# Semiconcave functions in Alexandrov's geometry

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**Abstract**

The following is a compilation of some techniques in Alexandrov's geometry which are directly connected to convexity.

## 0 Introduction

This paper is not about results, it is about available techniques in Alexandrov's geometry which are linked to semiconcave functions. We consider only spaces with lower curvature bound, but most techniques described here also work for upper curvature bound and even in more general settings.

Many proofs are omitted, I include only those which necessary for a continuous story and some easy ones. The proof of the existence of quasigeodesics is included in appendix A (otherwise it would never be published).

I did not bother with rewriting basics of Alexandrov's geometry but I did change notation, so it does not fit exactly in any introduction. I tried to make it possible to read starting from any place. As a result the dependence of statements is not linear, some results in the very beginning depend on those in the very end and vice versa (but there should not be any cycle).

Here is a list of available introductions to Alexandrov's geometry:

◇ [BGP] and its extension [Perelman 1991] is the first introduction to Alexandrov's geometry. I use it as the main reference.

  Some parts of it are not easy to read. In the English translation of [BGP] there were invented some militaristic terms, which no one ever used, mainly *burst point* should be *strained point* and *explosion* should be *collection of strainers*.

◇ [Shiohama] intoduction to Alexandrov's geometry, designed to be reader friendly.

◇ [Plaut 2002] A survey in Alexandrov's geometry written for topologists. The first 8 sections can be used as an introduction. The material covered in my paper is closely related to sections 7–10 of this survey.

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⋄ [BBI, Chapter 10] is yet an other reader friendly introduction.

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# Contents

## 0.1 Notation and conventions

- By $\mathit{Alex}^m(\kappa)$ we will denote the class of $m$-dimensional Alexandrov's spaces with curvature $\geqslant \kappa$. In this notation we may omit $\kappa$ and $m$, but if not stated otherwise we assume that dimension is finite.
- Gromov–Hausdorff convergence is understood with fixed sequence of approximations. I.e. once we write $X_n \xrightarrow{\mathrm{GH}} X$ that means that we fixed a sequence of Hausdorff approximations $f_n : X_n \to X$ (or equivalently $g_n : X \to X_n$).

  This makes possible to talk about limit points in $X$ for a sequence $x_n \in X_n$, limit of functions $f_n : X_n \to \mathbb{R}$, Hausdorff limit of subsets $S_n \subset X_n$ as well as weak limit of measures $\mu_n$ on $X_n$.
- regular fiber — see page 31
- $\measuredangle xyz$ — angle at $y$ in a geodesic triangle $\triangle xyz \subset A$
- $\measuredangle(\xi, \eta)$ — an angle between two directions $\xi, \eta \in \Sigma_p$
- $\tilde\measuredangle_\kappa xyz$ — a comparison angle, i.e. angle of the model triangle $\tilde\triangle xyz$ in $\text{Л}_\kappa$ at $y$.
- $\tilde\measuredangle_\kappa(a, b, c)$ — an angle opposite $b$ of a triangle in $\text{Л}_\kappa$ with sides $a, b$ and $c$. In case $a + b < c$ or $b + c < a$ we assume $\tilde\measuredangle_\kappa(a, b, c) = 0$.
- $\uparrow_p^q$ — a direction at $p$ of a minimazing geodesic from $p$ to $q$
- $\Uparrow_p^q$ — the set of all directions at $p$ of minimazing geodesics from $p$ to $q$
- $A$ — usually an Alexandrov's space
- argmax — see page 48
- $\partial A$ — boundary of $A$
- $\operatorname{dist}_x(y) = |xy|$ — distance between $x$ and $y$
- $d_p f$ — differential of $f$ at $p$, see page 6
- $\operatorname{gexp}_p$ — see section 3
- $\operatorname{gexp}_p(\kappa; v)$ — see section 3.2
- $\gamma^\pm$ — right/left tangent vector, see 2.1

- ◇ $\text{Л}_\kappa$ — model plane see page 5
- ◇ $\text{Л}_\kappa^+$ — model halfplane see page 21
- ◇ $\text{Л}_\kappa^m$ — model $m$-space see page 38
- ◇ $\log_p$ — see page 7
- ◇ $\nabla_p f$ — gradient of $f$ at $p$, see definition 1.3.2
- ◇ $\rho_\kappa$ — see page 5.
- ◇ $\Sigma(X)$ — the spherical suspension over $X$ see [BGP, 4.3.1], in [Plaut 2002, 89] and [Berestovskii] it is called *spherical cone*.
- ◇ $\sigma_\kappa$ — see footnote 15 on page 20.
- ◇ $T_p = T_p A$ — tangent cone at $p \in A$, see page 6.
- ◇ $T_p E$ — see page 28
- ◇ $\Sigma_p = \Sigma_p A$ — see footnote 4 on page 7.
- ◇ $\Sigma_p E$ — see page 28
- ◇ $f^\pm$ — see page 10

# 1 Semi-concave functions.

## 1.1 Definitions

**1.1.1. Definition for a space without boundary.** *Let $A \in \mathsf{Alex}$, $\partial A = \varnothing$ and $\Omega \subset A$ be an open subset.*

*A locally Lipschitz function $f\colon \Omega \to \mathbb{R}$ is called* $\lambda$-concave *if for any unit-speed geodesic $\gamma$ in $\Omega$, the function*

$$f \circ \gamma(t) - \lambda t^2/2$$

*is concave.*

If $A$ is an Alexandrov's space with non-empty boundary[1], then its doubling[2] $\tilde{A}$ is also an Alexandrov's space (see [Perelman 1991, 5.2]) and $\partial \tilde{A} = \varnothing$.

Set $\mathtt{p} : \tilde{A} \to A$ to be the canonical map.

**1.1.2. Definition for a space with boundary.** *Let $A \in \mathsf{Alex}$, $\partial A \neq \varnothing$ and $\Omega \subset A$ be an open subset.*

*A locally Lipschitz function $f\colon \Omega \to \mathbb{R}$ is called* $\lambda$-concave *if $f \circ \mathtt{p}$ is $\lambda$-concave in $\mathtt{p}^{-1}(\Omega) \subset \tilde{A}$.*

**Remark.** Note that the restriction of a linear function on $\mathbb{R}^n$ to a ball is not 0-concave in this sense.

## 1.2 Variations of definition.

A function $f : A \to \mathbb{R}$ is called *semiconcave* if for any point $x \in A$ there is a neighborhood $\Omega_x \ni x$ and $\lambda \in \mathbb{R}$ such that the restriction $f|_{\Omega_x}$ is $\lambda$-concave.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function. A function $f : A \to \mathbb{R}$ is called $\varphi(f)$*-concave* if for any point $x \in A$ and any $\varepsilon > 0$ there is a neighborhood $\Omega_x \ni x$ such that $f|_{\Omega_x}$ is $(\varphi \circ f(x) + \varepsilon)$-concave

For the Alexandrov's spaces with curvature $\geqslant \kappa$, it is natural to consider the class of $(1 - \kappa f)$-concave functions. The advantage of such functions comes from the fact that on the model space[3] $Л_\kappa$, one can construct model $(1-\kappa f)$-concave functions which are equally concave in all directions at any fixed point. The most important example of $(1-\kappa f)$-concave function is $\rho_\kappa \circ \mathrm{dist}_x$, where $\mathrm{dist}_x(y) = |xy|$ denotes distance function from $x$ to $y$ and

$$\rho_\kappa(x) = \left[\begin{array}{ccc} \frac{1}{\kappa}(1-\cos(x\sqrt{\kappa})) & \text{if} & \kappa > 0 \\ x^2/2 & \text{if} & \kappa = 0 \\ \frac{1}{\kappa}(\mathrm{ch}(x\sqrt{-\kappa}) - 1) & \text{if} & k < 0 \end{array}\right.$$

In the above definition of $\lambda$-concave function one can exchange Lipschitz continuity for usual continuity. Then it will define the same set of functions, see corollary 3.3.2.

[1] Boundary of Alexandrov's space is defined in [BGP, 7.19].

[2] i.e. two copies of $A$ glued along their boundaries.

[3] i.e. the simply connected 2-manifold of constant curvature $\kappa$ (the Russian Л is for Lobachevsky)

## 1.3 Differential

Given a point $p$ in an Alexandrov's space $A$, we denote by $T_p = T_pA$ the tangent cone at $p$.

For an Alexandrov's space, the tangent cone can be defined in two equivalent ways (see [BGP, 7.8.1]):

- ◇ As a cone over space of directions at a point and
- ◇ As a limit of rescalings of the Alexandrov's space, i.e.:

  Given $s > 0$, we denote the space $(A, s \cdot d)$ by $sA$, where $d$ denotes the metric of an Alexandrov's space $A$, i.e. $A = (A, d)$. Let $i_s : sA \to A$ be the canonical map. The limit of $(sA, p)$ for $s \to \infty$ is the tangent cone $(T_p, o_p)$ at $p$ with marked origin $o_p$.

**1.3.1. Definition.** *Let $A \in \mathsf{Alex}$ and $\Omega \subset A$ be an open subset.*

*For any function $f : \Omega \to \mathbb{R}$ the function $d_pf \colon T_p \to \mathbb{R}$, $p \in \Omega$ defined by*

$$d_pf = \lim_{s\to\infty} s(f \circ i_s - f(p)), \quad f \circ i_s : sA \to \mathbb{R}$$

*is called the* differential *of $f$ at $p$.*

It is easy to see that the differential $d_pf$ is well defined for any semiconcave function $f$. Moreover, $d_pf$ is a concave function on the tangent cone $T_p$ which is positively homogeneous, i.e. $d_pf(r \cdot v) = r \cdot d_pf(v)$ for $r \geqslant 0$.

**Gradient.** With a slight abuse of notation, we will call elements of the tangent cone $T_p$ the "tangent vectors" at $p$. The origin $o = o_p$ of $T_p$ plays the role of a "zero vector". For a tangent vector $v$ at $p$ we define its absolute value $|v|$ as the distance $|ov|$ in $T_p$. For two tangent vectors $u$ and $v$ at $p$ we can define their "scalar product"

$$\langle u, v\rangle \stackrel{\text{def}}{=} (|u|^2 + |v|^2 - |uv|^2)/2 = |u| \cdot |v| \cos\alpha,$$

where $\alpha = \measuredangle uov = \tilde{\measuredangle}_0 uov$ in $T_p$.

It is easy to see that for any $u \in T_p$, the function $x \mapsto -\langle u, x\rangle$ on $T_p$ is concave.

**1.3.2. Definition.** *Let $A \in \mathsf{Alex}$ and $\Omega \subset A$ be an open subset. Given a $\lambda$-concave function $f : \Omega \to \mathbb{R}$, a vector $g \in T_p$ is called a* gradient *of $f$ at $p \in \Omega$ (in short: $g = \nabla_p f$ ) if*

*(i) $d_pf(x) \leqslant \langle g, x\rangle$ for any $x \in T_p$, and*

*(ii) $d_pf(g) = \langle g, g\rangle$.*

It is easy to see that any $\lambda$-concave function $f : \Omega \to \mathbb{R}$ has a uniquely defined gradient vector field. Moreover, if $d_pf(x) \leqslant 0$ for all $x \in T_p$, then $\nabla_p f = o_p$; otherwise,

$$\nabla_p f = d_pf(\xi_{\max}) \cdot \xi_{\max}$$

where $\xi_{\max} \in \Sigma_p$[4] is the (necessarily unique) unit vector for which the function $d_p f$ attains its maximum.

For two points $p, q \in A$ we denote by $\uparrow_p^q \in \Sigma_p$ a direction of a minimizing geodesic from $p$ to $q$. Set $\log_p q = |pq| \cdot \uparrow_p^q \in T_p$. In general, $\uparrow_p^q$ and $\log_p q$ are not uniquely defined.

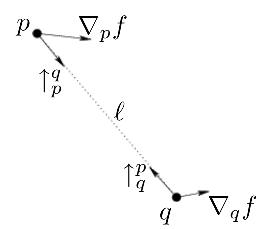
The following inequalities describe an important property of the "gradient vector field" which will be used throughout this paper.

**1.3.3. Lemma.** *Let $A \in \mathrm{Alex}$ and $\Omega \subset A$ be an open subset, $f : \Omega \to \mathbb{R}$ be a $\lambda$-concave function. Assume all minimizing geodesics between $p$ and $q$ belong to $\Omega$, set $\ell = |pq|$. Then*

$$\langle \uparrow_p^q, \nabla_p f \rangle \geqslant \{f(q) - f(p) - \frac{\lambda}{2}\ell^2\}/\ell,$$

*and in particular*

$$\langle \uparrow_p^q, \nabla_p f \rangle + \langle \uparrow_q^p, \nabla_q f \rangle \geqslant -\lambda \ell.$$




*Proof.* Let $\gamma : [0, \ell] \to \Omega$ be a unit-speed minimizing geodesic from $p$ to $q$, so

$$\gamma(0) = p, \quad \gamma(\ell) = q, \quad \gamma^+(0) = \uparrow_p^q.$$

From definition 1.3.2 and the $\lambda$-concavity of $f$ we get

$$\langle \uparrow_p^q, \nabla_p f \rangle = \langle \gamma^+(0), \nabla_p f \rangle \geqslant d_p f(\gamma^+(0)) =$$
$$= (f \circ \gamma)^+(0) \geqslant \frac{f \circ \gamma(\ell) - f \circ \gamma(0) - \lambda \ell^2/2}{\ell}$$

and the first inequality follows (for definition of $\gamma^+$ and $(f \circ \gamma)^+$ see 2.1).

The second inequality is just a sum of two of the first type. $\square$

**1.3.4. Lemma.** *Let $A_n \xrightarrow{\mathrm{GH}} A$, $A_n \in \mathrm{Alex}^m(\kappa)$.*

*Let $f_n : A_n \to \mathbb{R}$ be a sequence of $\lambda$-concave functions and $f_n \to f : A \to \mathbb{R}$.*

*Let $x_n \in A_n$ and $x_n \to x \in A$.*

*Then*

$$|\nabla_x f| \leqslant \liminf_{n \to \infty} |\nabla_{x_n} f_n|.$$

In particular we have lower-semicontinuity of the function $x \mapsto |\nabla_x f|$:

**1.3.5. Corollary.** *Let $A \in \mathrm{Alex}$ and $\Omega \subset A$ be an open subset.*

*If $f : \Omega \to \mathbb{R}$ is a semiconcave function then the function*

$$x \mapsto |\nabla_x f|$$

*is lower-semicontinuos, i.e. for any sequence $x_n \to x \in \Omega$, we have*

$$|\nabla_x f| \leqslant \liminf_{n \to \infty} |\nabla_{x_n} f|.$$

[4] By $\Sigma_p \subset T_p$ we denote the set of unit vectors, which we also call directions at $p$. The space $(\Sigma_p, \measuredangle)$ with angle metric is an Alexandrov's space with curvature $\geqslant 1$. $(\Sigma_p, \measuredangle)$ it is also path-isometric to the subset $\Sigma_p \subset T_p$.

*Proof of lemma 1.3.4.* Fix an $\varepsilon > 0$ and choose $q$ near $p$ such that

$$\frac{f(q) - f(p)}{|pq|} > |\nabla_p f| - \varepsilon.$$

Now choose $q_n \in A_n$ such that $q_n \to q$. If $|pq|$ is sufficiently small and $n$ is sufficiently large, the $\lambda$-concavity of $f_n$ then implies that

$$\liminf_{n\to\infty} d_{p_n} f_n(\uparrow_{p_n}^{q_n}) \geqslant |\nabla_p f| - 2\varepsilon.$$

Hence,

$$\liminf_{n\to\infty} |\nabla_{p_n} f_n| \geqslant |\nabla_p f| - 2\varepsilon \quad \text{for any} \quad \varepsilon > 0$$

and therefore

$$\liminf_{n\to\infty} |\nabla_{p_n} f_n| \geqslant |\nabla_p f|.$$

□

**Supporting and polar vectors.**

**1.3.6. Definition.** *Assume $A \in \mathrm{Alex}$ and $\Omega \subset A$ is an open subset, $p \in \Omega$, let $f : \Omega \to \mathbb{R}$ be a semiconcave function.*

*A vector $s \in T_p$ is called a* supporting vector *of $f$ at $p$ if*

$$d_p f(x) \leqslant -\langle s, x\rangle \quad \textit{for any} \quad x \in T_p$$

The set of supporting vectors is not empty, i.e.

**1.3.7. Lemma.** *Assume $A \in \mathrm{Alex}$ and $\Omega \subset A$ is an open subset, $f : \Omega \to \mathbb{R}$ is a semiconcave function, $p \in \Omega$. Then set of supporting vectors of $f$ at $p$ form a non-empty convex subset of $T_p$.*

*Proof.* Convexity of the set of supporting vectors follows from concavity of the function $x \to -\langle u, x\rangle$ on $T_p$. To show existence, consider a minimum point $\xi_{\min} \in \Sigma_p$ of the function $d_p f|_{\Sigma_p}$. We will show that the vector

$$s = [-d_p f(\xi_{\min})] \cdot \xi_{\min}$$

is a supporting vector for $f$ at $p$. Assume that we know the existence of supporting vectors in dimension $< m$. Applying it to $d_p f|_{\Sigma_p}$ at $\xi_{\min}$, we get $d_{\xi_{\min}}(d_p f|_{\Sigma_p}) \equiv 0$. Therefore, since $d_p f|_{\Sigma_p}$ is $(-d_p f)$-concave (see section 1.2) for any $\eta \in \Sigma_p$ we have

$$d_p f(\eta) \leqslant d_p f(\xi_{\min}) \cdot \cos \measuredangle(\xi_{\min}, \eta)$$

hence the result. □

In particular, it follows that if the space of directions $\Sigma_p$ has a diameter[5] $\leqslant \pi/2$ then $\nabla_p f = o$ for any $\lambda$-concave function $f$.

Clearly, for any vector $s$, supporting $f$ at $p$ we have

$$|s| \geqslant |\nabla_p f|.$$

**1.3.8. Definition.** *Two vectors $u, v \in T_p$ are called* polar *if for any vector $x \in T_p$ we have*

$$\langle u, x\rangle + \langle v, x\rangle \geqslant 0.$$

*More generally, a vector $u \in T_p$ is called polar to a set of vectors $\mathcal{V} \subset T_p$ if*

$$\langle u, x\rangle + \sup_{v\in\mathcal{V}} \langle v, x\rangle \geqslant 0.$$

Note that if $u, v \in T_p$ are polar to each other then

$$d_p f(u) + d_p f(v) \leqslant 0. \tag{$*$}$$

Indeed, if $s$ is a supporting vector then

$$d_p f(u) + d_p f(v) \leqslant -\langle s, u\rangle - \langle s, v\rangle \leqslant 0.$$

Similarly, if $u$ is polar to a set $\mathcal{V}$ then

$$d_p f(u) + \inf_{v\in\mathcal{V}} d_p f(v) \leqslant 0. \tag{$**$}$$

*Examples of pairs of polar vectors.*

(i) If two vectors $u, v \in T_p$ are *antipodal*, i.e. $|u| = |v|$ and $\measuredangle u o_p v = \pi$ then they are polar to each other.

In general, if $|u| = |v|$ then they are polar if and only if for any $x \in T_p$ we have $\measuredangle u o_p x + \measuredangle x o_p v \leqslant \pi$.

(ii) If $\uparrow_q^p$ is uniquely defined then $\uparrow_q^p$ is polar to $\nabla_q \operatorname{dist}_p$.

More generally, if $\Uparrow_p^q \subset \Sigma_p$ denotes the set of all directions from $p$ to $q$ then $\nabla_q \operatorname{dist}_p$ is polar to the set $\Uparrow_q^p$.

Both statement follow from the identity

$$d_q(v) = \min_{\xi\in\Uparrow_q^p} -\langle \xi, v\rangle$$

and the definition of gradient (see 1.3.2).

Given a vector $v \in T_p$, applying above property (ii) to the function $\operatorname{dist}_v : T_p \to \mathbb{R}$ we get that $\nabla_o f_v$ is polar to $\uparrow_o^v$. Since there is a natural isometry $T_o T_p \to T_p$ we have

**1.3.9. Lemma.** *Given any vector $v \in T_p$ there is a polar vector $v^* \in T_p$. Moreover, one can assume that $|v^*| \leqslant |v|$*

In A.3.2 using quasigeodesics we will show that in fact one can assume $|v^*| = |v|$

[5] We always consider $\Sigma_p$ with angle metric.

# 2 Gradient curves.

The technique of gradient curves was influenced by Sharafutdinov's retraction, see [Sharafutdinov]. These curves were designed to simplify Perelman's proof of existence of quasigeodesics. However, it turned out that gradient curves themselves provide a superior tool, which is in fact almost universal in Alexandrov's geometry. Unlike most of Alexandrov's techniques, gradient curves work equally well for infinitely dimensional Alexandrov's spaces (the proof requires some quasifications, but essentially is the same), for spaces with curvature bounded above and for locally compact spaces with well defined tangent cone at each point, see [Lytchak]. It was pointed out to me that some traces of these properties can be found even in general metric spaces see [AGS].

## 2.1 Definition and main properties

Given a curve $\gamma(t)$ in an Alexandrov's space $A$, we denote by $\gamma^+(t)$ the right, and by $\gamma^-(t)$ the left, tangent vectors to $\gamma(t)$, where, respectively,

$$\gamma^\pm(t) \in T_{\gamma(t)}, \quad \gamma^\pm(t) = \lim_{\varepsilon\to 0+} \frac{\log_{\gamma(t)} \gamma(t\pm\varepsilon)}{\varepsilon}.$$

This sign convention is not quite standard; in particular, for a function $f:\mathbb{R}\to\mathbb{R}$, its right derivative is equal to $f^+$ and its left derivative is equal to $-f^-(t)$. For example

$$\text{if} \;\; f(t)=t \;\; \text{then} \;\; f^+(t)\equiv 1 \;\; \text{and} \;\; f^-(t)\equiv -1.$$

**2.1.1. Definition.** *Let $A \in \mathsf{Alex}$ and $f:A\to\mathbb{R}$ be a semiconcave function.*
*A curve $\alpha(t)$ is called* $f$-gradient curve *if for any $t$*

$$\alpha^+(t) = \nabla_{\alpha(t)} f.$$

**2.1.2. Proposition.** *Given a $\lambda$-concave function $f$ on an Alexandrov's space $A$ and a point $p\in A$ there is a unique gradient curve $\alpha\colon[0,\infty)\to A$ such that $\alpha(0)=p$.*

The gradient curve can be constructed as a limit of broken geodesics, made up of short segments with directions close to the gradient. Convergence, uniqueness, follow from lemma 1.3.3, while corollary 1.3.5 guarantees that the limit is indeed a gradient curve.

