenomenon which is referred to as collapse. We will not elaborate further on this here, only describe the fact that when manifolds with definite curvature and diameter bounds have sufficiently small volume, then *additional structure emerges*.

In the context of bounded curvature there is a well developed theory for collapse due to Cheeger, Fukaya, and Gromov. This is anchored in Gromov's milestone theorem for *almost flat manifolds*, i.e., manifolds with bounded diameter and (arbitrary) small curvature bounds [29] (or equivalently having bounded curvature and (arbitrary) small diameter): Any such manifold is up to a finite cover a quotient of a nilpotent Lie group by a discrete subgroup.For the ultimate result see Ruh [61]. In general, the presence of nilpotent groups is immanent when collapse occurs with bounded curvature. In vague terms such collapse yield a decomposition of the manifold into submanifolds, a singular foliation, whose leaves in local covers are orbits by actions of nilpotent groups. Moreover, the collapse takes place along these infra-nilmanifolds (see [15]). Aside from the so-called Margulis lemma, which is behind the nilpotency properties, the center of mass described above provides a useful tool in piecing this structure together.

When the collapsed manifold M is simply connected, the structure above is much simpler to describe. In fact, in this case one has a global almost isometric and fixed point free torus action, which becomes isometric under a small perturbation of the metric (cf. [59]). This structure and additional Gromov-Hausdorff convergence techniques has recently been used to obtain the following remarkable analogue of Cheeger's finiteness result for two-connected manifolds with bounded curvature and diameter, but no restrictions on volume [52]:

THEOREM 4.6. *The class of simply connected closed Riemannian n-manifolds, M with finite* $\pi_2(M)$*,* $|\sec M| \le C$ *and* $\operatorname{diam} M \le D$ *contains at most finitely many diffeomorphism types.*

The center of mass conjugation theorem and a modification of it applied to the induced torus and principal group action on the principal bundle also plays a central role in the proof of the result. When combined with Gromov's Betti number theorem [30], one arrives at the following amazing result [52]:

THEOREM 4.7. *For each* n, C*, and* D*, there exist a finite number of manifolds* $M_1, \dots, M_{k(n,C,D)}$*, such that any simply connected n-manifold,* M *with* $|\sec M| \le C$ *and* $\operatorname{diam} M \le D$ *is diffeomorphic to a torus quotient of one of the* M_i*'s.*

If in (4.6) the lower curvature bound is positive, the same conclusion was obtained independently by Fang and Rong [21] using a totally different approach. In their approach the principal bundle is not used, but the maximal rank theorem (2.6) enters at an essential induction step. An extension of (4.6) has recently been carried out by Fang and Rong [22].

We would also like to point out that collapsing methods have been applied by Rong to investigate the structure of fundamental groups of positively curved manifolds (cf. e.g. [59]). His work actually provides some partial support for the original Chern conjecture discussed earlier in this section.

5. Open Problems

We would like to exhibit and discuss a list of problems some of which are directly related to the philosophy presented here, others only in spirit.

PROBLEM 5.1. *Determine the lowest dimensional positively curved manifolds on which* $\mathrm{E}_6, \mathrm{E}_7$ *or* E_8 *can act isometrically and (almost) effectively.*

For the other simple Lie groups the answer is known (cf. [37]) and corresponds to the real, complex, quaternionic or Cayley projective space of the lowest dimensional linear representation. All of these spaces are homogeneous in contrast to the answer to (5.1). Similarly, the same problem for *products* of the classical groups has not been analyzed, and might well lead to interesting examples.

PROBLEM 5.2. *Classify positively curved manifolds with fixed point cohomogeneity two.*

In this formulation, we are primarily thinking of those manifold where the G-action has nonempty fixed point set, i.e., we are looking for the next step of (2.8). However, in the case of empty fixed point set, this problem is equivalent to the following

PROBLEM 5.3. *Classify positively curved manifolds of cohomogeneity one.*

This was done in dimensions at most 6 in [62], and the answer is that only rank one symmetric spaces occur. For additional classification approaches above dimesion 6 see [54, 55]. In dimension 7 infinitely many of the Eschenburg spaces are of cohomogeneity one (cf. [39]). Of course cohomogeneity one manifolds will not in general have a homogeneous structure. It is worth pointing out, however, that a homogeneous manifold need not have a cohomogeneity one structure either.

As mentioned earlier, so far all known manifolds of positive curvature can be exhibited as biquotients, even as biquotients of Lie groups G with biinvariant metrics. To solve the above problem it is clear that new constructions are needed. On the other hand, it is natural to wonder how general the biquotient construction is. Some partial result about the following question can be found in [57].

PROBLEM 5.4. *Determine the structure of all Riemannian fibrations of G, where G is a Lie group equipped with a bi-invariant metric.*

If G is replaced by the constant curvature sphere the answer is known as we saw in section 4. When the group G is abelian, the work of Walschap and Gromoll in [69] and [28] should provide a complete answer as well. In analogy to the case of the Gromoll-Meyer sphere [27], which is an Sp(1) biquotient of Sp(2), the case where $G = \mathrm{Spin}(9)$ is particularly interesting. Potentially an exotic 15-sphere could be the base of a Riemannian fibration of $G = \mathrm{Spin}(9)$.