**Distance estimates.**

**2.1.3. Lemma.** *Let $A \in \mathsf{Alex}$ and $f:A\to\mathbb{R}$ be a $\lambda$-concave function and $\alpha(t)$ be an $f$-gradient curve.*
*Assume $\bar\alpha(s)$ is the reparametrization of $\alpha(t)$ by arclength. Then $f\circ\bar\alpha$ is $\lambda$-concave.*

*Proof.* For $s > s_0$,

$$(f \circ \bar\alpha)^+(s_0) = |\nabla_{\bar\alpha(s_0)} f| \geqslant d_{\bar\alpha(s_0)} f \left(\uparrow_{\bar\alpha(s_0)}^{\bar\alpha(s)}\right) \geqslant$$

$$\geqslant \frac{f(\bar\alpha(s)) - f(\bar\alpha(s_0)) - \lambda |\bar\alpha(s)\,\bar\alpha(s_0)|^2/2}{|\bar\alpha(s)\,\bar\alpha(s_0)|}.$$

Therefore, since $s - s_0 \geqslant |\bar\alpha(s)\,\bar\alpha(s_0)| = s - s_0 - o(s - s_0)$, we have

$$(f \circ \bar\alpha)^+(s_0) \geqslant \frac{f(\bar\alpha(s)) - f(\bar\alpha(s_0)) - \lambda(s - s_0)^2/2}{s - s_0} + o(s - s_0)$$

i.e. $f \circ \bar\alpha$ is $\lambda$-concave. □

The following lemma states that there is a nice parametrization of a gradient curve (by $\vartheta_\lambda$) which makes them behave as a geodesic in some respects.

**2.1.4. Lemma.** *Let $A \in \mathcal{A}lex$, $f : A \to \mathbb{R}$ be a $\lambda$-concave function and $\alpha, \beta : [0, \infty) \to A$ be two $f$-gradient curves with $\alpha(0) = p$, $\beta(0) = q$.*
*Then*

*(i) for any $t \geqslant 0$,*

$$|\alpha(t)\beta(t)| \leqslant e^{\lambda t} |pq|$$

*(ii) for any $t \geqslant 0$,*

$$|\alpha(t)q|^2 \leqslant |pq|^2 + \left\{2f(p) - 2f(q) + \lambda|pq|^2\right\} \cdot \vartheta_\lambda(t) + |\nabla_p f|^2 \cdot \vartheta_\lambda^2(t),$$

*where*

$$\vartheta_\lambda(t) = \int_0^t e^{\lambda t} dt = \left[\begin{array}{ll} t & \text{if} \quad \lambda = 0 \\ \frac{e^{\lambda t} - 1}{\lambda} & \text{if} \quad \lambda \neq 0 \end{array}\right.$$

*(iii) if $t_p \geqslant t_q \geqslant 0$ then*

$$\begin{aligned} |\alpha(t_p)\beta(t_q)|^2 \leqslant e^{2\lambda t_q} \big[ &|pq|^2 + \\ &+ \left\{2f(p) - 2f(q) + \lambda|pq|^2\right\} \cdot \vartheta_\lambda(t_p - t_q) + \\ &+ |\nabla_p f|^2 \cdot \vartheta_\lambda^2(t_p - t_q) \big]. \end{aligned}$$

In case $\lambda > 0$, this lemma can also be reformulated in a geometer-friendly way:

**2.1.4′. Lemma.** *Let $\alpha$, $\beta$, $p$ and $q$ be as in lemma 2.1.4 and $\lambda > 0$. Consider points $\tilde o, \tilde p, \tilde q \subset \mathbb{R}^2$ defined by the following:*

$$|\tilde p \tilde q| = |pq|, \quad \lambda |\tilde o \tilde p| = |\nabla_p f|,$$

$$\frac{\lambda}{2}\left(|\tilde o \tilde q|^2 - |\tilde o \tilde p|^2\right) = f(q) - f(p)$$

*Let $\tilde\alpha(t)$ and $\tilde\beta(t)$ be $\left(\frac{\lambda}{2}\operatorname{dist}_{\tilde o}^2\right)$-gradient curves in $\mathbb{R}^2$ with $\tilde\alpha(0) = \tilde p$, $\tilde\beta(0) = \tilde q$. Then,* *(i)* $|\alpha(t)q| \leqslant |\tilde{\alpha}(t)\tilde{q}|$*. for any* $t > 0$

*(ii)* $|\alpha(t)\beta(t)| \leqslant |\tilde{\alpha}(t)\tilde{\beta}(t)|$

*(iii) if* $t_p \geqslant t_q$ *then* $|\alpha(t_p)\beta(t_q)| \leqslant |\tilde{\alpha}(t_p)\tilde{\beta}(t_q)|$

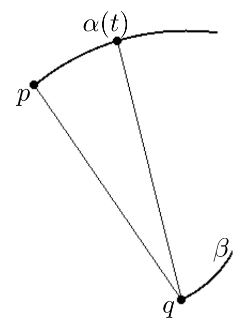
*Proof.* (ii). If $\lambda = 0$, from lemma 2.1.3 it follows that[6]

$$f \circ \alpha(t) - f \circ \alpha(0) \leqslant \left|\nabla_{\bar{\alpha}(0)} f\right|^2 \cdot t.$$

Therefore from lemma 1.3.3, setting $\ell = \ell(t) = |q\alpha(t)|$, we get[7]

$$\left(\ell^2/2\right)' \leqslant f(p) - f(q) + |\nabla_p f|^2 \cdot t,$$

hence the result.
(i) follows from the second inequality in lemma 1.3.3;
(iii) follows from (i) and (ii). □




**Passage to the limit.** The next lemma states that gradient curves behave nicely with Gromov–Hausdorff convergence, i.e. a limit of gradient curves is a gradient curve for the limit function.

**2.1.5. Lemma.** *Let* $A_n \xrightarrow{\text{GH}} A$, $A_n \in \mathrm{Alex}^m(\kappa)$, $A_n \ni p_n \to p \in A$.
*Let* $f_n : A_n \to \mathbb{R}$ *be a sequence of* $\lambda$*-concave functions and* $f_n \to f : A \to \mathbb{R}$.
*Let* $\alpha_n : [0, \infty) \to A_n$ *be the sequence of* $f_n$*-gradient curves with* $\alpha_n(0) = p_n$ *and let* $\alpha : [0, \infty) \to A$ *be the* $f$*-gradient curve with* $\alpha(0) = p$.
*Then* $\alpha_n \to \alpha$ *as* $n \to \infty$.

*Proof.* Let $\bar{\alpha}_n(s)$ denote the reparametrization of $\alpha_n(t)$ by arc length. Since all $\bar{\alpha}_n$ are 1-Lipschitz, we can choose a partial limit, say $\bar{\alpha}(s)$ in $A$. Note that we may assume that $f$ has no critical points and so $d(f \circ \bar{\alpha}) \neq 0$. Otherwise consider instead the sequence $A'_n = A_n \times \mathbb{R}$ with $f'_n(a \times x) = f_n(a) + x$.

Clearly, $\bar{\alpha}$ is also 1-Lipschitz and hence, by Lemma 1.3.4,

$$\lim_{n\to\infty} f_n \circ \bar{\alpha}_n|_a^b = \lim_{n\to\infty} \int_a^b |\nabla_{\bar{\alpha}_n(s)} f_n| ds \geqslant$$

$$\geqslant \int_a^b |\nabla_{\bar{\alpha}(s)} f| ds \geqslant \int_a^b d_{\bar{\alpha}(s)} f(\bar{\alpha}^+(s)) ds = f \circ \bar{\alpha}|_a^b,$$

where $\bar{\alpha}^+(s)$ denotes any partial limit of $\log_{\bar{\alpha}(s)} \bar{\alpha}(s+\varepsilon)/\varepsilon$, $\varepsilon \to 0+$.

---

[6]For $\lambda \neq 0$ it will be $f \circ \alpha(t) - f \circ \alpha(0) \leqslant \left|\nabla_{\bar{\alpha}(0)} f\right|^2 \cdot [\vartheta_\lambda(t) + \lambda\vartheta_\lambda^2(t)/2]$.
[7]For $\lambda \neq 0$ it will be $\left(\ell^2/2\right)' - \lambda\ell^2/2 \leqslant f(p) - f(q) + |\nabla_p f|^2 \cdot [\vartheta_\lambda(t) + \lambda\vartheta_\lambda^2(t)/2]$.

On the other hand, since $\bar\alpha_n \to \bar\alpha$ and $f_n \to f$ we have $f_n\circ\bar\alpha_n|_a^b \to f\circ\bar\alpha|_a^b$, i.e. equality holds in both of these inequalities. Hence

$$|\nabla_{\bar\alpha(s)} f| = \lim_{n\to\infty} |\nabla_{\bar\alpha_n(s)} f_n|, \quad |\bar\alpha^+(s)| = 1 \quad \text{a.e.}$$

and the directions of $\bar\alpha^+(s)$ and $\nabla_{\bar\alpha(s)} f$ coincide almost everywhere.

This implies that $\bar\alpha(s)$ is a gradient curve reparametrized by arc length. It only remains to show that the original parameter $t_n(s)$ of $\alpha_n$ converges to the original parameter $t(s)$ of $\alpha$.

Notice that $|\nabla_{\bar\alpha_n(s)} f_n| dt_n = ds$ or $dt_n/ds = ds/d(f_n\circ\bar\alpha_n)$. Likewise, $dt/ds = ds/d(f\circ\bar\alpha)$. Then the convergence $t_n \to t$ follows from the $\lambda$-concavity of $f_n\circ\bar\alpha_n$ (see Lemma 2.1.3) and the convergence $f_n\circ\bar\alpha_n \to f\circ\bar\alpha$. $\square$

## 2.2 Gradient flow

Let $f$ be a semi-concave function on an Alexandrov's space $A$. We define the $f$-*gradient flow* to be the one parameter family of maps

$$\Phi_f^t : A \to A, \quad \Phi_f^t(p) = \alpha_p(t),$$

where $t \geqslant 0$ and $\alpha_p : [0,\infty) \to A$ is the $f$-gradient curve which starts at $p$ (i.e. $\alpha_p(0) = p$). [8] Obviously

$$\Phi_f^{t+\tau} = \Phi_f^t \circ \Phi_f^\tau.$$

This map has the following main properties:

1. $\Phi_f^t$ is locally Lipschitz (in the domain of definition). Moreover, if $f$ is $\lambda$-concave then it is $e^{\lambda t}$-Lipschitz.

   This follows from lemma 2.1.4(i).

2. Gradient flow is stable under Gromov–Hausdorff convergence, namely:

   If $A_n \in \mathit{Alex}^m(\kappa)$, $A_n \xrightarrow{\mathrm{GH}} A$, $f_n : A_n \to \mathbb{R}$ is a sequence of $\lambda$-concave functions which converges to $f : A \to \mathbb{R}$ then $\Phi_{f_n}^t : A_n \to A_n$ converges pointwise to $\Phi_f^t : A \to A$.

   This follows from lemma 2.1.5.

3. For any $x \in A$ and all sufficiently small $t \geqslant 0$, there is $y \in A$ so that $\Phi_f^t(y) = x$.

   For spaces without boundary this follows from [Grove–Petersen 1993, lemma 1]. For spaces with boundary one should consider its doubling.

[8] In general the domain of definition of $\Phi_f^t$ can be smaller than $A$, but it is defined on all $A$ for a reasonable type of function, say for $\lambda$-concave and for $(1-\kappa f)$-concave functions.

Gradient flow can be used to deform a mapping with target in $A$. For example, if $X$ is a metric space, then given a Lipschitz map $F : X \to A$ and a positive Lipschitz function $\tau : X \to \mathbb{R}_+$ one can consider the map $F'$ called *gradient deformation* of $F$ which is defined by

$$F'(x) = \Phi_f^{\tau(x)} \circ F(x), \quad F' : X \to A.$$

From lemma 2.1.4 it is easy to see that the *dilation*[9] of $F'$ can be estimated in terms of $\lambda$, $\sup_x \tau(x)$, dilation of $F$ and the Lipschitz constants of $f$ and $\tau$.

Here is an optimal estimate for the length element of a curve which follows from lemma 2.1.4:

**2.2.1. Lemma.** *Let $A \in \mathrm{Alex}$. Let $\gamma_0(s)$ be a curve in $A$ parametrized by arc-length, $f : A \to \mathbb{R}$ be a $\lambda$-concave function, and $\tau(s)$ be a non-negative Lipschitz function. Consider the curve*

$$\gamma_1(s) = \Phi_f^{\tau(s)} \circ \gamma_0(s).$$

*If $\sigma = \sigma(s)$ is its arc-length parameter then*

$$d\sigma^2 \leqslant e^{2\lambda\tau} \left[ ds^2 + 2d(f \circ \gamma_0)d\tau + |\nabla_{\gamma_0(s)} f|^2 d\tau^2 \right]$$

## 2.3 Applications

Gradient flow gives a simple proof to the following result which generalizes a key lemma in [Liberman]. This generalization was first obtained in [Perelman–Petrunin 1993, 5.3], a simplified proof was given in [Petrunin 1997, 1.1]. See sections 4 and 5 for definition of extremal subset and quasigeodesic.

**2.3.1. Generalized Lieberman's Lemma.** *Any unit-speed geodesic for the induced intrinsic metric on an extremal subset is a quasigeodesic in the ambient Alexandrov's space.*

*Proof.* Let $\gamma : [a, b] \to E$ be a unit-speed minimizing geodesic in an extremal subset $E \subset A$ and $f$ be a $\lambda$-concave function defined in a neighborhood of $\gamma$. Assume $f \circ \gamma$ is not $\lambda$-concave, then there is a non-negative Lipschitz function $\tau$ with support in $(a, b)$ such that

$$\int_a^b [(f \circ \gamma)'\tau' + \lambda\tau]\, ds < 0$$

Then as follows from lemma 2.2.1, for small $t \geqslant 0$

$$\gamma_t(s) = \Phi_f^{t \cdot \tau(s)} \circ \gamma_0(s)$$

[9] i.e. its optimal Lipschitz constant.

gives a length-contracting homotopy of curves relative to ends and according to definition 4.1.1, it stays in $E$ — this is a contradiction. $\square$

The fact that gradient flow is stable with respect to collapsing has the following useful consequence: Let $M_n$ be a collapsing sequence of Riemannian manifolds with curvature $\geqslant \kappa$ and $M_n \xrightarrow{\text{GH}} A$. For a regular point $p$ let us denote by $F_n(p)$ the *regular fiber*[10] over $p$, it is well defined for all large $n$. Let $f : A \to \mathbb{R}$ be a $\lambda$-concave function. If $\alpha(t)$ is an $f$-gradient curve in $A$ which passes only through regular points, then for any $t_0 < t_1$ there is a homotopy equivalence $F_n(\alpha(t_0)) \to F_n(\alpha(t_1))$ with dilation $\approx e^{\lambda(t_1-t_0)}$.

This observation was used in [KPT] to prove some properties of almost nonnegatively curved manifolds. In particular, it gave simplified proofs of the results in [Fukaya–Yamaguchi]):

**2.3.2. Nilpotency theorem.** *Let $M$ be a closed almost nonnegatively curved manifold. Then a finite cover of $M$ is a* nilpotent space*, i.e. its fundamental group is nilpotent and it acts nilpotently on higher homotopy groups.*

**2.3.3. Theorem.** *Let $M$ be an almost nonnegatively curved $m$-manifold. Then $\pi_1(M)$ is* $\mathrm{Const}(m)$*-nilpotent, i.e., $\pi_1(M)$ contains a nilpotent subgroup of index at most* $\mathrm{Const}(m)$.

Gradient flow also gives an alternative proof of the homotopy lifting theorem 4.2.3. To explain the idea let us start with definition:

Given a topological space $X$, a map $F : X \to A$, a finite sequence of $\lambda$-concave functions $\{f_i\}$ on $A$ and continuous functions $\tau_i : X \to \mathbb{R}_+$ one can consider a composition of gradient deformations (see 2.2)

$$F'(x) = \Phi_{f_N}^{\tau_N(x)} \circ \cdots \circ \Phi_{f_2}^{\tau_2(x)} \circ \Phi_{f_1}^{\tau_1(x)} \circ F(x), \quad F' : X \to A,$$

which we also call *gradient deformation* of $F$.

Let us define *gradient homotopy* to be a gradient deformation of trivial homotopy

$$F : [0,1] \times X \to A, \quad F_t(x) = F_0(x)$$

with the functions

$$\tau_i : [0,1] \times X \to \mathbb{R}_+ \quad \text{such that} \quad \tau_i(0,x) \equiv 0.$$

If $Y \subset X$, then to define *gradient homotopy relative to $Y$* we assume in addition

$$\tau_i(t,y) = 0 \quad \text{for any} \quad y \in Y, \;\; t \in [0,1].$$

Then theorem 4.2.3 follows from lemma 2.1.5 and the following lemma:

[10] see footnote 31 on page 31

**2.3.4. Lemma [Petrunin-GH].** *Let $A$ be an Alexandrov's space without proper extremal subsets and $K$ be a finite simplicial complex. Then, given $\varepsilon > 0$, for any homotopy*

$$F_t : K \to A, \quad t \in [0,1]$$

*one can construct an $\varepsilon$-close gradient homotopy*

$$G_t : K \to A$$

*such that $G_0 \equiv F_0$.*

# 3 Gradient exponent

One of the technical difficulties in Alexandrov's geometry comes from nonextendability of geodesics. In particular, the exponential map, $\exp_p : T_p \to A$, if defined the usual way, can be undefined in an arbitrary small neighborhood of origin. Here we construct its analog, the *gradient exponential map* $\mathrm{gexp}_p : T_p \to A$, which practically solves this problem. It has many important properties of the ordinary exponential map, and is even "better" in certain respects.

Let $A$ be an Alexandrov's space and $p \in A$, consider the function $f = \mathrm{dist}_p^2/2$. Recall that $i_s : sA \to A$ denotes canonical maps (see page 6). Consider the one parameter family of maps

$$\Phi_f^t \circ i_{e^t} : e^t A \to A \quad \text{as} \quad t \to \infty \quad \text{so} \quad (e^t A, p) \xrightarrow{\text{GH}} (T_p, o_p)$$

where $\Phi_f^t$ denotes gradient flow (see section 2.2). Let us define the *gradient exponential map* as the limit

$$\mathrm{gexp}_p : T_p A \to A, \quad \mathrm{gexp}_p = \lim_{t\to\infty} \Phi_f^t \circ i_{e^t}.$$

*Existence and uniqueness of gradient exponential.* If $A$ is an Alexandrov's space with curvature $\geqslant 0$, then $f$ is 1-concave, and from lemma 2.1.4, $\Phi_f^t$ is an $e^t$-Lipschitz and therefore compositions $\Phi_f^t \circ i_{e^t} : e^t A \to A$ are *short*[11]. Hence a partial limit $\mathrm{gexp}_p : T_p A \to A$ exists, and it is a short map.[12]

[11] i.e. maps with Lipschitz constant 1.

[12] For general lower curvature bound, $f$ is only $(1 + O(r^2))$-concave in the ball $B_r(p)$. Therefore $\Phi_f^1 : B_{r/e}(p) \to B_r(p)$ is $e(1 + O(r^2))$-Lipschitz. By taking compositions of these maps for different $r$ we get that $\Phi_f^N : B_{r/e^N}(p) \to B_r(p)$ is $e^N(1 + O(r^2))$-Lipschitz. Obviously, the same is true for any $t \geqslant 0$, i.e. $\Phi_f^t : B_{r/e^t}(p) \to B_r(p)$ is $e^t(1 + O(r^2))$-Lipschitz, or

$$\Phi_f^t \circ i_{e^t} : e^t A \to A$$

is $(1 + O(r^2))$-Lipschitz on $B_r(p) \subset e^t A$. This is sufficient for existence of partial limit $\mathrm{gexp}_p : T_p A \to A$, which turns out to be $(1 + O(r^2))$-Lipschitz on a central ball of radius $r$ in $T_p$.

Clearly for any partial limit we have

$$\Phi_f^t \circ \operatorname{gexp}_p(v) = \operatorname{gexp}_p(e^t \cdot v) \tag{$*$}$$

and since $\Phi^t$ is $e^t$-Lipschitz, it follows that $\operatorname{gexp}_p$ is uniquely defined.

**3.1.1. Property.** *If $E \in A$ is an extremal subset, $p \in E$ and $\xi \in \Sigma_p E$ then $\operatorname{gexp}_p(t \cdot \xi) \in E$ for any $t \geqslant 0$.*

It follows from above and from definition of extremal subset (4.1.1).

**Radial curves.** From identity $(*)$, it follows that for any $\xi \in \Sigma_p$, curve

$$\alpha_\xi : t \mapsto \operatorname{gexp}_p(t \cdot \xi)$$

satisfies the following differention equation

$$\alpha_\xi^+(t) = \frac{|p\,\alpha_\xi(t)|}{t} \nabla_{\alpha_\xi(t)} \operatorname{dist}_p \quad \text{for all} \;\; t > 0 \quad \text{and} \quad \alpha_\xi^+(0) = \xi \tag{$\diamond$}$$

We will call such a curve *radial curve* from $p$ in the direction $\xi$. From above, such radial curve exists and is unique in any direction.

Clearly, for any radial curve from $p$, $|p\alpha_\xi(t)| \leqslant t$; and if this inequality is exact for some $t_0$ then $\alpha_\xi : [0, t_0] \to A$ is a unit-speed minimizing geodesic starting at $p$ in the direction $\xi \in \Sigma_p$. In other words,

$$\operatorname{gexp}_p \circ \log_p = \mathrm{id}_A .^{13}$$

Next lemma gives a comparison inequality for radial curves.

**3.1.2. Lemma.** *Let $A \in \mathrm{Alex}$, $f : A \to \mathbb{R}$ be a $\lambda$-concave function $\lambda \geqslant 0$ then for any $p \in A$ and $\xi \in \Sigma_p$*

$$f \circ \operatorname{gexp}_p(t \cdot \xi) \leqslant f(p) + t \cdot d_p f(\xi) + t^2 \cdot \lambda/2.$$

*Moreover, the function*

$$\vartheta(t) = \{f \circ \operatorname{gexp}_p(t \cdot \xi) - f(p) - t^2 \cdot \lambda/2\}/t$$

*is non-increasing.*

In particular, applying this lemma for $f = \operatorname{dist}_q^2/2$ we get

**3.1.3. Corollary.** *If $A \in \mathrm{Alex}(0)$ then for any $p, q, \in A$ and $\xi \in \Sigma_p$,*

$$\tilde\measuredangle_0(t, |\operatorname{gexp}_p(t\xi) q|, |pq|)$$

*is non-increasing in $t$.*[14] *In particular,*

$$\tilde\measuredangle_0(t, |\operatorname{gexp}_p(t\xi)\, q|, |pq|) \leqslant \measuredangle(\xi, \uparrow_p^q).$$

---

[13] In proposition 3.3.6 we will show that $\alpha_\xi((0, t_0))$ does not meet any other radial curve from $p$.

[14] $\tilde\measuredangle_\kappa(a, b, c)$ denotes angle opposite to $b$ in a triangle with sides $a, b, c$ in $\Pi_\kappa$.

In 3.2 you can find a version of this corollary for arbitrary lower curvature bound.

*Proof of lemma 3.1.2.* Recall that $\nabla_q \operatorname{dist}_p$ is polar to the set $\Uparrow_q^p \subset T_q$ (see example (ii) on page 9). In particular, from inequality $(**)$ on page 9,

$$d_q f(\nabla_q \operatorname{dist}_p) + \inf_{\zeta \in \Uparrow_q^p} \{d_q f(\zeta)\} \leqslant 0$$

On the other hand, since $f$ is $\lambda$-concave,

$$d_q f(\zeta) \geqslant \frac{f(p) - f(q) - \lambda |pq|^2/2}{|pq|} \quad \text{for any} \quad \zeta \in \Uparrow_q^p,$$

therefore

$$d_q f(\nabla_q \operatorname{dist}_p) \leqslant \frac{f(q) - f(p) + \lambda |pq|^2/2}{|pq|}.$$

Set $\alpha_\xi(t) = \operatorname{gexp}(t \cdot \xi)$, $q = \alpha_\xi(t_0)$, then $\alpha_\xi^+(t_0) = \frac{|pq|}{t} \nabla_q \operatorname{dist}_p$ as in $(\diamond)$. Therefore,

$$(f \circ \alpha_\xi)^+(t_0) = d_q f(\alpha_\xi^+(t_0)) \leqslant$$

$$\leqslant \frac{|pq|}{t_0} \left[ \frac{f(q) - f(p) + \lambda |pq|^2/2}{|pq|} \right] = \frac{f(q) - f(p) + \lambda |pq|^2/2}{t_0} \leqslant$$

since $|pq| \leqslant t_0$ and $\lambda \geqslant 0$,

$$\leqslant \frac{f(q) - f(p) + \lambda t_0^2/2}{t_0} = \frac{f(\alpha_\xi(t_0)) - f(p) + \lambda t_0^2/2}{t_0}.$$

Substituting this inequality in the expression for derivative of $\vartheta$,

$$\vartheta^+(t_0) = \frac{(f \circ \alpha_\xi)^+(t)}{t_0} - \frac{f \circ \operatorname{gexp}_p(t_0 \cdot \xi) - f(p)}{t_0^2} - \lambda/2,$$

we get $\vartheta^+ \leqslant 0$, i.e. $\vartheta$ is non-increasing.

Clearly, $\vartheta(0) = d_p f(\xi)$ and so the first statement follows. $\square$

## 3.2 Spherical and hyperbolic gradient exponents

The gradient exponent described above is sufficient for most applications. It works perfectly for non-negatively curved Alexandrov's spaces and where one does not care for the actual lower curvature bound. However, for fine analysis on spaces with curvature $\geqslant \kappa$, there is a better analog of this map, which we denote $\operatorname{gexp}_p(\kappa; v)$; $\operatorname{gexp}_p(0; v) = \operatorname{gexp}_p(v)$.

In addition to case $\kappa = 0$, it is enough to consider only two cases: $\kappa = \pm 1$, the rest can be obtained by rescalings. We will define two maps: $\operatorname{gexp}_p(-1, *)$ and $\operatorname{gexp}_p(1, *)$, and list their properties, leaving calculations to the reader. These properties are analogous to the following properties of the ordinary gradient exponent:

◇ if $A \in \mathrm{Alex}(0)$, then $\operatorname{gexp}_p : T_p \to A$ is distance non-increasing.

Moreover, for any $q \in A$, the angle

$$\tilde{\measuredangle}_0(t, |\operatorname{gexp}_p(t \cdot \xi)\, q|, |pq|)$$

is non-increasing in $t$ (see corollary 3.1.3). In particular

$$\tilde{\measuredangle}_0(t, |\operatorname{gexp}_p(t \cdot \xi)\, q|, |pq|) \leqslant \measuredangle(\xi, \uparrow_p^q).$$

The calculations for the case $\kappa = 1$ are more complicated than for $\kappa = -1$. Note that formulas in definitions of these two cases are really different; the formulas for $\kappa \geqslant 0$ and $\kappa \leqslant 0$ are not analytic extension of each other.

**3.2.1. Case $\kappa = -1$.**

The hyperbolic radial curves are defined by the following differential equation

$$\alpha_\xi^+(t) = \frac{\operatorname{sh}|p\alpha_\xi(t)|}{\operatorname{sh} t} \nabla_{\alpha_\xi(t)} \operatorname{dist}_p \quad \text{and} \quad \alpha_\xi^+(0) = \xi.$$

These radial curves are defined for all $t \in [0, \infty)$. Let us define

$$\operatorname{gexp}_p(-1; t \cdot \xi) = \alpha_\xi(t).$$

This map is defined on tangent cone $T_p$. Let us equip the tangent cone with a hyperbolic metric $\mathfrak{h}(u, v)$ defined by the hyperbolic rule of cosines

$$\operatorname{ch}(\mathfrak{h}(u, v)) = \operatorname{ch}|u| \operatorname{ch}|v| - \operatorname{sh}|u| \operatorname{sh}|v| \cos\alpha,$$

where $u, v \in T_p$ and $\alpha = \measuredangle u o_p v$. $(T_p, \mathfrak{h}) \in \mathrm{Alex}(-1)$, this is a so called *elliptic cone* over $\Sigma_p$; see [BGP, 4.3.2], [Alexander–Bishop 2004]. Here are the main properties of $\operatorname{gexp}(-1; *)$:

◇ if $A \in \mathrm{Alex}(-1)$, then $\operatorname{gexp}(-1; *) : (T_p, \mathfrak{h}) \to A$ is distance non-increasing.

Moreover, the function

$$t \mapsto \tilde{\measuredangle}_{-1}(t, |\operatorname{gexp}(-1; t \cdot \xi)\, q|, |pq|)$$

is non-increasing in $t$. In particular for any $t > 0$,

$$\tilde{\measuredangle}_{-1}(t, |\operatorname{gexp}(-1; t \cdot \xi)\, q|, |pq|) \leqslant \measuredangle(\xi, \uparrow_p^q).$$

**3.2.2. Case $\kappa = 1$.**

For unit tanget vector $\xi \in \Sigma_p$, the spherical radial curve is defined to satisfy the following identity:

$$\alpha_\xi^+(t) = \frac{\operatorname{tg}|p\alpha_\xi(t)|}{\operatorname{tg} t} \nabla_{\alpha_\xi(t)} \operatorname{dist}_p \quad \text{and} \quad \alpha_\xi^+(0) = \xi.$$

These radial curves are defined for all $t \in [0, \pi/2]$. Let us define the spherical gradient exponential map by

$$\mathrm{gexp}_p(1; t \cdot \xi) = \alpha_\xi(t).$$

This map is well defined on $\bar{B}_{\pi/2}(o_p) \subset T_p$. Let us equip $\bar{B}_{\pi/2}(o_p)$ with a spherical distance $\mathfrak{s}(u, v)$ defined by the spherical rule of cosines

$$\cos(\mathfrak{s}(u, v)) = \cos|u| \cos|v| + \sin|u| \sin|v| \cos\alpha,$$

where $u, v \in B_\pi(o_p) \subset T_p$ and $\alpha = \measuredangle u o_p v$. $(\bar{B}_\pi(o_p), \mathfrak{s}) \in \mathrm{Alex}(1)$, this is isometric to *spherical suspension* $\Sigma(\Sigma_p)$, see [BGP, 4.3.1], [Alexander–Bishop 2004]. Here are the main properties of $\mathrm{gexp}(1; *)$:

◇ If $A \in \mathrm{Alex}(1)$ then $\mathrm{gexp}_p(1, *) : (\bar{B}_{\pi/2}(o_p), \mathfrak{s}) \to A$ is distance non-increasing. Moreover, if $|pq| \leqslant \pi/2$, then function

$$t \mapsto \tilde{\measuredangle}_1(t, |\,\mathrm{gexp}_p(1; t \cdot \xi)\, q|, |pq|)$$

is non-increasing in $t$. In particular, for any $t > 0$

$$\tilde{\measuredangle}_1(t, |\,\mathrm{gexp}_p(1; t \cdot \xi)\, q|, |pq|) \leqslant \measuredangle(\xi, \uparrow_p^q).$$

## 3.3 Applications

One of the main applications of gradient exponent and radial curves is the proof of existence of quasigeodesics; see property 4 page 34 and appendix A for the proof.

An infinite-dimensional generalization of gradient exponent was introduced by Perelman to make the last step in the proof of equality of Hausdorff and topological dimension for Alexandrov's spaces, see [Perelman–Petrunin QG, A.4]. According to [Plaut 1996] (or [Plaut 2002, 151]), if $\dim_H A \geqslant m$, then there is a point $p \in A$, the tangent cone of which contains a subcone $W \subset T_p$ isometric to Euclidean $m$-space. Then infinite-dimensional analogs of properties in section 3.2 ensure that image $\mathrm{gexp}_p(W)$ has topological dimension $\geqslant m$ and therefore $\dim A \geqslant m$.

The following statement has been proven in [Perelman 1991], then its formulation was made more exact in [Alexander–Bishop 2003]. Here we give a simplified proof with the use of a gradient exponent.

**3.3.1. Theorem.** *Let $A \in \mathrm{Alex}(\kappa)$ and $\partial A \neq \varnothing$; then the function $f = \sigma_\kappa \circ \mathrm{dist}_{\partial A}$*[15] *is $(-\kappa f)$-concave in $\Omega = A \backslash \partial A$.*[16]

*In particular,*

---

[15] $\sigma_\kappa : \mathbb{R} \to \mathbb{R}$ is defined by

$$\sigma_\kappa(x) = \sum_{n=0}^{\infty} \frac{(-\kappa)^n}{(2n+1)!} x^{2n+1} = \left[ \begin{array}{ll} \frac{1}{\sqrt{\kappa}} \sin(x\sqrt{\kappa}) & \text{if } \kappa > 0 \\ x & \text{if } \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \mathrm{sh}(x\sqrt{-\kappa}) & \text{if } \kappa < 0 \end{array} \right. .$$

[16] Note that by definition 1.1.2, $f$ is not semiconcave in $A$.

*(i) if* $\kappa = 0$*,* $\operatorname{dist}_{\partial A}$ *is concave in* $\Omega$*;*

*(ii) if* $\kappa > 0$*, the level sets* $L_x = \operatorname{dist}^{-1}_{\partial A}(x) \subset A$*,* $x > 0$ *are strictly concave hypersurfaces.*

*Proof.* We have to show that for any unit-speed geodesic $\gamma$, the function $f \circ \gamma$ is $(-\kappa f)$-concave; i.e. for any $t_0$,

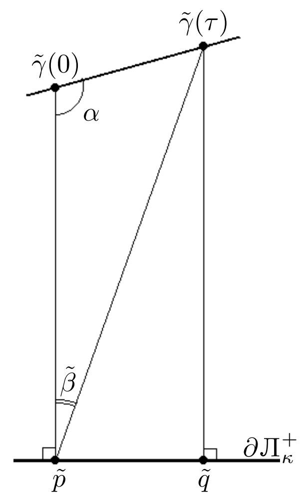
$$(f \circ \gamma)''(t_0) \leqslant -\kappa f \circ \gamma(t_0)$$

*in a barrier sense*[17]. Without loss of generality we can assume $t_0 = 0$.

Direct calculations show that the statement is true for $A = \text{Л}^+_\kappa$, the halfspace of the model space $\text{Л}_\kappa$.

Let $p \in \partial A$ be a closest point to $\gamma(0)$ and $\alpha = \measuredangle(\gamma^+(0), \uparrow^p_{\gamma(0)})$.

Consider the following picture in the model halfspace $\text{Л}^+_\kappa$: Take a point $\tilde p \in \partial \text{Л}^+_\kappa$ and consider the geodesic $\tilde\gamma$ in $\text{Л}^+_\kappa$ such that




$$|\gamma(0)p| = |\tilde\gamma(0)\tilde p| = |\tilde\gamma(0)\partial\text{Л}^+_\kappa|,$$

so $\tilde p$ is the closest point to $\tilde\gamma(0)$ on the boundary[18] and

$$\measuredangle(\tilde\gamma^+(0), \uparrow^{\tilde p}_{\tilde\gamma(0)}) = \alpha.$$

Then it is enough to show that

$$\operatorname{dist}_{\partial A} \gamma(\tau) \leqslant \operatorname{dist}_{\partial \text{Л}^+_\kappa} \tilde\gamma(\tau) + o(\tau^2).$$

Set

$$\beta(\tau) = \measuredangle \gamma(0)\, p\, \gamma(\tau)$$

and

$$\tilde\beta(\tau) = \measuredangle \tilde\gamma(0)\, \tilde p\, \tilde\gamma(\tau).$$

From the comparison inequalities

$$|p\gamma(\tau)| \leqslant |\tilde p \tilde\gamma(\tau)|$$

and

$$\vartheta(\tau) = \max\left\{0,\, \tilde\beta(\tau) - \beta(\tau)\right\} = o(\tau). \qquad (*)$$

[17] For a continuous function $f$, $f''(t_0) \leqslant c$ *in a barrier sense* means that there is a smooth function $\bar f$ such that $f \leqslant \bar f$, $f(t_0) = \bar f(t_0)$ and $\bar f''(t_0) \leqslant c$

[18] in case $\kappa > 0$ it is possible only if $|\gamma(0)p| \leqslant \frac{\pi}{2\sqrt{\kappa}}$, but this is always the case since otherwise any small variation of $p$ in $\partial A$ decreases distance $|\gamma(0)p|$.

Note that the tangent cone at $p$ splits: $T_pA = \mathbb{R}_+ \times T_p\partial A$.[19] Therefore we can represent $v = \log_p \gamma(\tau) \in T_pA$ as $v = (s, w) \in \mathbb{R}_+ \times T_p\partial A$. Let $\tilde{q} = \tilde{q}(\tau) \in \partial\text{Л}_\kappa$ be the closest point to $\tilde{\gamma}(\tau)$, so

$$\measuredangle(\uparrow_p^{\gamma(\tau)}, w) = \frac{\pi}{2} - \beta(\tau) \leqslant \frac{\pi}{2} - \tilde{\beta}(\tau) - \vartheta(\tau) = \measuredangle\tilde{\gamma}(\tau)\tilde{p}\tilde{q} + o(\tau).$$

Set $q = \operatorname{gexp}_p\left(\kappa; |\tilde{p}\tilde{q}|\frac{w}{|w|}\right)$.[20] Since gradient curves preserve extremal subsets $q \in \partial A$ (see property 3.1.1 on page 17). Clearly $|\tilde{p}\tilde{q}| = O(\tau)$, therefore applying the comparison from section 3.2 (or Corollary 3.1.3 if $\kappa = 0$) together with $(*)$, we get

$$\operatorname{dist}_{\partial A}\gamma(\tau) \leqslant |q\gamma(\tau)| \leqslant |\tilde{q}\tilde{\gamma}(\tau)| + O\left(|\tilde{p}\tilde{q}| \cdot \vartheta(\tau)\right) = \operatorname{dist}_{\partial\text{Л}_\kappa^+}\tilde{\gamma}(\tau) + o(\tau^2).$$

□

The following corollary implies that the Lipschitz condition in the definition of convex function 1.1.2– 1.1.1 can be relaxed to usual continuity.

**3.3.2. Corollary.** *Let $A \in \mathrm{Alex}$, $\partial A = \varnothing$, $\lambda \in \mathbb{R}$ and $\Omega \subset A$ be open.*

*Assume $f : \Omega \to \mathbb{R}$ is a continuous function such that for any unit-speed geodesic $\gamma$ in $\Omega$ we have that the function*

$$t \mapsto f \circ \gamma - \lambda t^2/2$$

*is concave; then $f$ is locally Lipschitz.*

*In particular, $f$ is $\lambda$-concave in the sense of definition 1.1.2.*

*Proof.* Assume $f$ is not Lipschitz at $p \in \Omega$. Without loss of generality we can assume that $\Omega$ is convex[21] and $\lambda < 0$[22]. Then, since $f$ is continuous, sub-graph

$$X_f = \{(x, y) \in \bar{\Omega} \times \mathbb{R} | y \leqslant f(x)\}$$

is closed convex subset of $A \times \mathbb{R}$, therefore it forms an Alexandrov's space.

Since $f$ is not Lipschitz at $p$, there is a sequence of pairs of points $(p_n, q_n)$ in $A$, such that

$$p_n, q_n \to p \quad \text{and} \quad \frac{f(p_n) - f(q_n)}{|p_nq_n|} \to +\infty.$$

Consider a sequence of radial curves $\alpha_n$ in $X_f$ which extend shortest paths from $(p_n, f(p_n))$ to $(q_n, f(q_n))$. Since the boundary $\partial X_f \subset X_f$ is an extremal subset, we have $\alpha_n(t) \in \partial X_f$ for all

$$t \geqslant \ell_n = |(p_n, f(p_n))(q_n, f(q_n))| = \sqrt{|p_nq_n|^2 + (f(p_n) - f(q_n))^2}.$$

[19] This follows from the fact that $p$ lies on a shortest path between two preimages of $\gamma(0)$ in the doubling $\tilde{A}$ of $A$, see [BGP, 7.15].

[20] Alternatively, one can set $q = \gamma(|\tilde{p}\tilde{q}|)$, where $\gamma$ is a quasigeodesic in $\partial A$ starting at $p$ in direction $\frac{w}{|w|} \in \Sigma_p$ (it exists by second part of property 4 on page 34).

[21] Otherwise, pass to a small convex neighborhood of $p$ which exists by by corollary 7.1.2.

[22] Otherwise, add a very concave (Lipschitz) function which exists by theorem 7.1.1

Clearly, the function $h : X_f \to \mathbb{R}$, $h : (x,y) \mapsto y$ is concave. Therefore, from 3.1.2, there is a sequence $t_n > \ell_n$, so $\alpha_n(t_n) \to (p, f(p) - 1)$. Therefore, $(p, f(p)-1) \in \partial X_f$ thus $p \in \partial A$, i.e. $\partial A \neq \varnothing$, a contradiction. $\square$

**3.3.3. Corollary.** *Let $A \in \mathsf{Alex}^m(\kappa)$, $m \geqslant 2$ and $\gamma$ be a unit-speed curve in $A$ which has a convex $\kappa$-developing with respect to any point. Then $\gamma$ is a quasigeodesic, i.e. for any $\lambda$-concave function $f$, function $f \circ \gamma$ is $\lambda$-concave.*

*Proof.* Let us first note that in the proof of theorem 3.3.1 we used only two properties of curve $\gamma$: $|\gamma^{\pm}| = 1$ and the convexity of the $\kappa$-development of $\gamma$ with respect to $p$.

Assume $\kappa = \lambda = 0$ then sub-graph of $f$

$$X_f = \{(x,y) \in A \times \mathbb{R} \mid y \leqslant f(x)\}$$

is a closed convex subset, therefore it forms an Alexandrov's space.

Applying the above remark, we get that if $\gamma$ is a unit-speed curve in $X_f \backslash \partial X_f$ with convex 0-developing with respect to any point then $\mathrm{dist}_{\partial X_f} \circ \gamma$ is concave. Hence, for any $\varepsilon > 0$, the function $f_\varepsilon$, which has the level set $\mathrm{dist}^{-1}_{\partial X_f}(\varepsilon) \subset \mathbb{R} \times A$ like the graph, has a concave restriction to any curve $\gamma$ in $A$ with a convex 0-developing with respect to any point in $A \backslash \gamma$. Clearly, $f_\varepsilon \to f$ as $\varepsilon \to 0$, hence $f \circ \gamma$ is concave.

For $\lambda$-concave function the set $X_f$ is no longer convex, but it becomes convex if one changes metric on $A \times \mathbb{R}$ to *parabolic cone*[23] and then one can repeat the same arguments. $\square$

**Remark** One can also get this corollary from the following lemma:

**3.3.4. Lemma.** *Let $A \in \mathsf{Alex}^m(\kappa)$, $\Omega$ be an open subset of $A$ and $f : \Omega \to \mathbb{R}$ be a $\lambda$-concave $L$-Lipschitz function. Then function*

$$f_\varepsilon(y) = \min_{x \in \Omega}\{f(x) + \tfrac{1}{\varepsilon}|xy|^2\}$$

*is $(\lambda + \delta)$-concave in the domain of definition*[24] *for some*[25] $\delta = \delta(L, \lambda, \kappa, \varepsilon)$, $\delta \to 0$ *as* $\varepsilon \to 0$.

*Moreover, if $m \geqslant 2$ and $\gamma$ is a unit-speed curve in $A$ with $\kappa$-convex developing with respect to any point then $f_\varepsilon \circ \gamma$ is also $(\lambda + \delta)$-concave.*

*Proof.* It is analogous to theorem 3.3.1. We only indicate it in the simplest case, $\kappa = \lambda = 0$. In this case $\delta$ can be taken to be $0$.

[23] i.e. warped-product $\mathbb{R} \times_{\exp(\mathrm{Const}\, t)} A$, which is an Alexandrov's space, see [BGP, 4.3.3], [Alexander–Bishop 2004]

[24] i.e. at the set where the minimum is defined.

[25] this function $\delta(L, \lambda, \kappa, \varepsilon)$ is achieved for the model space $\Lambda_\kappa$

Let $\gamma$ be a unit-speed geodesic (or it satisfies the last condition in the lemma). It is enough to show that for any $t_0$

$$(f_\varepsilon\circ\gamma)''(t_0)\leqslant 0$$

in a barrier sense.

Let $y=\gamma(t_0)$ and $x\in\Omega$ be a point for which $f_\varepsilon(y)=f(x)+\frac{1}{\varepsilon}|xy|^2$. The tangent cone $T_x$ splits in direction $\uparrow_y^x$, i.e. there is an isometry $T_x\to\mathbb{R}\times\mathrm{Cone}$ such that $\uparrow_x^y\mapsto(1,o)$, where $o\in\mathrm{Cone}$ is its origin. Let

$$\log_x\gamma(t)=(a(t),v(t))\in\mathbb{R}\times\mathrm{Cone}=T_x.$$

Consider vector

$$w(t)=(a(t)-|xy|,v(t))\in\mathbb{R}\times\mathrm{Cone}=T_x.$$

Clearly $|w(t)|\geqslant|x\gamma(t)|$. Set $x(t)=\mathrm{gexp}_y(w(t))$ then lemma 3.1.2 gives an estimate for $f\circ x(t))$ while corollary 3.1.3 gives an estimate for $|\gamma(t)x(t)|^2$. Hence the result. □

Here is yet another illustration for the use of gradient exponents. At first sight it seems very simple, but the proof is not quite obvious. In fact, I did not find any proof of this without applying the gradient exponent.