Recall that in general, if M is a compact Riemannian G-manifold, then there is a family of Riemannian metrics on M which collapse to M/G under a lower curvature bound [73]. Here, in analogy to earlier, collapse means that in the (Gromov-Hausdorff) limit the limiting object is of lower dimension. For manifolds with a lower curvature bound and an upper diameter bound this is equivalent to having volume going to zero. Obviously, manifolds which scale to a point with a lower curvature bound are precisely manifolds with nonnegative curvature. Manifolds which collapse to a point under a lower curvature bound are by definition manifolds with *almost nonnegative* curvature (Equivalently, such manifolds have metrics with lower curvature bound arbitrarily close to zero and bounded diameter). It is tempting to

CONJECTURE (Collapsing Conjecture). *Simply connected manifolds of nonnegative or more generally almost nonnegative curvature admit non-trivial collapse with a lower curvature bound.*

Here by non-trivial we mean that the collapse is not to a point. Note that all known simply connected examples with nonnegative curvature admit non-finite isometry groups. An affirmative answer to the following question would of course resolve the conjecture.

PROBLEM 5.5. *Do simply connected manifolds of nonnegative or more generally almost nonnegative curvature have positive symmetry degree?*

In this formulation, the degree of symmetry of M is the maximal dimension of any compact subgroup G of the diffeomorphism group of M. It should be pointed out that the classes of manifolds with positive, nonnegative or almost nonnegative curvature are strictly contained in each other. However, for simply connected manifolds no obstructions are known that can distinguish these classes!

In view of this discussion and a possible strategy outlined below, we propose to extend the so-called Bott conjecture (cf. [31] for a discussion of remarkable consequences) for manifolds of positive/ nonnegative curvature to include almost nonnegative curvature.

CONJECTURE (Ellipticity Conjecture). *All simply connected manifolds M of almost nonnegative curvature are rationally elliptic, i.e.,* $\dim \pi_*(M) \otimes Q < \infty$.

It is a simple fact in rational homotopy theory that rational ellipticity is equivalent to polynomial growth of the rational Betti numbers of the loop space. The original conjecture of Bott seemed to be based on Morse theory via the belief that on a positively curved manifold the number of geodesics between generic points should grow at most polynomially with the length (for a special case cf. [9]). The collapsing conjecture above could provide the initial step in a totally different approach to the Bott conjecture, where the extension suggested above is crucial. Suppose for instance that the collapsing conjecture could be solved by giving a positive answer to the following:

Problem 5.6. *Do simply connected manifolds of almost nonnegative curvature collapse to an interval with a lower curvature bound?*

It is plausible that manifolds with this property, in analogy to the case of cohomogeneity one manifolds, can be exhibited as a union of two discbundles, where their common sphere bundle is almost nonnegatively curved. If this were true, then the main result of [32] (the union of two disc bundles is elliptic if and only if their common sphere bundle is elliptic) in conjunction with induction by dimension would almost provide a proof of the ellipticity conjecture. At the same time it would provide a proof of the following

Conjecture (Double Soul Conjecture). *Any closed simply connected manifold of nonnegative (almost nonneagtive) curvature is the union of two discbundles.*

Note that the latter conjecture is independent of the ellipticity conjecture. It is easy to check that this is indeed true for all known simply connected examples of positively curved manifolds. To check it for all homogeneous spaces, or more generally for all biquotients would be very interesting.

The proper context for the discussion above concerning simply connected manifolds might be manifolds with nilpotent fundamantal group. We point out that the double soul conjecture is false for general non-simply connected manifolds. For example, it is not difficult to see that the Poincare homology 3-sphere S^3/I^* is not the union of two disc bundles. It is, however, the union of $S^1 \times D^2$ and a 3-manifold W with boundary $S^1 \times S^1$ and $H_*(X) \cong H_*(S^1)$. Also note that, according to the comprihensive treatment of 3-manifold collapse in [66], S^3/I^* does not collapse to a one dimensional space, but since it is a Seifert fiber bundle with three exceptional fibers it does collapses to an S^2 with three singular points. This collapse most likely cannot occur under a positive lower curvature bound in view of Hamilton's work [42], since the Ricci flow preserves isometries, and no linear action on S^3 induces the Seifert structure on S^3/I^*.

Let us close our discussion by recalling that except for the locally rank one symmetric spaces, no manifolds of positive curvature is known in dimensions above 24. One of the remarkable facts in Wilking's examples of almost positive curvature is indeed that they occur in infinitely many dimensions. The following is of obvious fundamental importance:

Problem 5.7. *Are there manifolds M^n other than locally rank one symmetric spaces with* $\sec M > 0$ *and* $n \to \infty$*?*

This question motivates the study of infinite dimensional manifolds with positive curvature.

Problem 5.8. *Are there nontrivial infinite dimensional manifolds with positive curvature?*

The precise notion of dimension and of curvature will depend on circumstances yet to be investigated, as well as on its potential impact on the previous problem.

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