**3.3.5. Lytchak's problem.** *Let $A\in\mathcal{A}lex^m(1)$. Show that*

$$\mathrm{vol}_{m-1}\partial A\leqslant\mathrm{vol}_{m-1}S^{m-1}$$

*where $\partial A$ denotes the boundary of $A$ and $S^{m-1}$ the unit $(m-1)$-sphere.*

The problem would have followed from conjecture 9.1.1 (that boundary of an Alexandrov's space is an Alexandrov's space), but before this conjecture has been proven, any partial result is of some interest. Among other corollaries of conjecture 9.1.1, it is expected that if $A\in\mathcal{A}lex(1)$ then $\partial A$, equipped with induced intrinsic metric, admits a noncontracting map to $S^{m-1}$. In particular, its intrinsic diameter is at most $\pi$, and perimeter of any triangle in $\partial A$ is at most $2\pi$. This does not follow from the proof below, since in general $\mathrm{gexp}_z(1;\partial B_{\pi/2}(o_z))\not\subset\partial A$, i.e. $\mathrm{gexp}_z(1;\partial B_{\pi/2}(o_z))$ might have some creases left inside of $A$, which might be used as a shortcut for curves with ends in $\partial A$.

Let us first prepare a proposition:

**3.3.6. Proposition.** *The inverse of the gradient exponential map $\mathrm{gexp}_p^{-1}(\kappa;*)$ is uniquely defined inside any minimizing geodesic starting at $p$.*

*Proof.* Let $\gamma:[0,t_0]\to A$ be a unit-speed minimizing geodesic, $\gamma(0)=p$, $\gamma(t_0)=q$. From the angle comparison we get that $|\nabla_x\mathrm{dist}_p|\geqslant-\cos\tilde\measuredangle_\kappa pxq$. Therefore, for any $\zeta$ we have

$$|p\alpha_\zeta(t)|_t^+\geqslant-|\alpha_\zeta^+(t)|\cos\tilde\measuredangle_\kappa p\,\alpha_\zeta(t)\,q\ \text{ and }\ |\alpha_\zeta(t)q|_t^+\geqslant-|\alpha_\zeta^+(t)|.$$

Therefore, $\tilde{\measuredangle}_\kappa p\, q\, \alpha_\zeta(t)$ is nondecreasing in $t$, hence the result. $\square$

*Proof of 3.3.5.* Let $z \in A$ be the point at maximal distance from $\partial A$, in particular it realizes maximum of $f = \sigma_1 \circ \mathrm{dist}_{\partial A} = \sin \circ \mathrm{dist}_{\partial A}$. From theorem 3.3.1, $f$ is $(-f)$-concave and $f(z) \leqslant 1$.

Note that $A \subset \bar{B}_{\pi/2}(z)$, otherwise if $y \in A$ with $|yz| > \pi/2$, then since $f$ is $(-f)$-concave and $f(y) \geqslant 0$, we have $df(\uparrow_z^y) > 0$, i.e. $z$ is not a maximum of $f$.

From this it follows that gradient exponent

$$\mathrm{gexp}_z(1; *) : (\bar{B}_{\pi/2}(o_z), \mathfrak{s}) \to A$$

is a short onto map.

Moreover,

$$\partial A \subset \mathrm{gexp}_z(\partial B_{\pi/2}(o_z)).$$

Indeed, gexp gives a homotopy equivalence $\partial B_{\pi/2}(o_z) \to A \backslash z$. Clearly, $\Sigma_z = \partial(B_{\pi/2}(o_z), \mathfrak{s})$ has no boundary, therefore $H_{m-1}(\partial A, \mathbb{Z}_2) \neq 0$, see [Grove–Petersen 1993, lemma 1]. Hence for any point $x \in \partial A$, any minimizing geodesic $zx$ must have a point of the image $\mathrm{gexp}(1; \partial B_{\pi/2}(o))$ but, as it is shown in proposition 3.3.6, it can only be its end $x$.

Now since

$$\mathrm{gexp}_z(1; *) : (\bar{B}_{\pi/2}(o_z), \mathfrak{s}) \to A$$

is short and $(\partial B_{\pi/2}(o), \mathfrak{s})$ is isometric to $\Sigma_z A$ we get $\mathrm{vol}\, \partial A \leqslant \mathrm{vol}\, \Sigma_z A$ and clearly, $\mathrm{vol}\, \Sigma_z A \leqslant \mathrm{vol}\, S^{m-1}$. $\square$

# 4 Extremal subsets

Imagine that you want to move a heavy box inside an empty room by pushing it around. If the box is located in the middle of the room, you can push it in any direction. But once it is pushed against a wall you can not push it back to the center; and once it is pushed into a corner you cannot push it anywhere anymore. The same is true if one tries to move a point in an Alexandrov's space by pushing it along a gradient flow, but the role of walls and corners is played by extremal subsets.

Extremal subsets first appeared in the study of their special case — the boundary of an Alexandrov's space. They were introduced in [Perelman–Petrunin 1993], and were studied further in [Petrunin 1997], [Perelman 1997].

An Alexandrov's space without extremal subsets resembles a very non-smooth Riemannian manifold. The presence of extremal subsets makes it behave as something new and maybe intersting; it gives an interesting additional combinatoric structure which reflects geometry and topology of the space itself, as well as of nearby spaces.

## 4.1 Definition and properties.

It is best to define extremal subsets as "ideals" of the gradient flow, i.e.

**4.1.1. Definition.** *Let $A \in \mathcal{Alex}$.*
$E \subset A$ *is an* extremal subset*, if for any semiconcave function $f$ on $A$, $t \geqslant 0$ and $x \in E$, we have $\Phi_f^t(x) \in E$.*

Recall that $\Phi_f^t$ denotes the $f$-gradient flow for time $t$, see 2.2. Here is a quick corollary of this definition:

1. Extremal subsets are closed. Moreover:
   (i) For any point $p \in A$, there is an $\varepsilon > 0$, such that if an extremal subset intersects $\varepsilon$-neighborhood of $p$ then it contains $p$.
   (ii) On each extremal subset the intrinsic metric is locally finite.

   These properties follow from the fact that the gradient flow for a $\lambda$-concave function with $d_p f|_{\Sigma_p} < 0$ pushes a small ball $B_\varepsilon(p)$ to $p$ in time proportionate to $\varepsilon$.

*Examples.*

(i) An Alexandrov's space itself, as well as the empty set, forms an extremal subsets.

(ii) A point $p \in A$ forms a one-point extremal subset if its space of directions $\Sigma_p$ has a diameter $\leqslant \pi/2$

(iii) If one takes a subset of points of an Alexandrov's space with tangent cones homeomorphic[26] to each other then its closure[27] forms an extremal subset.

In particular, if in this construction we take points with tangent cone homeomorphic to $\mathbb{R}_+ \times \mathbb{R}^{m-1}$ then we get the boundary of an Alexandrov's space.

This follows from theorem 4.1.2 and the Morse lemma (property 7 page 45).

(iv) Let $A/G$ be a factor of an Alexandrov's space by an isometry group, and $S_H \subset A$ be the set of points with stabilizer conjugate to a subgroup $H \subset G$ (or its connected component). Then the closure of the projection of $S_H$ in $A/G$ forms an extremal subset.

For example: A cube can be presented as a quotient of a flat torus by a discrete isometry group, and each face of the cube forms an extremal subset.

The following theorem gives an equivalence of our definition of extremal subset and the definition given in [Perelman–Petrunin 1993]:

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[26] Equivalently, with homeomorphic small spherical neigborhoods. The equivalence follows from Perelman's stability theorem.

[27] As well as the closure of its connected component.

**4.1.2. Theorem.** *A closed subset $E$ in an Alexandrov's space $A$ is extremal if and only if for any $q \in A \backslash E$, the following condition is fulfilled:*

*If $\operatorname{dist}_q$ has a local minimum on $E$ at a point $p$, then $p$ is a critical point of $\operatorname{dist}_q$ on $A$, i.e., $\nabla_p \operatorname{dist}_q = o_p$.*

*Proof.* For the "only if" part, note that if $p \in E$ is not a critical point of $\operatorname{dist}_q$, then one can find a point $x$ close to $p$ so that $\uparrow_p^x$ is uniquely defined and close to the direction of $\nabla_p \operatorname{dist}_q$, so $d_p \operatorname{dist}_q(\uparrow_p^x) > 0$. Since $\nabla_p \operatorname{dist}_x$ is polar to $\uparrow_p^x$ (see page 8) we get

$$d_p \operatorname{dist}_q(\nabla_p \operatorname{dist}_x) < 0,$$

see inequality 1.3 on page 9. Hence, the gradient flow $\Phi^t_{\operatorname{dist}_x}$ pushes the point $p$ closer to $q$, which contradicts the fact that $p$ is a minimum point $\operatorname{dist}_q$ on $E$.

To prove the "if" part, it is enough to show that if $F \subset A$ satisfies the condition of the theorem, then for any $p \in F$, and any semiconcave function $f$, either $\nabla_p f = o_p$ or $\frac{\nabla_p f}{|\nabla_p f|} \in \Sigma_p F$. If so, an $f$-gradient curve can be obtained as a limit of broken lines with vertexes on $F$, and from uniqueness, any gradient curve which starts at $F$ lives in $F$.

Let us use induction on $\dim A$. Note that if $F \subset A$ satisfies the condition, then the same is true for $\Sigma_p F \subset \Sigma_p$, for any $p \in F$. Then using the inductive hypothesis we get that $\Sigma_p F \subset \Sigma_p$ is an extremal subset.

If $p$ is isolated, then clearly $\operatorname{diam} \Sigma_p \leqslant \pi/2$ and therefore $\nabla_p f = o$, so we can assume $\Sigma_p F \neq \varnothing$.

Note that $d_p f$ is $(-d_p f)$-concave on $\Sigma_p$ (see 1.2, page 5). Take $\xi = \frac{\nabla_p f}{|\nabla_p f|}$, so $\xi \in \Sigma_p$ is the maximal point of $d_p f$. Let $\eta \in \Sigma_p F$ be a direction closest to $\xi$, then $\measuredangle(\xi, \eta) \leqslant \pi/2$; otherwise $F$ would not satisfy the condition in the theorem for a point $q$ with $\uparrow_p^q \approx \xi$. Hence, since $\Sigma_p F \subset \Sigma_p$ is an extremal subset, $\nabla_\eta d_p f \in \Sigma_\eta \Sigma_p F$ and therefore

$$d_\eta d_p f(\uparrow_\eta^\xi) \leqslant \langle \nabla_\eta d_p f, \uparrow_\eta^\xi \rangle \leqslant 0.$$

Hence, $d_p f(\eta) \geqslant d_p f(\xi)$, and therefore $\xi = \eta$, i.e. $\frac{\nabla_p f}{|\nabla_p f|} \in \Sigma_p F$. □

From this theorem it follows that in the definition of extremal subset (4.1.1), one has to check only squares of distance functions. Namely: *Let $A \in \mathrm{Alex}$, then $E \subset A$ is an extremal subset, if for any point $p \in A$, and any $x \in E$, we have $\Phi^t_{\operatorname{dist}_p^2}(x) \in E$ for any $t \geqslant 0$.*

In particular, applying lemma 2.1.5 we get

**4.1.3. Lemma.** *The limit of extremal subsets is an extremal subset.*

*Namely, if $A_n \in \mathrm{Alex}^m(\kappa)$, $A_n \xrightarrow{\mathrm{GH}} A$ and $E_n \subset A_n$ is a sequence of extremal subsets such that $E_n \to E \subset A$ then $E$ is an extremal subset of $A$.*

The following is yet another important technical lemma:

**4.1.4. Lemma.** [Perelman–Petrunin 1993, 3.1(2)] *Let $A \in \mathrm{Alex}$ be compact, then there is $\varepsilon > 0$ such that $\operatorname{dist}_E$ has no critical values in $(0, \varepsilon)$. Moreover,*

$$|\nabla_x \operatorname{dist}_E| > \varepsilon \quad \text{if} \quad 0 < \operatorname{dist}_E(x) < \varepsilon.$$

*For a non-compact $A$, the same is true for the restriction $\operatorname{dist}_E|_\Omega$ to any bounded open $\Omega \subset A$.*

*Proof.* Follows from lemma 4.1.5 and theorem 4.1.2. □

**4.1.5. Lemma about an obtuse angle.** *Given $v > 0$, $r > 0$, $\kappa \in \mathbb{R}$ and $m \in \mathbb{N}$, there is $\varepsilon = \varepsilon(v, r, \kappa, m) > 0$ such that if $A \in \mathrm{Alex}^m(\kappa)$, $p \in A$, $\operatorname{vol}_m B_r(p) > v$, then for any two points $x, y \in B_r(p)$, $|xy| < \varepsilon$ there is point $z \in B_r(p)$ such that $\measuredangle zxy > \pi/2 + \varepsilon$ or $\measuredangle zyx > \pi/2 + \varepsilon$.*

The proof is based on a volume comparison for $\log_x : A \to T_x$ similar to [Grove–Petersen 1988, lemma 1.3].

Note that the tangent cone $T_pE$ of an extremal subset $E \subset A$ is well defined; i.e. for any $p \in E$, subsets $sE$ in $(sA, p)$ converge to a subcone of $T_pE \subset T_pA$ as $s \to \infty$. Indeed, assume $E \subset A$ is an extremal subset and $p \in E$. For any $\xi \in \Sigma_p E$[28], the radial curve $\operatorname{gexp}(t\cdot\xi)$ lies in $E$.[29] In particular, there is a curve which goes in any tangent direction of $E$. Therefore, as $s \to \infty$, $(sE \subset sA, p)$ converges to a subcone $T_pE \subset T_pA$, which is simply cone over $\Sigma_pE$ (see also [Perelman–Petrunin 1993, 3.3])

Next we list some properties of tangent cones of extremal subsets:

2. A closed subset $E \subset A$ is extremal if and only if the following condition is fulfilled:

   ◇ At any point $p \in E$, its tangent cone $T_pE \subset T_pA$ is well defined, and it is an extremal subset of the tangent cone $T_pA$.(compare [Perelman–Petrunin 1993, 1.4])

   (Here is an equivalent formulation in terms of the space of directions: For any $p \in E$, either (a) $\Sigma_pE = \varnothing$ and $\operatorname{diam} \Sigma_p \leqslant \pi/2$ or (b) $\Sigma_pE = \{\xi\}$ is one point extremal subset and $\bar{B}_{\pi/2}(\xi) = \Sigma_p$ or (c) $\Sigma_pE$ is extremal subset of $\Sigma_p$ with at least two points.)

   $T_pE$ is extremal as a limit of extremal subsets, see lemma 4.1.3. On the other hand for any semiconcave function $f$ and $p \in E$, the differential $d_pf : T_p \to \mathbb{R}$ is concave and since $T_pE \subset T_p$ is extremal we have $\nabla_p f \in T_pE$. I.e. gradient curves can be approximated by broken geodesics with vertices on $E$, see page 10.

[28] For a closed subset $X \subset A$, and $p \in X$, $\Sigma_pX \subset \Sigma_p$ denotes the set of tangent directions to $X$ at $p$, i.e. the set of limits of $\uparrow_p^{q_n}$ for $q_n \to p$, $q_n \in X$.

[29] that follows from the fact that the curves $t \mapsto \operatorname{gexp}(t\cdot\uparrow_p^{q_n})$ starting with $q_n$ belong to $E$ and their converge to $\operatorname{gexp}(t\cdot\xi)$

3. [Perelman–Petrunin 1993, 3.4–5] If $E$ and $F$ are extremal subsets then so are

   (i) $E \cap F$ and for any $p \in E \cap F$ we have $T_p(E \cup F) = T_pE \cup \Sigma_pF$

   (ii) $E \cup F$ and for any $p \in E \cup F$ we have $T_p(E \cap F) = T_pE \cap \Sigma_pF$

   (iii) $\overline{E \backslash F}$ and for any $p \in \overline{E \backslash F}$ we have $T_p(\overline{E \backslash F}) = \overline{T_pE \backslash T_pF}$

   In particular, if $T_pE = T_pF$ then $E$ and $F$ coincide in a neighborhood of $p$.

   The properties (i) and (ii) are obvious. The property (iii) follows from property 2 and lemma 4.1.4.

We continue with properties of the intrinsic metric of extremal subsets:

4. [Perelman–Petrunin 1993, 3.2(3)] Let $A \in \mathit{Alex}^m(\kappa)$ and $E \subset A$ be an extremal subset. Then the induced metric of $E$ is locally bi-Lipschitz equivalent to its induced intrinsic metric. Moreover, the local Lipschitz constant at point $p \in E$ can be expressed in terms of $m$, $\kappa$ and volume of a ball $v = \operatorname{vol} B_r(p)$ for some (any) $r > 0$.

   From lemma 4.1.5, it follows that for two sufficiently close points $x, y \in E$ near $p$ there is a point $z$ so that $\langle \nabla_x \operatorname{dist}_z, \uparrow_x^y \rangle > \varepsilon$ or $\langle \nabla_y \operatorname{dist}_z, \uparrow_y^x \rangle > \varepsilon$. Then, for the corresponding point, say $x$, the gradient curve $t \to \Phi^t_{\operatorname{dist}_z}(x)$ lies in $E$, it is 1-Lipschitz and the distance $|\Phi^t_{\operatorname{dist}_z}(x)\, y|$ is decreasing with the speed of at least $\varepsilon$. Hence the result.

5. Let $A_n \in \mathit{Alex}^m(\kappa)$, $A_n \xrightarrow{\mathrm{GH}} A$ without collapse (i.e. $\dim A = m$) and $E_n \subset A_n$ be extremal subsets. Assume $E_n \to E \subset A$ as subsets. Then

   (i) [Kapovitch 2007, 9.1] For all large $n$, there is a homeomorphism of pairs $(A_n, E_n) \to (A, E)$. In particular, for all large $n$, $E_n$ is homeomorphic to $E$,

   (ii) [Petrunin 1997, 1.2] $E_n \xrightarrow{\mathrm{GH}} E$ as length metric spaces (with the intrinsic metrics induced from $A_n$ and $A$).

   The first property is a coproduct of the proof of Perelman's stability theorem. The proof of the second is an application of quasigeodesics.

6. [Petrunin 1997, 1.4] *The first variation formula.* Assume $A \in \mathit{Alex}$ and $E \subset A$ is an extremal subset, let us denote by $|**|_E$ its intrinsic metric. Let $p, q \in E$ and $\alpha(t)$ be a curve in $E$ starting from $p$ in direction $\alpha^+(0) \in \Sigma_pE$. Then

   $$|\alpha(t)\, q|_E = |pq|_E - \cos\varphi \cdot t + o(t).$$

   where $\varphi$ is the minimal (intrinsic) distance in $\Sigma_pE$ between $\alpha^+(0)$ and a direction of a shortest path in $E$ from $p$ to $q$ (if $\varphi > \pi$, we assume $\cos\varphi = -1$).

7. *Generalized Lieberman's Lemma.* Any minimizing geodesic for the induced intrinsic metric on an extremal subset is a quasigeodesic in the ambient space.

   See 2.3.1 for the proof and discussion.

Let us denote by $\operatorname{Ext}(x)$ the minimal extremal subset which contains a point $x \in A$. Extremal subsets which can be obtained this way will be called *primitive*. Set

$$\operatorname{Ext}^\circ(x) = \{y \in \operatorname{Ext}(x) | \operatorname{Ext}(y) = \operatorname{Ext}(x)\};$$

let us call $\operatorname{Ext}^\circ(x)$ the *main part* of $\operatorname{Ext}(x)$. $\operatorname{Ext}^\circ(x)$ is the same as $\operatorname{Ext}(x)$ with its proper extremal subsets removed. From property 3iii on page 29, $\operatorname{Ext}^\circ(x)$ is open and everywhere dense in $\operatorname{Ext}(x)$. Clearly the main parts of primitive extremal subsets form a disjoint covering of $M$.

8. [Perelman–Petrunin 1993, 3.8] *Stratification.* The main part of a primitive extremal subset is a topological manifold. In particular, the main parts of primitive extremal subsets stratify Alexandrov's space into topological manifolds.

   This follows from theorem 4.1.2 and the Morse lemma (property 7 page 45); see also example iii, page 26.

## 4.2 Applications

The notion of extremal subsets is used to make more precise formulations. Here is the simplest example, a version of the radius sphere theorem:

**4.2.1. Theorem.** *Let $A \in \mathsf{Alex}^m(1)$, $\operatorname{diam} A > \pi/2$ and $A$ have no extremal subsets. Then $A$ is homeomorphic to a sphere.*

From lemma 5.2.1 and theorem 4.1.2, we have $A \in \mathsf{Alex}(1)$, $\operatorname{rad} A > \pi/2$ implies that $A$ has no extremal subsets. I.e. this theorem does indeed generalize the radius sphere theorem 5.2.2(ii).

*Proof.* Assume $p, q \in A$ realize the diameter of $A$. Since $A$ has no extremal subsets, from example iii, page 26, it follows that a small spherical neighborhood of $p \in A$ is homeomorphic to $\mathbb{R}^m$. From angle comparison, $\operatorname{dist}_p$ has only two critical points $p$ and $q$. Therefore, this theorem follows from the Morse lemma (property 7 page 45) applied to $\operatorname{dist}_p$. □

The main result of such type is the result in [Perelman 1997]. It roughly states that a collapsing to a compact space without proper extremal subsets carries a natural Serre bundle structure.

This theorem is analogous to the following:

**4.2.2. Yamaguchi's fibration theorem [Yamaguchi].** *Let $A_n \in \mathsf{Alex}^m(\kappa)$ and $A_n \xrightarrow{\mathrm{GH}} M$, $M$ be a Riemannian manifold.*

*Then there is a sequence of locally trivial fiber bundles $\sigma_n : A_n \to M$. Moreover, $\sigma_n$ can be chosen to be* almost submetries[30] *and the diameters of its fibers converge to* $0$.

The conclusion in Perelman's theorem is weaker, but on the other hand it is just as good for practical purposes. In addition it is sharp, i.e. there are examples of a collapse to spaces with extremal subsets which do not have the homotopy lifting property. Here is a source of examples: take a compact Riemannian manifold $M$ with an isometric and non-free action by a compact connected Lie group $G$, then $(M \times \varepsilon G)/G \xrightarrow{\text{GH}} M/G$ as $\varepsilon \to 0$ and since the curvature of $G$ is non-negative, by O'Naill's formula, we get that the curvature of $(M \times \varepsilon G)/G$ is uniformly bounded below.

**4.2.3. Homotopy lifting theorem.** *Let $A_n \xrightarrow{\text{GH}} A$, $A_n \in \mathrm{Alex}^m(\kappa)$, $A$ be compact without proper extremal subsets and $K$ be a finite simplicial complex.*

*Then, given a homotopy*

$$F_t : K \to A, \quad t \in [0,1]$$

*and a sequence of maps $G_{0;n} : K \to A_n$ such that $G_{0,n} \to F_0$ as $n \to \infty$ one can extend $G_{0;n}$ by homotopies*

$$G_{t;n} : K \to A$$

*such that $G_{t;n} \to F_t$ as $n \to \infty$.*

An alternative proof is based on Lemma 2.3.4.

**4.2.4. Remark.** As a corollary of this theorem one obtains that for all large $n$ it is possible to write a homotopy exact sequence:

$$\cdots \pi_k(F_n) \longrightarrow \pi_k(A_n) \longrightarrow \pi_k(A) \longrightarrow \pi_{k-1}(F_n) \cdots ,$$

where the space $F_n$ can be obtained the following way: Take a point $p \in A$, and fix $\varepsilon > 0$ so that $\mathrm{dist}_p : A \to \mathbb{R}$ has no critical values in the interval $(0, 2\varepsilon)$. Consider a sequence of points $A_n \ni p_n \to p$ and take $F_n = B_\varepsilon(p_n) \subset A_n$. In particular, if $p$ is a regular point then for large $n$, $F_n$ is homotopy equivalent to a *regular fiber over* $p$[31].

Next we give two corollaries of the above remark. The last assertion of the following theorem was conjectured in [Shioya] and was proved in [Mendonça].

[30] i.e. Lipshitz and co-Lipschitz with constants almost 1.

[31] It is constructed the following way: take a distance chart $G : B_{2\varepsilon}(p) \to \mathbb{R}^k$, $k = \dim A$ around $p \in A$ and lift it to $A_n$. It defines a map $G_n : B_\varepsilon(p_n) \to \mathbb{R}^k$. Then take $F_n = G_n^{-1} \circ G(p)$ for large $n$. If $A_n$ are Riemannian then $F_n$ are manifolds and they do not depend on $p$ up to a homeomorphism. Moreover, $F_n$ are almost non-negatively curved in a generalized sense; see [KPT, definition 1.4].

**4.2.5. Theorem [Perelman 1997, 3.1].** *Let $M$ be a complete noncompact Riemannian manifold of nonnegative sectional curvature. Assume that its asymptotic cone* $\mathrm{Cone}_\infty(M)$ *has no proper extremal subsets, then $M$ splits isometrically into the product $L \times N$, where $L$ is a compact Riemannian manifold and $N$ is a non-compact Riemannian manifold of the same dimension as* $\mathrm{Cone}_\infty(M)$.

*In particular, the same conclusion holds if radius of the ideal boundary of $M$ is at least $\pi/2$.*

The proof is a direct application of theorem 4.2.3 and remark 4.2.4 for collapsing

$$\varepsilon M \xrightarrow{\mathrm{GH}} \mathrm{Cone}_\infty(M), \quad \text{as} \quad \varepsilon \to 0.$$

**4.2.6. Theorem [Perelman 1997, 3.2].** *Let $A_n \in \mathcal{A}lex^m(1)$, $A_n \xrightarrow{\mathrm{GH}} A$ be a collapsing sequence (i.e. $m > \dim A$), then* $\mathrm{Cone}(A)$ *has proper extremal subsets. In particular,* $\mathrm{rad}\, A \leqslant \pi/2$.

The last assertion of this theorem (in a stronger form) has been proven in [Grove–Petersen 1993, 3(3)].

The proof is a direct application of theorem 4.2.3 and remark 4.2.4 for collapsing of spherical suspensions

$$\Sigma(A_n) \xrightarrow{\mathrm{GH}} \Sigma(A), \quad n \to \infty.$$

# 5 Quasigeodesics

The class of quasigeodesics[32] generalizes the class of geodesics to nonsmooth metric spaces. It was first introduced in [Alexandrov 1945] for 2-dimensional convex hypersurfaces in the Euclidean space, as the curves which "turn" right and left simultaneously. This type of curves was studied further in [Alexandrov–Burago], [Pogorelov], [Milka 1971] and was generalized to surfaces with bounded integral curvature [Alexandrov 1949] and to multidimensional polyhedral spaces [Milka 1968], [Milka 1969]. For multi-dimensional Alexandrov's spaces they were introduced in [Perelman–Petrunin QG].

In Alexandrov's spaces, quasigeodesics behave more naturally than geodesics, mainly:

- ⋄ There is a quasigeodesic starting in any direction from any point;
- ⋄ The limit of quasigeodesics is a quasigeodesic.

Quasigeodesics have beauty on their own, but also due to the generalized Lieberman lemma (2.3.1), they are very useful in the study of intrinsic metric of extremal subsets, in particular the boundary of Alexandrov's space.

[32] It should be noted that the class of quasigeodesics described here has nothing to do with the Gromov's quasigeodesics in $\delta$-hyperbolic spaces.

Since quasigeodesics behave almost as geodesics, they are often used instead of geodesics in the situations when there is no geodesic in a given direction. In most of these applications one can instead use the radial curves of gradient exponent, see section 3; a good example is the proof of theorem 3.3.1, see footnote 20, page 22. In this type of argument, radial curves could be considered as a simpler and superior tool since they can be defined in a more general setting, in particular, for infinitely dimensional Alexandrov's spaces.

## 5.1 Definition and properties

In section 1, we defined $\lambda$-concave functions as those locally Lipschitz functions whose restriction to any unit-speed minimizing geodesic is $\lambda$-concave. Now consider a curve $\gamma$ in an Alexandrov's space such that restriction of any $\lambda$-concave function to $\gamma$ is $\lambda$-concave. It is easy to see that for any Riemannian manifold $\gamma$ has to be a unit-speed geodesic. In a general Alexandrov's space $\gamma$ should only be a quasigeodesic.

**5.1.1. Definition.** *A curve $\gamma$ in an Alexandrov's space is called* quasigeodesic *if for any $\lambda \in \mathbb{R}$, given a $\lambda$-concave function $f$ we have that $f \circ \gamma$ is $\lambda$-concave.*

Although this definition works for any metric space, it is only reasonable to apply it for the spaces where we have $\lambda$-concave functions, but not all functions are $\lambda$-concave, and Alexandrov's spaces seem to be the perfect choice.

The following is a list of corollaries from this definition:

1. Quasigeodesics are unit-speed curves. I.e., if $\gamma(t)$ is a quasigeodesic then for any $t_0$ we have
$$\lim_{t \to t_0} \frac{|\gamma(t)\gamma(t_0)|}{|t - t_0|} = 1.$$
To prove that quasigeodesic $\gamma$ is 1-Lipschitz at some $t = t_0$, it is enough to apply the definition for $f = \operatorname{dist}^2_{\gamma(t_0)}$ and use the fact that in any Alexandrov's space $\operatorname{dist}^2_p$ is $(2 + O(r^2))$-concave in $B_r(p)$. The lower bound is more complicated, see theorem 7.3.3.

2. For any quasigeodesic the right and left tangent vectors $\gamma^+$, $\gamma^-$ are uniquely defined unit vectors.
To prove, take a partial limits $\xi^\pm \in T_{\gamma(t_0)}$ for
$$\frac{\log_{\gamma(t_0)} \gamma(t_0 \pm \tau)}{\tau}, \quad \text{as } \tau \to 0+$$
It exists since quasigeodesics are 1-Lipschitz (see the previous property). For any semiconcave function $f$, $(f \circ \gamma)^\pm$ are well defined, therefore
$$(f \circ \gamma)^\pm(t_0) = d_{\gamma(t_0)} f(\xi^\pm).$$
Taking $f = \operatorname{dist}^2_q$ for different $q \in A$, one can see that $\xi^\pm$ is defined uniquely by this identity, and therefore $\gamma^\pm(t_0) = \xi^\pm$.

3. *Generalized Lieberman's Lemma.* Any unit-speed geodesic for the induced intrinsic metric on an extremal subset is a quasigeodesic in the ambient Alexandrov's space.

   See 2.3.1 for the proof and discussion.

4. For any point $x \in A$, and any direction $\xi \in \Sigma_x$ there is a quasigeodesic $\gamma : \mathbb{R} \to A$ such that $\gamma(0) = x$ and $\gamma^+(0) = \xi$.

   Moreover, if $E \subset A$ is an extremal subset and $x \in E$, $\xi \in \Sigma_x E$, then $\gamma$ can be chosen to lie completely in $E$.

   The proof is quite long, it is given in appendix A.

Applying the definition locally, we get that if $f$ is a $(1-\kappa f)$-concave function then $f \circ \gamma$ is $(1 - \kappa f \circ \gamma)$-concave (see section 1.2). In particular, if $A$ is an Alexandrov's space with curvature $\geqslant \kappa$, $p \in A$ and $h_p(t) = \rho_\kappa \circ \operatorname{dist}_p \circ \gamma(t)$[33] then we have the following inequality in the *barrier sense*

$$h_p'' \leqslant 1 - \kappa h_p.$$

This inequality can be reformulated in an equivalent way: Let $A \in \mathcal{A}lex^m(\kappa)$, $p \in A$ and $\gamma$ be a quasigeodesic, then function

$$t \mapsto \tilde\measuredangle_\kappa(|\gamma(0)p|, |\gamma(t)p|, t)$$

is decreasing for any $t > 0$ (if $\kappa > 0$ then one has to assume $t \leqslant \pi/\sqrt{\kappa}$).

In particular,

$$\tilde\measuredangle_\kappa(|\gamma(0)p|, |\gamma(t)p|, t) \leqslant \measuredangle(\uparrow_{\gamma(0)}^p, \gamma^+(0))$$

for any $t > 0$ (if $\kappa > 0$ then in addition $t \leqslant \pi/\sqrt{\kappa}$).

It also can be reformulated more geometrically using the notion of *developing* (see below):

*Any quasigeodesic in an Alexandrov's space with curvature $\geqslant \kappa$, has a convex $\kappa$-developing with respect to any point.*

**5.1.2. Definition of developing [Alexandrov 1957].** *Fix a real $\kappa$.*

*Let $X$ be a metric space, $\gamma : [a, b] \to X$ be a 1-Lipschitz curve and $p \in X \backslash \gamma$. If $\kappa > 0$, assume in addition that $|p\gamma(t)| < \pi/\sqrt{\kappa}$ for all $t \in [a, b]$.*

*Then there exists a unique (up to rotation) curve $\tilde\gamma : [a, b] \to \text{Л}_\kappa$, parametrized by the arclength, and such that $|o\tilde\gamma(t)| = |p\gamma(t)|$ for all $t$ and some fixed $o \in \text{Л}_\kappa$, and the segment $o\tilde\gamma(t)$ turns clockwise as $t$ increases (this is easy to prove). Such a curve $\tilde\gamma$ is called the* $\kappa$-development of $\gamma$ with respect to $p$.

*The development $\tilde\gamma$ is called* convex *if for every $t \in (a, b)$, for sufficiently small $\tau > 0$ the curvilinear triangle, bounded by the segments $o\tilde\gamma(t \pm \tau)$ and the arc $\tilde\gamma|_{t-\tau, t+\tau}$, is convex.*

In [Milka 1971], it has been proven that the developing of a quasigeodesic on a convex surface is convex.

[33] Function $\rho_\kappa : \mathbb{R} \to \mathbb{R}$ is defined on page 5

5. Let $A \in \mathrm{Alex}^m(\kappa)$, $m > 1$[34]. A curve $\gamma$ in $A$ is a quasigeodesic if and only if it is parametrized by arc-length and one of the following properties is fulfilled:

   (i) For any point $p \in A \backslash \gamma$ the $\kappa$-developing of $\gamma$ with respect to $p$ is convex.

   (ii) For any point $p \in A$, if $h_p(t) = \rho_\kappa \circ \mathrm{dist}_p \circ \gamma(t)$, then we have the following inequality in *a barrier sense*
   $$h_p'' \leqslant 1 - \kappa h_p.$$

   (iii) Function
   $$t \mapsto \tilde{\measuredangle}_\kappa(|\gamma(0)p|, |\gamma(t)p|, t)$$
   is decreasing for any $t > 0$ (if $\kappa > 0$ then in addition $t \leqslant \pi/\sqrt{\kappa}$).

   (iv) The inequality
   $$\measuredangle(\uparrow_{\gamma(0)}^{p}, \gamma^+(0)) \geqslant \tilde{\measuredangle}_\kappa(|\gamma(0)p|, |\gamma(t)p|, t)$$
   holds for all small $t > 0$.

   The "only if" part has already been proven above, and the "if" part follows from corollary 3.3.3

6. A pointwise limit of quasigeodesics is a quasigeodesic. More generally:

   *Assume $A_n \xrightarrow{\mathrm{GH}} A$, $A_n \in \mathrm{Alex}^m(\kappa)$, $\dim A = m$ (i.e. it is not a collapse). Let $\gamma_n : [a, b] \to A_n$ be a sequence of quasigeodesics which converges pointwise to a curve $\gamma : [a, b] \to A$. Then $\gamma$ is a quasigeodesic.*

   As it follows from lemma 7.2.3, the statement in the definition is correct for any $\lambda$-concave function $f$ which has controlled convexity type $(\lambda, \kappa)$. I.e. $\gamma$ satisfies the property 7.3.4. In particular, the $\kappa$-developing of $\gamma$ with respect to any point $p \in A$ is convex, and as it is noted in remark 7.3.5, $\gamma$ is a unit-speed curve. Therefore, from corollary 3.3.3 we get that it is a quasigeodesic.

Here is a list of open problems on quasigeodesics:

(i) Is there an analog of the Liouvile theorem for "quasigeodesic flow"?

(ii) Is it true that any finite quasigeodesic has bounded variation of turn?

   or

   Is it possible to approximate any finite quasigeodesic by sequence of broken lines with bounded variation of turn?

(iii) Is it true that in an Alexandrov's space without boundary there is an infinitely long geodesic?

[34]This condition is only needed to ensure that the set $A \backslash \gamma$ is everywhere dense.

As it was noted by A. Lytchak, the first and last questions can be reduced to the following: Assume $A$ is a compact Alexandrov's $m$-space without boundary. Let us set $V(r)=\int_A \mathrm{vol}_m(B_r(x))$, then $V(r)=\mathrm{vol}_m(A)\omega_m r^m+o(r^{m+1})$. The technique of *tight maps* makes it possible to prove only that $V(r)=\mathrm{vol}_m(A)\omega_m r^m+O(r^{m+1})$. Note that if $A$ is a Riemannian manifold with boundary then $V(r)=\mathrm{vol}_m(A)\omega_m r^m+\mathrm{vol}_{m-1}(\partial A)\omega'_m r^{m+1}+o(r^{m+1})$.

## 5.2 Applications.

The quasigeodesics is the main technical tool in the questions linked to the intrinsic metric of extremal subsets, in particular the boundary of Alexandrov's space. The main examples are the proofs of convergence of intrinsic metric of extremal subsets and the first variation formula (see properties 5ii and 6, on page 29).

Below we give a couple of simpler examples:

**5.2.1. Lemma.** *Let $A\in \mathrm{Alex}^m(1)$ and $\operatorname{rad} A>\pi/2$. Then for any $p\in A$ the space of directions $\Sigma_p$ has radius $>\pi/2$.*

*Proof.* Assume that $\Sigma_p$ has radius $\le\pi/2$, and let $\xi\in\Sigma_p$ be a direction, such that $\bar B_\xi(\pi/2)=\Sigma_p$. Consider a quasigeodesic $\gamma$ starting at $p$ in direction $\xi$.

Then for $q=\gamma(\pi/2)$ we have $\bar B_q(\pi/2)=A$. Indeed, for any point $x\in A$ we have $\measuredangle(\xi,\uparrow_p^x)\leqslant\pi/2$. Therefore, by the comparison inequality (property 5iv, page 35), $|xq|\leqslant\pi/2$. This contradicts our assumption that $\operatorname{rad} A>\pi/2$. □

**5.2.2. Corollary.** *Let $A\in\mathrm{Alex}^m(1)$ and $\operatorname{rad} A>\pi/2$ then*

*(i) $A$ has no extremal subsets.*

*(ii)* [Grove–Petersen 1993]*(radius sphere theorem) $A$ is homeomorphic to an $m$-sphere.*

Yet another proof of the radius sphere theorem follows immediately from [Perelman–Petrunin 1993, 1.2, 1.4.1]; theorem 4.2.1 gives a slight generalization.

*Proof.* Part (i) is obvious.

Part (ii): From lemma 5.2.1, $\operatorname{rad}\Sigma_p>\pi/2$. Since $\dim\Sigma_p<m$, by the induction hypothesis we have $\Sigma_p\simeq S^{m-1}$. Now the Morse lemma (see property 7, page 45) for $\mathrm{dist}_p: A\to\mathbb{R}$ gives that $A\simeq\Sigma(\Sigma_p)\simeq S^m$, here $\Sigma(\Sigma_p)$ denotes a spherical suspension over $\Sigma_p$. □

# 6 Simple functions

This is a short technical section. Here we introduce *simple functions*, a subclass of semiconcave functions which on one hand includes all functions we need and in addition is liftable; i.e. for any such function one can construct a nearby function on a nearby space with "similar" properties.

Our definition of simple function is a modification of two different definitions of so called "admissible functions" given in [Perelman 1993, 3.2] and [Kapovitch 2007, 5.1].

**6.1.1. Definition** *Let $A \in \mathrm{Alex}$, a function $f : A \to \mathbb{R}$ is called* simple *if there is a finite set of points $\{q_i\}_{i=1}^N$ and a semiconcave function $\Theta : \mathbb{R}^N \to \mathbb{R}$ which is non-decreasing in each argument such that*

$$f(x) = \Theta(\mathrm{dist}^2_{q_1}, \mathrm{dist}^2_{q_2}, \dots, \mathrm{dist}^2_{q_N})$$

It is straightforward to check that simple functions are semiconcave. Class of simple functions is closed under summation, multiplication by a positive constant[35] and taking the minimum.

In addition this class is liftable; i.e. given a converging sequence of Alexandrov's spaces $A_n \xrightarrow{\mathrm{GH}} A$ and a simple function $f : A \to \mathbb{R}$ there is a way to construct a sequence of functions $f_n : A_n \to \mathbb{R}$ such that $f_n \to f$. Namely, for each $q_i$ take a sequence $A_n \ni q_{i,n} \to q_i \in A$ and consider function $f_n : A_n \to \mathbb{R}$ defined by

$$f_n = \Theta(\mathrm{dist}^2_{q_{1,n}}, \mathrm{dist}^2_{q_{2,n}}, \dots, \mathrm{dist}^2_{q_{N,n}}).$$

## 6.2 Smoothing trick.

Here we present a trick which is very useful for doing local analysis in Alexandrov's spaces, it was introduced in [Otsu–Shioya, section 5].

Consider function

$$\widetilde{\mathrm{dist}}_p = \oint\limits_{B_\varepsilon(p)} \mathrm{dist}_x \, dx.$$

In this notation, we do not specify $\varepsilon$ assuming it to be very small. It is easy to see that $\widetilde{\mathrm{dist}}_p$ is semiconcave.

Note that

$$d_y \widetilde{\mathrm{dist}}_p = \oint\limits_{B_\varepsilon(p)} d_y \, \mathrm{dist}_x \, dx.$$

If $y \in A$ is regular, i.e. $T_y$ is isometric to Euclidean space, then for almost all $x \in B_\varepsilon(p)$ the differential $d_y \, \mathrm{dist}_x : T_y \to \mathbb{R}$ is a linear function. Therefore $\widetilde{\mathrm{dist}}_p$ is differentiable at every regular point, i.e.

$$d_y \widetilde{\mathrm{dist}}_p : T_y \to \mathbb{R}$$

is a linear function for any regular $y \in A$.

The same trick can be applied to any simple function

$$f(x) = \Theta(\mathrm{dist}^2_{q_1}, \mathrm{dist}^2_{q_2}, \dots, \mathrm{dist}^2_{q_N}).$$

[35] as well as multiplication by positive simple functions

This way we obtain function

$$\tilde{f}(x) = \oint_{B_\varepsilon(q_1)\times B_\varepsilon(q_2)\times\cdots\times B_\varepsilon(q_N)} \Theta(\mathrm{dist}^2_{x_1}, \mathrm{dist}^2_{x_2}, \dots, \mathrm{dist}^2_{x_N})\, dx_1 dx_2 \cdots dx_N,$$

which is differentiable at every regular point, i.e. if $T_y$ is isometric to the Euclidean space then

$$d_y\tilde{f} : T_y \to \mathbb{R}$$

is a linear function.

# 7 Controlled concavity

In this and the next sections we introduce a couple of techniques which use comparison of $m$-dimensional Alexandrov's space with a model space of the same dimension $Л^m_\kappa$ (i.e. simply connected Riemannian manifold with constant curvature $\kappa$). These techniques were introduced in [Perelman 1993] and [Perelman-DC].

We start with the local existence of a strictly concave function on an Alexandrov's space.

**7.1.1. Theorem [Perelman 1993, 3.6].** *Let $A \in \mathit{Alex}$.*

*For any point $p \in A$ there is a strictly concave function $f$ defined in an open neighborhood of $p$.*

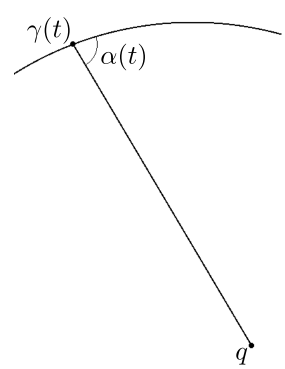
*Moreover, given $v \in T_p$, the differential, $d_p f(x)$, can be chosen arbitrarily close to $x \mapsto -\langle v, x\rangle$*

*Proof.* Consider the real function

$$\varphi_{r,c}(x) = (x - r) - c(x - r)^2/r,$$

so we have

$$\varphi_{r,c}(r) = 0,\ \ \varphi'_{r,c}(r) = 1\ \ \varphi''_{r,c}(r) = -2c/r.$$




Let $\gamma$ be a unit-speed geodesic, fix a point $q$ and set

$$\alpha(t) = \measuredangle(\gamma^+(t), \uparrow^q_{\gamma(t)}).$$

If $r > 0$ is sufficiently small and $|q\gamma(t)|$ is sufficiently close to $r$, then direct calculations show that

$$(\varphi_{r,c} \circ \mathrm{dist}_q \circ \gamma)''(t) \leqslant \frac{3 - c\cos^2\alpha(t)}{r}.$$

Now, assume $\{q_i\}$, $i = \{1, .., N\}$ is a finite set of points such that $|pq_i| = r$ for any $i$. For $x \in A$ and $\xi_x \in \Sigma_x$, set $\alpha_i(\xi_x) = \measuredangle(\xi_x, \uparrow^{q_i}_p)$. Assume we have a collection $\{q_i\}$ such that for any $x \in B_\varepsilon(p)$ and $\xi_x \in \Sigma_x$ we have $\max_i |\alpha_i(\xi_x) - \pi/2| \geqslant \varepsilon > 0$. Then taking in the above inequality $c > 3N/\cos^2\varepsilon$, we get that the function

$$f = \sum_i \varphi_{r,c} \circ \mathrm{dist}_{q_i}$$

is strictly concave in $B_{\varepsilon'}(p)$ for some positive $\varepsilon' < \varepsilon$.

To construct the needed collection $\{q_i\}$, note that for small $r > 0$ one can construct $N_\delta \geqslant \mathrm{Const}/\delta^{(m-1)}$ points $\{q_i\}$ such that $|pq_i| = r$ and $\tilde\measuredangle_\kappa q_i p q_j > \delta$ (here $\mathrm{Const} = \mathrm{Const}(\Sigma_p) > 0$). On the other hand, the set of directions which is orthogonal to a given direction is smaller than $S^{m-2}$ and therefore contains at most $\mathrm{Const}(m)/\delta^{(m-2)}$ directions with angles at least $\delta$. Therefore, for small enough $\delta > 0$, $\{q_i\}$ forms the needed collection.

If $r$ is small enough, points $q_i$ can be chosen so that all directions $\uparrow_p^{q_i}$ will be $\varepsilon$-close to a given direction $\xi$ and therefore the second property follows. □

Note that in the theorem 7.1.1 (as well as in theorem 7.2.2), the function $f$ can be chosen to have maximum value 0 at $p$, $f(p) = 0$ and with $d_p f(x)$ arbitrary close to $-|x|$. It can be constructed by taking the minimum of the functions in these theorems.

In particular it follows that

**7.1.2. Claim.** *For any point of an Alexandrov's space there is an arbitrary small closed convex neighborhood.*

By rescaling and passing to the limit one can even estimate the size of the convex hull in an Alexandrov's space in terms of the volume of a ball containing it:

**7.1.3. Lemma on strictly concave convex hulls [Perelman–Petrunin 1993, 4.3].** *For any $v > 0$, $r > 0$ and $\kappa \in \mathbb{R}$, $m \in \mathbb{N}$ there is $\varepsilon > 0$ such that, if $A \in \mathsf{Alex}^m(\kappa)$ and $\mathrm{vol}\, B_r(p) \geqslant v$ then for any $\rho < \varepsilon r$,*

$$\mathrm{diam}\,\mathrm{Conv}\, B_\rho(p) \leqslant \rho/\varepsilon.$$

*In particular, for any compact Alexandrov's $A$ space there is* Const $\in \mathbb{R}$ *such that for any subset $X \subset A$*

$$\mathrm{diam}\,(\mathrm{Conv}\, X) \leqslant \mathrm{Const} \cdot \mathrm{diam}\, X.$$

## 7.2 General definition.

The above construction can be generalized and optimized in many ways to fit particular needs. Here we introduce one such variation which is not the most general, but general enough to work in most applications.

Let $A$ be an Alexandrov's space and $f : A \to \mathbb{R}$,

$$f = \Theta(\mathrm{dist}^2_{q_1}, \mathrm{dist}^2_{q_2}, \dots, \mathrm{dist}^2_{q_N})$$

be a *simple function* (see section 6). If $A$ is $m$-dimensional, we say that such a function $f$ has *controlled concavity of type* $(\lambda,\kappa)$ at $p\in A$, if for any $\varepsilon>0$ there is $\delta>0$, such that for any collection of points $\{\tilde p,\tilde q_i\}$ in the *model $m$-space*[36] $\text{Л}^m_\kappa$ satisfying

$$|\tilde q_i\tilde q_j|>|q_iq_j|-\delta \ \text{ and } \ \big||\tilde p\tilde q_i|-|pq_i|\big|<\delta \ \text{ for all } \ i,j,$$

we have that the function $\tilde f:\text{Л}^m_\kappa\to\mathbb{R}$ defined by

$$\tilde f=\Theta(\operatorname{dist}^2_{\tilde q_1},\operatorname{dist}^2_{\tilde q_2},..,\operatorname{dist}^2_{\tilde q_n})$$

is $(\lambda-\varepsilon)$-concave in a small neighborhood of $\tilde p$.

The following lemma states that the conrolled concavity is stronger than the usual concavity.

**7.2.1. Lemma.** *Let $A\in\mathrm{Alex}^m(\kappa)$.*

*If a simple function*

$$f=\Theta(\operatorname{dist}^2_{q_1},\operatorname{dist}^2_{q_2},..,\operatorname{dist}^2_{q_N}),\quad f:A\to\mathbb{R}$$

*has a conrolled concavity type $(\lambda,\kappa)$ at each point $p\in\Omega$, then $f$ is $\lambda$-concave in $\Omega$.*

The proof is just a direct calculation similar to that in the proof of 7.1.1. Note also, that the function constructed in the proof of theorem 7.1.1 has controlled concavity. In fact from the same proof follows:

**7.2.2. Existence.** *Let $A\in\mathrm{Alex}$, $p\in A$, $\lambda,\kappa\in\mathbb{R}$. Then there is a function $f$ of controlled concavity $(\lambda,\kappa)$ at $p$.*

*Moreover, given $v\in T_p$, the function $f$ can be chosen so that its differential $d_pf(x)$ will be arbitrary close to $x\mapsto-\langle v,x\rangle$.*

Since functions with a conrolled concavity are simple they admit liftings, and from the definition it is clear that these liftings also have controlled concavity of the same type, i.e.

**7.2.3. Concavity of lifting.** *Let $A\in\mathrm{Alex}^m$.*

*Assume a simple function*

$$f:A\to\mathbb{R},\quad f=\Theta(\operatorname{dist}^2_{q_1},\operatorname{dist}^2_{q_2},..,\operatorname{dist}^2_{q_N})$$

*has controlled concavity type $(\lambda,\kappa)$ at $p$.*

*Let $A_n\in\mathrm{Alex}^m(\kappa)$, $A_n\xrightarrow{\mathrm{GH}}A$ (so, no collapse) and $\{p_n\},\{q_{i,n}\}\in A_n$ be sequences of points such that $p_n\to p\in A$ and $q_{i,n}\to q_i\in A$ for each $i$.*

*Then for all large $n$, the liftings of $f$,*

$$f_n:A_n\to\mathbb{R},\quad f_n=\Theta(\operatorname{dist}^2_{q_{1,n}},\operatorname{dist}^2_{q_{2,n}},..,\operatorname{dist}^2_{q_{N,n}})$$

*have controlled concavity type $(\lambda,\kappa)$ at $p_n$.*

In other words, if $f:A\to\mathbb{R}$ has controlled concavity type $(\lambda,\kappa)$ at all points of some open set $\Omega\subset A$, then $f_n:A_n\to\mathbb{R}$ have controlled concavity type $(\lambda,\kappa)$ at all points of some sequence of open sets $\Omega_n\subset A_n$, such that $\Omega_n$ complement-converges to $\Omega$ (i.e. $A_n\backslash\Omega_n\to A\backslash\Omega$ in Hausdorff sense).

[36]i.e. a simply connected $m$-manifold with constant curvature $\kappa$.

## 7.3 Applications

As was already noted, in the theorems 7.1.1 and 7.2.2, the function $f$ can be chosen to have a maximum value 0 at $p$, and with $d_p f(x)$ arbitrary close to $-|x|$. This observation was used in [Kapovitch 2002] to solve the second part of problem 32 from [Petersen 1996]:

**7.3.1. Petersen's problem.** *Let $A$ be a smoothable Alexandrov's $m$-space, i.e. there is a sequence of Riemannian $m$-manifolds $M_n$ with curvature $\geqslant \kappa$ such that $M_n \xrightarrow{\mathrm{GH}} A$.*

*Prove that the space of directions $\Sigma_x A$ for any point $x \in A$ is homeomorphic to the standard sphere.*

Note that Perelman's stability theorem (see [Perelman 1991], [Kapovitch 2007]) only gives that $\Sigma_x A$ has to be homotopically equivalent to the standard sphere.

*Sketch of the proof:* Fix a big negative $\lambda$ and construct a function $f : A \to \mathbb{R}$ with $d_p f(x) \approx -|x|$ and controlled concavity of type $(\lambda, \kappa)$. From 7.2.1, the liftings $f_n : M_n \to \mathbb{R}$ of $f$ (see 7.2.3) are strictly concave for large $n$. Let us slightly smooth the functions $f_n$ keeping them strictly concave. Then the level sets $f_n^{-1}(a)$, for values of $a$, which are little below the maximum of $f_n$, have strictly positive curvature and are diffeomorphic to the standard sphere[37].

Let us denote by $p_n \in M_n$ a maximum point of $f_n$. Then it is not hard to choose a sequence $\{a_n\}$ and a sequence of rescalings $\{s_n\}$ so that $(s_n M_n, p_n) \xrightarrow{\mathrm{GH}} (T_p, o_p)$ and $s_n f_n^{-1}(a_n) \subset s_n M_n$ converge to a convex hypersurface $S$ close to $\Sigma_p \subset T_p$. Then, from Perelman's stability theorem, it follows that $S$ and therefore $\Sigma_p$ is homeomorphic to the standard sphere. $\square$

**Remark.** From this proof it follows that $\Sigma_p$ is itself smoothable. Moreover, there is a non-collapsing sequence of Riemannian metrics $g_n$ on $S^{m-1}$ such that $(S^{m-1}, g_n) \xrightarrow{\mathrm{GH}} \Sigma_p$. This observation makes possible to proof a similar statement for iterated spaces of directions of smoothable Alexandrov space.

In the case of collapsing, the liftings $f_n$ of a function $f$ with controlled concavity type do not have the same controlled concavity type.

Nevertheless, the liftings are semiconcave and moreover, as was noted in [Kapovitch 2005], if $M_n$ is a sequence of $m+k$-dimensional Riemannian manifolds with curvature $\geqslant \kappa$, $M_n \xrightarrow{\mathrm{GH}} A$, $\dim A = m$, then one has a good control over the sum of $k+1$ maximal eigenvalues of their Hessians. In particular, a construction as in the proof of theorem 7.1.1 gives a strictly concave function on $A$ for which the liftings $f_n$ on $A_n$ have Morse index $\leqslant k$. It follows that one can retract an $\varepsilon$-neighborhood of $p_n$ to a $k$-dimensional CW-complex[38], where $p_n \in A_n$ is a maximum point of $f_n$ and $\varepsilon$ does not depend on $n$. This

[37] Since $f$ has only one critical value above $a$ and it is a local maximum.

[38] it is unknown whether it could be retracted to an $k$-submanifold. If true, it would give some interesting applications

observation gives a lower bound for the *codimension of a collapse*[39] to particular spaces. For example, for any lower curvature bound $\kappa$, the codimension of a collapse to $\Sigma(\mathbb{H}\mathrm{P}^m)$[40] is at least 3, and for $\Sigma(Ca\mathrm{P}^2)$ is at least 8 (it is expected to be $\infty$). In addition, it yields the following theorem, which seems to be the only sphere theorem which does not assume positiveness of curvature.

**7.3.2. Funny sphere theorem.** *If a $4(m+1)$ Riemannian manifold $M$ with sectional curvature $\geqslant \kappa$ is sufficiently close*[41] *to $\Sigma(\mathbb{H}\mathrm{P}^m)$, then it is homeomorphic to a sphere.*

The controlled concavity also gives a short proof of the following result:

**7.3.3. Theorem.** *Any quasigeodesic is a unit-speed curve.*

*Proof.* To prove that a quasigeodesic $\gamma$ is 1-Lipschitz at some $t = t_0$, it is enough to apply the definition for $f = \mathrm{dist}^2_{\gamma(t_0)}$ and use the fact that in any Alexandrov's space $\mathrm{dist}^2_p$ is $(2 + O(r^2))$-concave in $B_r(p)$.

Note that if $A_n, A \in \mathit{Alex}^m(\kappa)$, $A_n \xrightarrow{\mathrm{GH}} A$ without collapse, and $\gamma_n$ in $A_n$ is a sequence of quasigeodesics which converges to a curve $\gamma$ in $A$, then $\gamma$ has the following property[42]:

**7.3.4. Property.** *For any function $f$ on $A$ with controlled concavity type $(\lambda, \kappa)$ we have that $f \circ \gamma$ is $\lambda$-concave.*

If $\gamma$ is a quasigeodesic in $A$ with $\gamma(0) = p$, then the curves $\gamma(t/s)$ are quasigeodesics in $sA$. Therefore, as $s \to \infty$, the limit curve

$$\gamma_\infty(t) = \begin{bmatrix} |t|\gamma^+(0) & \text{if } t \geqslant 0 \\ |t|\gamma^-(0) & \text{if } t < 0 \end{bmatrix}$$

in $T_p$ has the above property. By a construction similar[43] to theorem 7.1.1, for any $\varepsilon > 0$ there is a function $f$ of controlled concavity type $(-2+\varepsilon, -\varepsilon)$ on a neighborhood of $\gamma^\pm \in T_p$ such that

$$f(t \cdot \gamma^\pm) = -(t-1)^2 + o((t-1)^2).$$

Applying the property above we get $|\gamma^\pm(0)| \geqslant 1$. □

[39] in our case, it is $k$; the difference between the dimension of spaces from the collapsing sequence and the dimension of the limit space

[40] i.e. a spherical suspension over $\mathbb{H}\mathrm{P}^m$

[41] i.e. $\varepsilon$-close for some $\varepsilon = \varepsilon(\kappa, m)$

[42] from statement 6, page 35, we that $\gamma$ is a quasigeodesic, but its proof is based on this theorem

[43] Setting $v = \gamma^\pm(0) \in T_p$ and $w = 2\gamma^\pm(0)$, this function can be presented as a sum

$$f = A(\varphi_{r,c} \circ \mathrm{dist}_o + \varphi_{r,c} \circ \mathrm{dist}_w) + B \sum_i \varphi_{r',c'} \circ \mathrm{dist}_{q_i},$$

for appropriately chosen positive reals $A$, $B$, $r$, $r'$, $c$, $c'$ and a collection of points $q_i$ such that, $\angle opq_i = \tilde{\angle}_0 opq_i = \pi/2$, $|pq_i| = r$ .

**7.3.5. Remark.** Note that we have proven a slightly stronger statement; namely, if a curve $\gamma$ satisfies the property 7.3.4 then it is a unit-speed curve.

**7.3.6. Question.** *Is it true that for any point $p \in A$ and any $\varepsilon > 0$, there is a $(-2+\varepsilon)$-concave function $f_p$ defined in a neighborhood of $p$, such that $f_p(p) = 0$ and $f_p \geqslant -\operatorname{dist}_p^2$?*

Existence of a such function would be a useful technical tool. In particular, it would allow for an easier proof of the above theorem.

# 8 Tight maps

The tight maps considered in this section give a more flexible version of distance charts.

Similar maps (so called *regular maps*) were used in [Perelman 1991] [Perelman 1993], and then they were modified to nearly this form in [Perelman-DC]. This technique is also useful for Alexandrov's spaces with upper curvature bound, see [Lytchak–Nagano].

**8.1.1. Definition.** *Let $A \in \mathrm{Alex}^m$ and $\Omega \subset A$ be an open subset. A collection of semiconcave functions $f_0, f_1, \dots, f_\ell$ on $A$ is called* tight in $\Omega$ *if*

$$\sup_{x\in\Omega,\, i\neq j} \{d_x f_i(\nabla_x f_j)\} < 0.$$

*In this case the map*

$$F : \Omega \to \mathbb{R}^{\ell+1}, \quad F : x \mapsto (f_0(x), f_1(x), \dots, f_\ell(x))$$

*is called* tight.

*A point $x \in \Omega$ is called a* critical point *of $F$ if $\min_i d_x f_i \leqslant 0$, otherwise the point $x$ is called* regular.

**8.1.2. Main example.** *If $A \in \mathrm{Alex}^m(\kappa)$ and $a_0, a_1, \dots, a_\ell, p \in A$ such that*

$$\tilde\measuredangle_\kappa a_i p a_j > \pi/2 \quad \textit{for all} \quad i \neq j$$

*then the map $x \mapsto (|a_0 x|, |a_1 x|, \dots, |a_\ell x|)$ is tight in a neighborhood of $p$.*

The inequality in the definition follows from inequality $(**)$ on page 9 and a subsequent to it example (ii).

This example can be made slightly more general. Let $f_0, f_1, ..., f_\ell$ be a collection of simple functions

$$f_i = \Theta_i(\operatorname{dist}^2_{a_{1,i}}, \operatorname{dist}^2_{a_{2,i}x}, \dots, \operatorname{dist}^2_{a_{n_i,i}x})$$

and the sets of points $K_i = \{a_{k,i}\}$ satisfy the following inequality

$$\tilde\measuredangle_\kappa xpy > \pi/2 \quad \text{for any} \quad x \in K_i, \quad y \in K_j, \quad i \neq j.$$

Then the map $x \mapsto (f_0(x), f_1(x), ..., f_\ell(x))$ is tight in a neighborhood of $p$. We will call such a map a *simple tight map*.

Yet further generalization is given in the property 1 below.

The maps described in this example have an important property, they are liftable and their lifts are tight. Namely, given a converging sequence $A_n \xrightarrow{\mathrm{GH}} A$, $A_n \in \mathrm{Alex}^m(\kappa)$ and a simple tight map $F : A \to \mathbb{R}^{\ell+1}$ around $p \in A$, the construction in section 6 gives simple tight maps $F_n : A_n \to \mathbb{R}^\ell$ for large $n$, $F_n \to F$.

I was unable to prove that tightness is a stable property in a sense formulated in the question below. It is not really important for the theory since all maps which appear naturally are simple (or, in the worst case they are as in the generalization and as in the property 1). However, for the beauty of the theory it would be nice to have a positive answer to the following question.

**8.1.3. Question.** *Assume $A_n \xrightarrow{\mathrm{GH}} A$, $A_n \in \mathrm{Alex}^m(\kappa)$, $f, g : A \to \mathbb{R}$ is a tight collection around $p$ and $f_n, g_n : A_n \to \mathbb{R}$, $f_n \to f$, $g_n \to g$ are two sequences of $\lambda$-concave functions and $A_n \ni p_n \to p \in A$. Is it true that for all large $n$, the collection $f_n, g_n$ must be tight around $p_n$?*

*If not, can one modify the definition of tightness so that*

*(i) it would be stable in the above sense,*

*(ii) the definition would make sense for all semiconcave functions*

*(iii) the maps described in the main example above are tight?*

Let us list some properties of tight maps with sketches of proofs:

1. Let $x \mapsto (f_0(x), f_1(x), ..., f_\ell(x))$ be a tight map in an open subset $\Omega \subset A$, then there is $\varepsilon > 0$ such that if $g_0, g_1, ..., g_n$ is a collection of $\epsilon$-Lipschitz semiconcave functions in $\Omega$ then the map
$$x \mapsto (f_0(x) + g_0(x), f_1(x) + g_1(x), ..., f_\ell(x) + g_\ell(x))$$
is also tight in $\Omega$.

2. The set of regular points of a tight map is open.

   Indeed, let $x \in \Omega$ be a regular point of tight map $F = (f_0, f_1, \dots, f_\ell)$. Take real $\lambda$ so that all $f_i$ are $\lambda$-concave in a neighborhood of $x$. Take a point $p$ sufficiently close to $x$ such that $d_x f_i(\uparrow_x^p) > 0$ and moreover $f_i(p) - f_i(x) > \lambda |xp|^2/2$ for each $i$. Then, from $\lambda$-concavity of $f_i$, there is a small neighborhood $\Omega_x \ni x$ such that for any $y \in \Omega_x$ and $i$ we have $d_y f_i(\uparrow_y^p) \geqslant \varepsilon$ for some fixed $\varepsilon > 0$.

3. If one removes one function from a tight collection (in $\Omega$) then (for the corresponding map) all points of $\Omega$ become regular. In other words, the projection of a tight map $F$ to any coordinate hyperplane is a tight map with all regular points (in $\Omega$).

   This follows from the property 3 on page 13 applied to the flow for the removed $f_i$.

4. The converse also holds, i.e. if $F$ is regular at $x$ then one can find a semiconcave function $g$ such that map $z \mapsto (F(z), g(z))$ is tight in a neighborhood of $x$. Moreover, $g$ can be chosen to have an arbitrary controlled concavity type.

   Indeed, one can take $g = \operatorname{dist}_p$, where $p$ as in the property 2. Then we have
$$d_x g(v) = -\max_{\xi \in \Uparrow_x^p} \langle \xi, v \rangle$$
   and therefore
$$d_x g(\nabla_x f_i) = -\max_{\xi \in \Uparrow_x^p} \langle \xi, \nabla_x f_i \rangle \leqslant -\max_{\xi \in \Uparrow_x^p} d_x f(\xi) \leqslant -\varepsilon.$$
   On the other hand, from inequality ($**$) on page 9 and example (ii) subsequent to it, we have
$$d_x f_i(\nabla_x g) + \min_{\xi \in \Uparrow_x^p} d_x f_i(\xi) \leqslant 0.$$
   The last statement follows from the construction in theorem 7.1.1.

5. A tight map is open and even *co-Lipschitz*[44] in a neighborhood of any regular point.

   This follows from lemma 8.1.4.

6. Let $A \in \mathcal{A}lex$, $\Omega \subset A$ be an open subset. If $F : \Omega \to \mathbb{R}^{\ell+1}$ is tight then $\ell \leqslant \dim A$.

   Follows from the properties 3 and 5.

7. *Morse lemma.* A tight map admits a local splitting in a neighborhood of its regular point, and a proper everywhere regular tight map is a locally trivial fiber bundle. Namely

   (i) If $F : \Omega \to \mathbb{R}^{\ell+1}$ is a tight map and $p \in \Omega$ is a regular point, then there is a neighborhood $\Omega \supset \Omega_p \ni p$ and homeomorphism
$$h : \Upsilon \times F(\Omega_p) \to \Omega_p,$$
   such that $F \circ h$ coincides with the projection to the second coordinate $\Upsilon \times F(\Omega_p) \to F(\Omega_p)$.

   (ii) If $F : \Omega \to \Delta \subset \mathbb{R}^{\ell+1}$ is a proper tight map and all points in $\Delta \subset \mathbb{R}^{\ell+1}$ are regular values of $F$, then $F$ is a locally trivial fiber bundle.

   The proof is a backward induction on $\ell$, see [Perelman 1993, 1.4], [Perelman 1991, 1.4.1] or [Kapovitch 2007, 6.7].

[44] A map $F : X \to Y$ between metric spaces is called $L$-co-Lipschitz in $\Omega \subset X$ if for any ball $B_r(x) \subset \Omega$ we have $F(B_r(x)) \supset B_{r/L}(F(x))$ in $Y$

The following lemma is an analog of lemmas [Perelman 1993, 2.3] and [Perelman-DC, 2.2].

**8.1.4. Lemma.** *Let $x$ be a regular point of a tight map*

$$F : x \mapsto (f_0(x), f_1(x), \dots, f_\ell(x)).$$

*Then there is $\varepsilon > 0$ and a neighborhood $\Omega_x \ni x$ such that for any $y \in \Omega_x$ and $i \in \{0, 1, \dots, \ell\}$ there is a unit vector $w_i \in \Sigma_x$ such that $d_x f_i(w_i) \geqslant \varepsilon$ and $d_x f_j(w_i) = 0$ for all $j \neq i$.*

*Moreover, if $E \subset A$ is an extremal subset and $y \in E$ then $w_i$ can be chosen in $\Sigma_y E$.*

*Proof.* Take $p$ as in the property 2 page 44. Then we can find a neighborhood $\Omega_x \ni x$ and $\varepsilon > 0$ so that for any $y \in \Omega_x$

(i) $d_y f_i(\uparrow_y^p) > \varepsilon$ for each $i$;

(ii) $-d_y f_i(\nabla_y f_j) > \varepsilon$. for all $i \neq j$.

Note that if $\alpha(t)$ is an $f_i$-gradient curve in $\Omega_x$ then

$$(f_i \circ \alpha)^+ > 0 \ \text{ and } \ (f_j \circ \alpha)^+ \leqslant -\varepsilon \ \text{ for any } \ j \neq i.$$

Applying lemma 2.1.5 for $(sA, y) \xrightarrow{\text{GH}} T_y$, $s[f_i - f_i(y)] \to d_y f_i$, we get the same inequalities for $d_y f_i$-gradient curves on $T_y$, i.e. if $\beta(t)$ is an $d_y f_i$-gradient curve in $T_y$ then

$$(d_y f_i \circ \beta)^+ > 0 \ \text{ and } \ (d_y f_j \circ \beta)^+ \leqslant -\varepsilon \ \text{ for any } \ j \neq i.$$

Moreover, $d_y f_i(v) > 0$ implies $\langle \nabla_v d_y f_i, \uparrow_v^o \rangle < 0$, therefore in this case $|\beta(t)|^+ > 0$.

Take $w_0 \in T_y$ to be a maximum point for $d_y f_0$ on the set

$$\{v \in T_y | f_i(v) \geqslant 0, |v| \leqslant 1\}.$$

Then

$$d_y f_0(w_0) \geqslant d_y f_0(\uparrow_y^p) > \varepsilon.$$

Assume for some $j \neq 0$ we have $f_j(w_0) > 0$. Then

$$\min_{i \neq j} \{d_{w_0} d_y f_i, d_{w_0} \nu\} \leqslant 0,$$

where the function $\nu$ is defined by $\nu : v \mapsto -|v|$; this is a concave function on $T_y$. Therefore, if $\beta_j(t)$ is a $d_y f_j$-gradient curve with an end[45] point at $w_0$, then moving along $\beta_j$ from $w_0$ backwards decreases only $d_y f_j$, and increases the other $d_y f_i$ and $\nu$ in the first order; this is a contradiction.

To prove the last statement it is enough to show that $w_0 \in T_y E$, which follows since $T_y E \subset T_y$ is an extremal subset (see property 2 on page 28). $\square$

[45] it does exist by property 3 on page 13

**8.1.5. Main theorem.** *Let $A \in \mathrm{Alex}^m(\kappa)$, $\Omega \subset A$ be the interior of a compact convex subset, and*

$$F : \Omega \to \mathbb{R}^{\ell+1}, \quad F : x \mapsto (f_0(x), f_1(x), \dots, f_\ell(x))$$

*be a tight map. Assume all $f_i$ are strictly concave. Then*

(i) *the set of critical points of $F$ in $\Omega$ forms an $\ell$-submanifold $M$*

(ii) *$F : M \to \mathbb{R}^{\ell+1}$ is an embedding.*

(iii) *$F(M) \subset \mathbb{R}^{\ell+1}$ is a convex hypersurface which lies in the boundary of $F(\Omega)$[46].*

**8.1.6. Remark.** The condition that all $f_i$ are strictly concave seems to be very restrictive, but that is not really so; if $x$ is a regular point of a tight map $F$ then, using properties 1 and 4 on page 44, one can find $\varepsilon > 0$ and $g$ such that

$$F' : y \mapsto (f_0(y) + \varepsilon g(y), \dots, f_\ell(y) + \varepsilon g(y), g(y))$$

is tight in a small neighborhood of $x$ and all its coordinate functions are strictly concave. In particular, in a neighborhood of $x$ we have

$$F = L \circ F'$$

where $L : \mathbb{R}^{\ell+2} \to \mathbb{R}^{\ell+1}$ is linear.

**8.1.7. Corollary.** *In the assumptions of theorem 8.1.5, if in addition $m = \ell$ then $M = \Omega$, $F(\Omega)$ is a convex hypersurface in $\mathbb{R}^{m+1}$ and $F : \Omega \to \mathbb{R}^{m+1}$ is a locally bi-Lipschitz embedding. Moreover, each projection of $F$ to a coordinate hyperplane is a locally bi-Lipschitz homeomorphism.*

*Proof of theorem 8.1.5.* Let $\gamma : [0, s] \to A$ be a minimal unit-speed geodesic connecting $x, y \in \Omega$, so $s = |xy|$. Consider a straight segment $\bar\gamma$ connecting $F(x)$ and $F(y)$:

$$\bar\gamma : [0, s] \to \mathbb{R}^{\ell+1}, \quad \bar\gamma(t) = F(x) + \tfrac{t}{s}\,[F(y) - F(x)]\,.$$

Each function $f_i \circ \gamma$ is concave, therefore all coordinates of

$$F \circ \gamma(t) - \bar\gamma(t)$$

are non-negative. This implies that the Minkowski sum[47]

$$Q = F(\Omega) + (\mathbb{R}_-)^{\ell+1}$$

is a convex set.

[46] In fact $F(M) = \partial F(\Omega) \cap F(\Omega)$.

[47] equivalently $Q = \{(x_0, x_1, \dots, x_\ell) \in \mathbb{R}^{\ell+1} | \exists (y_0, y_1, \dots, y_\ell) \in F(\Omega) \forall i\ x_i \leqslant y_i\}$.

Let $x_0 \in \Omega$ be a critical point of $F$. Since $\min_i d_{x_0} f_i \leqslant 0$, at least one of coordinates of $F(x)$ is smaller than the corresponding coordinate of $F(x_0)$ for any $x \in \Omega$. In particular, $F$ sends its critical point to the boundary of $Q$.

Consider map

$$G : \mathbb{R}^{\ell+1} \to A, \quad G : (y_0, y_1, \dots, y_\ell) \mapsto \operatorname{argmax}\{\min_i \{f_i - y_i\}\}$$

where $\operatorname{argmax}\{f\}$ denotes a maximum point of $f$. The function $\min_i\{f_i - y_i\}$ is strictly concave; therefore $\operatorname{argmax}\{\min_i\{f_i - y_i\}\}$ is uniquely defined and $G$ is continuous in the domain of definition.[48] The image of $G$ coincides with the set of critical points of $F$ and moreover $G \circ F|_M = \mathrm{id}_M$. Therefore $F|_M$ is a homeomorphism[49]. □

*Proof of corollary 8.1.7.* It only remains to show that $F$ is locally bi-Lipschitz.

Note that for any point $x \in \Omega$, one can find $\varepsilon > 0$ and a neighborhood $\Omega_x \ni x$, so that for any direction $\xi \in \Sigma_y$, $y \in \Omega_x$ one can choose $f_i$, $i \in \{0, 1, \dots, m\}$, such that $d_x f_i(\xi) \leqslant -\varepsilon$. Otherwise, by a slight perturbation[50] of collection $\{f_i\}$ we get a map $F : A^m \to \mathbb{R}^{m+1}$ regular at $y$, which contradicts property 5.

Therefore applying it for $\xi = \uparrow_z^y$ and $\uparrow_y^z$, $z, y \in \Omega$, we get two values $i, j$ such that

$$f_i(y) - f_i(z) \geqslant \varepsilon |yz| \quad \text{and} \quad f_j(z) - f_j(y) \geqslant \varepsilon |yz|.$$

Therefore $F$ is bi-Lipschits.

Clearly $i \neq j$ and therefore at least one of them is not zero. Hence the projection map $F' : x \mapsto (f_1(x), \dots, f_m(x))$ is also locally bi-Lipschitz. □

## 8.2 Applications.

One series of applications of tight maps is Morse theory for Alexandrov's spaces, it is based on the main theorem 8.1.5. It includes Morse lemma (property 7 page 45) and

- ⋄ *Local structure theorem [Perelman 1993].* Any small spherical neighborhood of a point in an Alexandrov's space is homeomorphic to a cone over its boundary.

[48] We do not need it, but clearly

$$G(y_0, y_1, \dots, y_\ell) = G(y_0 + h, y_1 + h, \dots, y_\ell + h)$$

for any $h \in \mathbb{R}$.

[49] In general, $G$ is not Lipschitz (even on $F(M)$); even in the case when all functions $f_i$ are $(-1)$-concave it is only possible to prove that $G$ is Hölder continuous of class $C^{0;\frac{1}{2}}$. (In fact the statement in [Perelman 1991], page 20, lines 23–25 is wrong but the proposition 3.5 is still OK.)

[50] as in the property 1 on page 44

◇ *Stability theorem [Perelman 1991].* For any compact $A \in \mathrm{Alex}^m(\kappa)$ there is $\varepsilon > 0$ such that if $A' \in \mathrm{Alex}^m(\kappa)$ is $\varepsilon$-close to $A$ then $A$ and $A'$ are homeomorphic.

The other series is the regularity results on an Alexandrov's space. These results obtained in [Perelman-DC] are improvements of earlier results in [Otsu–Shioya], [Otsu]. It use mainly the corollary 8.1.7 and the smoothing trick; see subsection 6.2.

◇ Components of metric tensor of an Alexandrov's space in a chart are continuous at each regular point[51]. Moreover they have bounded variation and are differentiable almost everywhere.

◇ The Christoffel symbols in a chart are well defined as signed Radon measures.

◇ Hessian of a semiconcave function on an Alexandrov's space is defined almost everywhere. I.e. if $f : \Omega \to \mathbb{R}$ is a semiconcave function, then for almost any $x_0 \in \Omega$ there is a symmetric bi-linear form $\mathrm{Hess}_f$ such that

$$f(x) = f(x_0) + d_{x_0} f(v) + \mathrm{Hess}_f(v, v) + o(|v|^2),$$

where $v = \log_{x_0} x$. Moreover, $\mathrm{Hess}_f$ can be calculated using standard formulas in the above chart.

Here is yet another, completely Riemannian application. This statement has been proven by Perelman, a sketch of its proof is included in an appendix to [Petrunin 2003]. The proof is based on the following observation: if $\Omega$ is an open subset of a Riemannian manifold and $F : \Omega \to \mathbb{R}^{\ell+1}$ is a tight map with strictly concave coordinate functions, then its level sets $F^{-1}(x)$ inherit the lower curvature bound.

◇ *Continuity of the integral of scalar curvature.* Given a compact Riemannian manifold $M$, let us define $\mathcal{F}(M) = \int_M \mathrm{Sc}$. Then $\mathcal{F}$ is continuous on the space of Riemannian $m$-dimensional manifolds with uniform lower curvature and upper diameter bounds.[52]

# 9 Please deform an Alexandrov's space.

In this section we discuss a number of related open problems. They seem to be very hard, but I think it is worth to write them down just to indicate the border between known and unknown things.

The main problem in Alexandrov's geometry is to find a way to vary Alexandrov's space, or simply to find a nearby Alexandrov's space to a given Alexandrov's space. Lack of such variation procedure makes it impossible to use Alexandrov's geometry in the way it was designed to be used:

[51] i.e. at each point with Euclidean tangent space

[52] In fact $\mathcal{F}$ is also bounded on the set of Riemannian $m$-dimensional manifolds with uniform lower curvature, this is proved in [Petrunin 2007] by a similar method.

For example, assume you want to solve the Hopf conjecture[53]. Assume it is wrong, then there is a volume maximizing Alexandrov's metrics $d$ on $S^2 \times S^2$ with curvature $\geqslant 1$[54]. Provided we have a procedure to vary $d$ while keeping its curvature $\geqslant 1$, we could find some special properties of $d$ and in ideal situation show that $d$ does not exist.

Unfortunately, at the moment, except for boring rescaling, there is no variation procedure available. The following conjecture (if true) would give such a procedure. Although it will not be sufficient to solve the Hopf conjecture, it will give some nontrivial information about the critical Alexandrov's metric.

**9.1.1. Conjecture.** *The boundary of an Alexandrov's space equipped with induced intrinsic metric is an Alexandrov's space with the same lower curvature bound.*

This also can be reformulated as:

**9.1.1′. Conjecture.** *Let $A$ be an Alexandrov space without boundary. Then a convex hypersurface in $A$ equipped with induced intrinsic metric is an Alexandrov's space with the same lower curvature bound.*

This conjecture, if true, would give a variation procedure. For example if $A$ is a non-negatively curved Alexandrov's space and $f : A \to \mathbb{R}$ is concave (so $A$ is necessarily open) then for any $t$ the graph

$$A_t = \{(x, tf(x)) \in A \times \mathbb{R}\}$$

with induced intrinsic metric would be an Alexandrov's space. Clearly $A_t \xrightarrow{\text{GH}} A$ as $t \to 0$. An analogous construction exists for semiconcave functions on closed manifolds, but one has to take a *parabolic cone*[55] instead of the product.

It seems to be hopeless to attack this problem with purely synthetic methods. In fact, so far, even for a convex hypersurface in a Riemannian manifold, there is only one proof available (see [Buyalo]) which uses smoothing and the Gauss formula[56]. There is one beautiful synthetic proof (see [Milka 1979]) for a convex surface in the Euclidian space, but this proof heavily relies on Euclidean structure and it seems impossible to generalize it even to the Riemannian case.

There is a chance of attacking this problem by proving a type of the Gauss formula for Alexandrov's spaces. One has to start with defining a curvature tensor of Alexandrov's spaces (it should be a measure-valued tensor field), then prove that the constructed tensor is really responsible for the geometry of the space. Such things were already done in the two-dimensional case and for spaces with bilaterly bounded curvature, see [Reshetnyak] and [Nikolaev] respectively.

[53] i.e. you want to find out if $S^2 \times S^2$ carries a metric with positive sectional curvature.

[54] There is no reason to believe that this metric $d$ is Riemannian, but from Gromov's compactness theorem such Alexandrov's metric should exist.

[55] see footnote 23 on page 23

[56] In fact in this paper the curvature bound is not optimal, but the statement follows from nearly the same idea; see **??**.

So far the best results in this direction are given in [Perelman-DC], see also section 8.2 for more details. This approach, if works, would give something really new in the area.

Almost everything that is known so far about the intrinsic metric of a boundary is also known for the intrinsic metric of a general extremal subset. In [Perelman–Petrunin 1993], it was conjectured that an analog of conjecture 9.1.1 is true for any *primitive extremal subset*, but it turned out to be wrong; a simple example was constructed in [Petrunin 1997]. All such examples appear when codimension of extremal subset is $\geqslant 3$. So it still might be true that

**9.1.2. Conjecture.** *Let* $A \in \mathsf{Alex}(\kappa)$, $E \subset A$ *be a primitive extremal subset and* $\operatorname{codim} E = 2$ *then* $E$ *equipped with induced intrinsic metric belongs to* $\mathsf{Alex}(\kappa)$

The following question is closely related to conjecture 9.1.1.

**9.1.3. Question.** *Assume* $A_n \xrightarrow{\mathrm{GH}} A$, $A_n \in \mathsf{Alex}^m(\kappa)$, $\dim A = m$ *(i.e. it is not a collapse).*

*Let* $f$ *be a* $\lambda$*-concave function of an Alexandrov's space* $A$. *Is it always possible to find a sequence of* $\lambda$*-concave functions* $f_n : A_n \to \mathbb{R}$ *which converges to* $f : A \to \mathbb{R}$*?*

Here is an equivalent formulation:

**9.1.3.′ Question.** *Assume* $A_n \xrightarrow{\mathrm{GH}} A$, $A_n \in \mathsf{Alex}^m(\kappa)$, $\dim A = m$ *(i.e. it is not a collapse) and* $\partial A = \varnothing$.

*Let* $S \subset A$ *be a convex hypersurface. Is it always possible to find a sequence of convex hypersurfaces* $S_n \subset A_n$ *which converges to* $S$*?*

If true, this would give a proof of conjecture 9.1.1 for the case of a *smoothable Alexandrov's space* (see page 41).

In most of (possible) applications, Alexandrov's spaces appear as limits of Riemannian manifolds of the same dimension. Therefore, even in this reduced generality, a positive answer would mean enough.

The question of whether an Alexandrov space is smoothable is also far from being solved. From Perelamn's stability theorem, if an Alexandrov's space has topological singularities then it is not smoothable. Moreover, from [Kapovitch 2002] one has that any space of directions of a smoothable Alexandrov's space is homeomorphic to the sphere. Except for the 2-dimensional case, it is only known that any polyhedral metric of non-negative curvature on a 3-manifold is smoothable (see [Matveev–Shevchishin]). There is yet no procedure of smoothing an Alexandrov's space even in a neighborhood of a regular point.

Maybe a more interesting question is whether smoothing is unique up to a diffeomorphism. If the answer is positive it would imply in particular that any Riemannian manifold with curvature $\geqslant 1$ and $\operatorname{diam} > \pi/2$ is diffeomorphic(!) to the standard sphere, see [Grove–Wilhelm] for details. Again, from Perelman's stability theorem ([Perelman 1991]), it follows that any two smoothings must be homeomorphic. In fact it seems likely that any two smoothings are PL-homeomorphic; see [Kapovitch 2007, question 1.3] and discussion right before it. It seems that today there is no technique which might approach the general uniqueness problem (so maybe one should try to construct a counterexample).

One may also ask similar questions in the collapsing case. In [PWZ] there were constructed Alexandrov's spaces with curvature $\geqslant 1$ which can not be presented as a limit of an (even collapsing) sequence of Riemannian manifolds with curvature $\geqslant \kappa > 1/4$. In [Kapovitch 2005] there were found some lower bounds for codimension of collapse with arbitrary lower curvature bound to some special Alexandrov's spaces, see section 7.3 for more discussion. It is expected that the same spaces (for example, the spherical suspension over the Cayley plane) can not be approximated by sequence of Riemannian manifolds of any fixed dimension and any fixed lower curvature bound, but so far this question remains open.

# A Existence of quasigeodesics

This appendix is devoted to the proof of property 4 on page 34, i.e.

**A.0.1. Existence theorem.** *Let $A \in \mathsf{Alex}^m$, then for any point $x \in A$, and any direction $\xi \in \Sigma_x$ there is a quasigeodesic $\gamma : \mathbb{R} \to A$ such that $\gamma(0) = x$ and $\gamma^+(0) = \xi$.*

*Moreover if $E \subset A$ is an extremal subset and $x \in E$, $\xi \in \Sigma_x E$ then $\gamma$ can be chosen to lie completely in $E$.*

The proof is quite long; it was obtained by Perelman around 1992; here we present a simplified proof similar to [Perelman–Petrunin QG] which is based on the gradient flow technique. We include a complete proof here, since otherwise it would never be published.

Quasigeodesics will be constructed in three big steps.

**A.1** Monotonic curves $\longrightarrow$ convex curves.

**A.2** Convex curves $\longrightarrow$ pre-quasigeodesics.

**A.3** Pre-quasigeodesics $\longrightarrow$ quasigeodesics.

In each step, we construct a better type of curves from a given type of curves by an extending-and-chopping procedure and then passing to a limit. The last part is most complicated.

The second part of the theorem is proved in the subsection A.4.

## A.0 Step 0: Monotonic curves

As a starting point we use radial curves, which do exist for any initial data (see section 3), and by lemma 3.1.2 are monotonic in the sense of the following definition:

**A.0.1. Definition.** *A curve $\alpha(t)$ in an Alexandrov's space $A$ is called* monotonic *with respect to a parameter value $t_0$ if for any $\lambda$-concave function $f$, $\lambda \geqslant 0$, we have that function*

$$t \mapsto \frac{f \circ \alpha(t+t_0) - f \circ \alpha(t_0) - \lambda t^2/2}{t}$$

*is non-increasing for $t > 0$.*

Here is a construction which gives a new monotonic curve out of two. It will be used in the next section to construct *convex curves.*

**A.0.2. Extention.** *Let $A \in \mathrm{Alex}$, $\alpha_1 [a, \infty) \to A$ and $\alpha_2 : [b, \infty) \to A$ be two monotonic curves with respect to $a$ and $b$ respectively.*

*Assume*

$$a \leqslant b, \quad \alpha_1(b) = \alpha_2(b) \quad \textit{and} \quad \alpha_1^+(b) = \alpha_2^+(b).$$

*Then its joint*

$$\beta : [a, \infty) \to A, \quad \beta(t) = \begin{bmatrix} \alpha_1(t) & \textit{if} & t < b \\ \alpha_2(t) & \textit{if} & t \geqslant b \end{bmatrix}$$

*is monotonic with respect to $a$ and $b$.*

*Proof.* It is enough to show that

$$t \mapsto \frac{f \circ \alpha_2(t+a) - f \circ \alpha_1(a) - \lambda t^2/2}{t}$$

is non-increasing for $t \geqslant b - a$. By simple algebra, it follows from the following two facts:

- ⋄ $\alpha_2$ is monotonic and therefore

$$t \mapsto \frac{f \circ \alpha_2(t+b) - f \circ \alpha_2(b) - \lambda t^2/2}{t}$$

is non-increasing for $t > 0$.

- ⋄ From monotonicity of $\alpha_1$,

$$(f \circ \alpha_2)^+(b) = d_{\alpha_1(b)} f(\alpha_1^+(b)) = (f \circ \alpha_1)^+(b) \leqslant$$

$$\leqslant \frac{f \circ \alpha_1(b) + f \circ \alpha_1(a) - \lambda (b-a)^2/2}{b-a}.$$

□

## A.1 Step 1: Convex curves.

In this step we construct *convex curves* with arbitrary initial data.

**A.1.1. Definition.** *A curve $\beta : [0, \infty) \to A$ is called* convex *if for any $\lambda$-concave function $f$, $\lambda \geqslant 0$, we have that function*

$$t \mapsto f \circ \beta(t) - \lambda t^2/2$$

*is concave.*

**Properties of convex curves.** Convex curves have the following properties; the proofs are either trivial or the same as for quasigeodesics:

1. A curve is convex if and only if it is monotonic with respect to any value of parameter.
2. Convex curves are 1-Lipschitz.
3. Convex curves have uniquely defined right and left tangent vectors.
4. A limit of convex curves is convex and the natural parameter converges to the natural parmeter of the limit curves (the proof the last statement is based on the same idea as theorem 7.3.3).

The next is a construction similar to A.0.2 which gives a new convex curve out of two. It will be used in the next section to construct *pre-quasigeodesics.*

**A.1.2. Extention.** *Let $A \in \mathsf{Alex}$, $\beta_1 : [a, \infty) \to A$ and $\beta_2 : [b, \infty) \to A$ be two convex curves. Assume*

$$a \leqslant b, \quad \beta_1(b) = \beta_2(b) \quad \textit{and} \quad \beta_1^+(b) = \beta_2^+(b)$$

*then its joint*

$$\gamma : [a, \infty) \to A, \quad \gamma(t) = \begin{bmatrix} \beta_1(t) & \textit{if} & t \leqslant b \\ \beta_2(t) & \textit{if} & t \geqslant b \end{bmatrix}$$

*is a convex curve.*

*Proof.* Follows immidetely from A.0.2 and property 1 above.

**A.1.3. Existence.** *Let $A \in \mathsf{Alex}$, $x \in A$ and $\xi \in \Sigma_x$. Then there is a convex curve $\beta_\xi : [0, \infty) \to A$ such that $\beta_\xi(0) = x$ and $\beta_\xi^+(0) = \xi$.*

*Proof.* For $v \in T_x A$, consider the radial curve

$$\alpha_v(t) = \mathrm{gexp}_x(tv)$$

According to lemma 3.1.2 if $|v| = 1$ then $\alpha_v$ is 1-Lipschitz and monotonic. Moreover, straightforward calculations show that the same is true for $|v| \leqslant 1$.

Fix $\varepsilon > 0$. Given a direction $\xi \in \Sigma_x$, let us consider the following recursively defined sequence of radial curves $\alpha_{v_n}(t)$ such that $v_0 = \xi$ and $v_n = \alpha^+_{v_{n-1}}(\varepsilon)$. Then consider their joint

$$\beta_{\xi,\varepsilon}(t) = \alpha_{v_{\lfloor t/\varepsilon \rfloor}}(t - \varepsilon\lfloor t/\varepsilon \rfloor).$$

Applying an extension procedure A.0.2 we get that $\beta_{\xi,\varepsilon} : [0,\infty) \to A$ is monotonic with respect to any $t = n\varepsilon$.

By property 1 on page 54, passing to a partial limit $\beta_{\xi,\varepsilon} \to \beta_\xi$ as $\varepsilon \to 0$ we get a convex curve $\beta_\xi : [0,\infty) \to A$.

It only remains to show that $\beta^+_\xi(0) = \xi$.

Since $\beta_\xi$ is convex, its right tangent vector is well defined and $|\beta^+_\xi(0)| \leqslant 1$[57]. On the other hand, since $\beta_{\xi,\varepsilon}$ are monotonic with respect to $0$, for any semiconcave function $f$ we have

$$d_x f(\beta^+_\xi(0)) = (f \circ \beta_\xi)^+(0) \leqslant \lim_{\varepsilon_i \to 0} (f \circ \beta_{\xi,\varepsilon})^+(0) = d_x f(\xi).$$

Substituting in this inequality $f = \mathrm{dist}_y$ with $\measuredangle(\uparrow^y_x, \xi) < \varepsilon$, we get

$$\langle \beta^+_\xi(0), \uparrow^y_x \rangle > 1 - \varepsilon$$

for any $\varepsilon > 0$. Together with $|\beta^+_\xi(0)| \leqslant 1$ (property 2 on page 54), it implies that

$$\beta^+(0) = \xi.$$

□

## A.2 Step 2: Pre-quasigeodesics

In this step we construct a *pre-quasigeodesic* with arbitrary initial data.

**A.2.1. Definition.** *A convex curve $\gamma : [a,b) \to A$ is called a pre-quasigeodesic if for any $s \in [a,b)$ such that $|\gamma^+(s)| > 0$, the curve $\gamma^s$ defined by*

$$\gamma^s(t) = \gamma\left(s + \frac{t}{|\gamma^+(s)|}\right)$$

*is convex for $t \geqslant 0$, and if $|\gamma^+(s)| = 0$ then $\gamma(t) = \gamma(s)$ for all $t \geqslant s$.*

Let us first define entropy of pre-quasigeodesic, which measures "how far" a given pre-quasigeodesic is from being a quasigeodesic.

**A.2.2. Definition.** *Let $\gamma$ be a pre-quasigeodesic in an Alexandrov's space.*

*The* entropy *of $\gamma$, $\mu_\gamma$ is the measure on the set of parameters defined by*

$$\mu_\gamma((a,b)) = \ln|\gamma^+(a)| - \ln|\gamma^-(b)|.$$

[57] see properties 3 and 2, page 54

Here are its main properties:

1. The entropy of a pre-quasigeodesic $\gamma$ is zero if and only if $\gamma$ is a quasigedesic.
2. For a converging sequence of pre-quasigeodesics $\gamma_n \to \gamma$, the entropy of the limit is a weak limit of entropies, $\mu_{\gamma_n} \rightharpoonup \mu_\gamma$.

   It follows from property 4 on page 54.

The next statement is similar to A.0.2 and A.1.2; it makes a new pre-quasigeodesic out of two. It will be used in the next section to construct *quasi-geodesics.*

**A.2.3. Extention.** *Let $A \in \mathrm{Alex}$, $\gamma_1 : [a,\infty) \to A$ and $\gamma_2 : [b,\infty) \to A$ be two pre-quasigeodesics. Assume*

$$a \leqslant b, \quad \gamma_1(b) = \gamma_2(b), \quad \gamma_1^-(b) \ \textit{ is polar to } \ \gamma_2^+(b) \ \textit{ and } \ |\gamma_2^+(b)| \leqslant |\gamma_1^-(b)|$$

*then its joint*

$$\gamma : [a,\infty) \to A, \quad \gamma(t) = \begin{bmatrix} \gamma_1(t) & \textit{if} & t \leqslant b \\ \gamma_2(t) & \textit{if} & t \geqslant b \end{bmatrix}$$

*is a pre-quasigeodesic. Moreover, its entropy is defined by*

$$\mu_\gamma|_{(a,b)} = \mu_{\gamma_1}, \quad \mu_\gamma|_{(b,c)} = \mu_{\gamma_2} \quad \textit{and} \quad \mu_\gamma(\{b\}) = \ln|\gamma^+(b)| - \ln|\gamma^-(b)|.$$

*Proof.* The same as for A.0.2. □

**A.2.4. Existence.** *Let $A \in \mathrm{Alex}$, $x \in A$ and $\xi \in \Sigma_x$. Then there is a pre-quasigeodesic $\gamma : [0,\infty) \to A$ such that $\gamma(0) = x$ and $\gamma^+(0) = \xi$.*

*Proof.* Let us choose for each point $x \in A$ and each direction $\xi \in \Sigma_x$ a convex curve $\beta_\xi : [0,\infty) \to A$ such that $\beta_\xi(0) = x$, $\beta_\xi^+(0) = \xi$. If $v = r\xi$, then set

$$\beta_v(t) = \beta_\xi(rt).$$

Clearly $\beta_v$ is convex if $0 \leqslant r \leqslant 1$.

Let us construct a convex curve $\gamma_\varepsilon : [0,\infty) \to M$ such that there is a representation of $[0,\infty)$ as a countable union of disjoint half-open intervals $[a_i, \bar a_i)$, such that $|\bar a_i - a_i| \leqslant \varepsilon$ and for any $t \in [a_i, \bar a_i)$ we have

$$|\gamma_\varepsilon^+(a_i)| \geqslant |\gamma_\varepsilon^+(t)| \geqslant (1-\varepsilon)|\gamma_\varepsilon^+(a_i)|. \qquad (*)$$

Moreover, for each $i$, the curve $\gamma_\varepsilon^{a_i} : [0,\infty) \to A$,

$$\gamma_\varepsilon^{a_i}(t) = \gamma_\varepsilon\left(a_i + \frac{t}{|\gamma_\varepsilon^+(a_i)|}\right)$$

is also convex.

Assume we already can construct $\gamma_\varepsilon$ in the interval $[0, t_{\max})$, and cannot do it any further. Since $\gamma_\varepsilon$ is 1-Lipschitz, we can extend it continuously to $[0, t_{\max}]$. Use lemma 1.3.9 to construct a vector $v^*$ polar to $\gamma_\varepsilon^-(t_{\max})$ with $|v^*| \leqslant |\gamma_\varepsilon^-(t_{\max})|$. Consider the joint of $\gamma_\varepsilon$ with a short half-open segment of $\beta_v$, a longer curve with the desired property. This is a contradiction.

Let $\gamma$ be a partial limit of $\gamma_\varepsilon$ as $\varepsilon \to 0$. From property 4 on page 54, we get that for almost all $t$ we have $|\gamma^+(t)| = \lim |\gamma_{\varepsilon_n}^+(t)|$. Combining this with inequality $(*)$ shows that for any $a \geqslant 0$

$$\gamma^a(t) = \gamma\left(a + \frac{t}{|\gamma^+(a)|}\right)$$

is convex. □

## A.3 Step 3: Quasigeodesics

We will construct quasigeodesics in an $m$-dimensional Alexandrov's space, assuming we already have such a construction in all dimensions $< m$. This construction is much easier for the case of an Alexandrov's space with only $\delta$-*strained points*; in this case we construct a sequence of special pre-quasigeodesics only by extending/chopping procedures (see below) and then pass to the limit. In a general Alexandrov's space we argue by contradiction, we assume that $\Omega$ is a maximal open set such that for any initial data one can construct an $\Omega$-quasigeodesic (i.e. a pre-quasigeodesic with zero *entropy* on $\Omega$, see A.2.2), and arrive at a contradiction with the assumption $\Omega \neq A$.

The following extention and chopping procedures are essential in the construction:

**A.3.1. Extention procedure.** *Given a pre-quasigeodesic $\gamma : [0, t_{\max}) \to A$ we can extend it as a pre-quasigeodesic $\gamma : [0, \infty) \to A$ so that*

$$\mu_\gamma(\{t_{\max}\}) = 0.$$

*Proof.* Let us set $\gamma(t_{\max})$ to be the limit of $\gamma(t)$ as $t \to t_{\max}$ (it exists since pre-quasigeodesics are Lipschitz).

From Milka's lemma A.3.2, we can construct a vector $\gamma^+(t_{\max})$ which is polar to $\gamma^-(t_{\max})$ and such that $|\gamma^+(t_{\max})| = |\gamma^-(t_{\max})|$. Then extend $\gamma$ by a pre-quasigeodesic in the direction $\gamma^+(t_{\max})$. By A.2.3, we get

$$\mu_\gamma\{t_{\max}\} = \ln|\gamma^+(t_{\max})| - \ln|\gamma^-(t_{\max})| = 0.$$

□

**A.3.2. Milka's lemma** *(existence of the polar direction). For any unit vector $\xi\in\Sigma_p$ there is a polar unit vector $\xi^*$, i.e. $\xi^*\in\Sigma_p$ such that*

$$\langle\xi,v\rangle+\langle\xi^*,v\rangle\geqslant 0$$

*for any $v\in T_p$.*

The proof is taken from [Milka 1968]. That is the only instance where we use existence of quasigeodesics in lower dimensional spaces.

*Proof.* Since $\Sigma_p$ is an Alexandrov's $(m-1)$-space with curvature $\geqslant 1$, given $\xi\in\Sigma_p$ we can construct a quasigeodesic in $\Sigma_p$ of length $\pi$, starting at $\xi$; the comparison inequality (theorem 5(5iv)) implies that the second endpoint $\xi^*$ of this quasigeodesic satisfies

$$|\xi\,\eta|_{\Sigma_q}+|\eta\,\xi^*|_{\Sigma_q}=\measuredangle(\xi,\eta)+\measuredangle(\eta,\xi^*)\leq\pi\quad\text{for all}\quad\eta\in\Sigma_p,$$

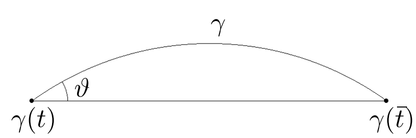
which is equivalent to the statement that $\xi$ and $\xi^*$ are polar in $T_p$. $\square$

**A.3.3. Chopping procedure.** *Given a pre-quasigeodesic $\gamma:[0,\infty)\to A$, for any $t\geqslant 0$ and $\varepsilon>0$ there is $\bar t>t$ such that*

$$\mu_\gamma\left((t,\bar t)\right)<\varepsilon[\vartheta+\bar t-t],\quad \bar t-t<\varepsilon,\quad \vartheta<\varepsilon,$$

*where*

$$\vartheta=\vartheta(t,\bar t)=\measuredangle\left(\gamma^+(t),\uparrow_{\gamma(t)}^{\gamma(\bar t)}\right).$$




*Proof.* For all sufficiently small $\tau>0$ we have

$$\vartheta(t,t+\tau)<\varepsilon$$

and from convexity of $\gamma^t$ it follows that

$$\mu\left((t,t+\tau/3)\right)<C\vartheta^2(t,t+\tau).$$

The following exercise completes the proof. $\square$

**A.3.4. Exercise.** *Let the functions $h, g : \mathbb{R}_+ \to \mathbb{R}_+$ be such that for any sufficiently small $s$,*

$$h(s/3) \leqslant g^2(s), \;\; s \leqslant g(s) \;\text{ and }\; \lim_{s\to 0} g(s) = 0.$$

*Show that for any $\varepsilon > 0$ there is $s > 0$ such that*

$$h(s) < 10g^2(s) \;\text{ and }\; g(s) \leqslant \varepsilon.$$

**Construction in the $\delta$-strained case.** From the extension procedure, it is sufficient to construct a quasigeodesic $\gamma : [0, T) \to A$ with any given initial data $\gamma^+(0) = \xi \in \Sigma_p$ for some positive $T = T(p)$.

*The plan:* Given $\varepsilon > 0$, we first construct a pre-quasigeodesic

$$\gamma_\varepsilon : [0, T) \to A, \quad \gamma_\varepsilon^+(0) = \xi$$

such that one can present $[0, T)$ as a countable union of disjoint half-open intervals $[a_i, \bar{a}_i)$ with the following property ($\vartheta$ is defined in the chopping procedure A.3.3):

$$\mu\left([a_i, \bar{a}_i)\right) < \varepsilon\vartheta(a_i, \bar{a}_i), \quad \bar{a}_i - a_i < \varepsilon, \quad \vartheta(a_i, \bar{a}_i) < \varepsilon. \tag{$\star$}$$

Then we show that the entropies $\mu_{\gamma_\varepsilon}([0, T)) \to 0$ as $\varepsilon \to 0$ and passing to a partial limit of $\gamma_\varepsilon$ as $\varepsilon \to 0$ we get a quasigeodesic.

*Existence of $\gamma_\varepsilon$:* Assume that we already can construct $\gamma_\varepsilon$ on an interval $[0, t_{\max})$, $t_{\max} < T$ and cannot construct it any further, then applying the extension procedure A.3.1 for $\gamma_\varepsilon : [0, t_{\max}) \to A$ and then chopping it (A.3.3) starting from $t_{\max}$, we get a longer curve with the desired property; that is a contradiction.

*Vanishing entropy:* From ($\star$) we have that

$$\mu_{\gamma_\varepsilon}([0, T)) < \varepsilon \left[ T + \sum_i \vartheta(a_i, \bar{a}_i) \right].$$

Therefore, to show that $\mu_{\gamma_\varepsilon}([0, T)) \to 0$, it only remains to show that $\sum_i \vartheta(a_i, \bar{a}_i)$ is bounded above by a constant independent of $\varepsilon$.

That will be the only instance, where we apply that $p$ is $\delta$-strained for a small enough $\delta$.

It is easy to see that there is $\varepsilon = \varepsilon(\delta) \to 0$ as $\delta \to 0$ and $T = T(p) > 0$ such that there is a finite collection of points $\{q_k\}$ which satisfy the following property: for any $x \in B_T(p)$ and $\xi \in \Sigma_x$ there is $q_k$ such that $\measuredangle(\xi, \uparrow_x^{q_k}) < \varepsilon$. Moreover, we can assume $\operatorname{dist}_{q_k}$ is $\lambda$-concave in $B_T(p)$ for some $\lambda > 0$.

Note that for any convex curve $\gamma : [0, T) \to B_T(p) \subset A$, the measures $\chi_k$ on $[0, T)$, defined by

$$\chi_k((a, b)) = (\operatorname{dist}_{q_k} \circ \gamma)^-(b) - (\operatorname{dist}_{q_k} \circ \gamma)^+(a) + \lambda(b - a),$$

are positive and their total mass is bounded by $\lambda T+2$ (this follows from the fact that $\operatorname{dist}_{q_k}$ is $\lambda$-concave and 1-Lipschitz).

Let $x\in B_T(p)$, and $\delta$ be small enough. Then for any two directions $\xi,\nu\in\Sigma_x$ there is $q_k$ which satisfies the following property:

$$\frac{1}{10}\measuredangle_x(\xi,\nu)\leqslant d_x\operatorname{dist}_{q_k}(\xi)-d_x\operatorname{dist}_{q_k}(\nu)\quad\text{and}\quad d_x\operatorname{dist}_{q_k}(\nu)\geqslant 0. \qquad (*)$$

Substituting in this inequality

$$\xi=\gamma^+(a_i)/|\gamma^+(a_i)|,\quad \nu={\uparrow}_{\gamma(a_i)}^{\gamma(\bar a_i)},$$

and applying lemma A.3.5, we get

$$\vartheta(a_i,\bar a_i)=\measuredangle(\xi,\nu)\leqslant 10\sum_n\chi_k([a_i,\bar a_i]).$$

Therefore

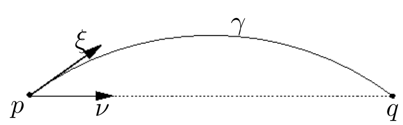
$$\sum_i\vartheta(a_i,\bar a_i)\leqslant 10N(\lambda T+2),$$

where $N$ is the number of points in the collection $\{q_k\}$. $\square$

**A.3.5. Lemma.** *Let $A\in\mathbb{A}\mathrm{lex}$, $\gamma:[0,t]\to A$ be a convex curve $|\gamma^+(0)|=1$ and $f$ be a $\lambda$-concave function, $\lambda\geqslant 0$. Set $p=\gamma(0)$, $q=\gamma(t)$, $\xi=(\gamma)^+(0)$ and $\nu={\uparrow}_p^q$. Then*

$$d_pf(\xi)-d_pf(\nu)\leqslant(f\circ\gamma)^+(0)-(f\circ\gamma)^-(t)+\lambda t,$$

*provided that $d_pf(\nu)\geqslant 0$.*




*Proof.* Clearly,

$$f(q)\leq f(p)+d_pf(\nu)|pq|+\lambda|pq|^2/2\leqslant f(p)+d_pf(\nu)t+\lambda t^2/2.$$

On the other hand,

$$f(p)\leqslant f(q)-(f\circ\gamma)^-(t)t+\lambda t^2/2.$$

Clearly, $d_pf(\xi)=(f\circ\gamma)^+(0)$, whence the result. $\square$

**What to do now?** We have just finished the proof for the case, where all points of $A$ are $\delta$-strained. From this proof it follows that if we denote by $\Omega_\delta$ the subset of all $\delta$-strained points of $A$ (which is an open everywhere dense set, see [BGP, 5.9]), then for any initial data one can construct a pre-quasigeodesic $\gamma$ such that $\mu_\gamma(\gamma^{-1}(\Omega_\delta)) = 0$. Assume $A$ has no boundary; set $\mathfrak{C} = A\backslash\Omega_\delta$. In this case it seems unlikely that we hit $\mathfrak{C}$ by shooting a pre-quasigeodesic in a generic direction. If we could prove that it almost never happens, then we obtain existence of quasigeodesics in all directions as the limits of quasigeodesics in generic directions (see property 6 on page 35) and passing to doubling in case $\partial A \neq \varnothing$. Unfortunately, we do not have any tools so far to prove such a thing[58]. Instead we generalize inequality $(*)$.

**A.3.6. The $(*)$ inequality.** *Let $A \in \mathrm{Alex}^m(\kappa)$ and $\mathfrak{C} \subset A$ be a closed subset. Let $p \in \mathfrak{C}$ be a point with $\delta$-maximal* $\mathrm{vol}_{m-1}\Sigma_p$*, i.e.*

$$\mathrm{vol}_{m-1}\Sigma_p + \delta > \inf_{x\in\mathfrak{C}} \mathrm{vol}_{m-1}\Sigma_p.$$

*Then, if $\delta$ is small enough, there is a finite set of points $\{q_i\}$ and $\varepsilon > 0$, such that for any $x \in \mathfrak{C}\cap \bar{B}_\varepsilon(p)$ and any pair of directions $\xi \in \Sigma_x\mathfrak{C}$[59] and $\nu \in \Sigma_x$ we can choose $q_i$ so that*

$$\frac{1}{10}\measuredangle_x(\xi,\nu) \leqslant d_x\,\mathrm{dist}_{q_i}(\xi) - d_x\,\mathrm{dist}_{q_i}(\nu) \quad \textit{and} \quad d_x\,\mathrm{dist}_{q_k}(\nu) \geqslant 0.$$

*Proof.* We can choose $\varepsilon > 0$ so small that for any $x \in \bar{B}_\varepsilon(p)$, $\Sigma_x$ is almost bigger than $\Sigma_p$.[60] Since $\mathrm{vol}_{m-1}\Sigma_p$ is almost maximal we get that for any $x \in \mathfrak{C}\cap\bar{B}_\varepsilon(p)$, $\Sigma_x$ is almost isometric to $\Sigma_p$. In particular, if one takes a set $\{q_i\}$ so that directions $\uparrow_p^{q_i}$ form a sufficiently dense set and $\measuredangle q_ipq_j \approx \tilde{\measuredangle}_\kappa q_ipq_j$, then directions $\uparrow_x^{q_i}$ will form a sufficiently dense set in $\Sigma_x$ for all $x \in \mathfrak{C}\cap\bar{B}_\varepsilon(p)$.

Note that for any $x \in \mathfrak{C}\cap\bar{B}_\varepsilon(p)$ and $\xi \in \Sigma_x\mathfrak{C}$, there is an almost isometry $\Sigma_x \to \Sigma(\Sigma_\xi\Sigma_x)$ such that $\xi$ goes to north pole of the spherical suspension $\Sigma(\Sigma_\xi\Sigma_x) = \Sigma_\xi T_x$. [61]

Using these two properties, we can find $q_i$ so that $\uparrow_\xi^\nu \approx \uparrow_\xi^{\uparrow_x^{q_i}}$ in $\Sigma_\nu\Sigma_xA$ and $\measuredangle(\xi,\uparrow_x^{q_i}) > \pi/2$, hence the statement follows. ☐

Now we are ready to finish construction in the general case. Let us define a subtype of pre-quasigeodesics:

**A.3.7. Definition.** *Let $A \in \mathrm{Alex}$ and $\Omega \subset A$ be an open subset. A pre-quasigeodesic $\gamma : [0,T) \to A$ is called $\Omega$-quasigeodesic if its entropy vanishes on*

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[58] It might be possible if we would have an analog of the Liouvile theorem for "pre-quasigeodesic flow"

[59] $\Sigma_x\mathfrak{C}$ is defined on page 28.

[60] i.e. for small $\delta > 0$ there is a map $f : \Sigma_p \to \Sigma_x$ such that $|f(x)f(y)| > |xy| - \delta$.

[61] Otherwise, taking a point $y \in \mathfrak{C}$, close to $x$ in direction $\xi$ we would get that $\mathrm{vol}_{m-1}\Sigma_y$ is essentially bigger than $\mathrm{vol}_{m-1}\Sigma_x$, which is impossible since both are almost equal to $\mathrm{vol}_{m-1}\Sigma_p$.

$\Omega$, *i.e.*

$$\mu_\gamma(\gamma^{-1}(\Omega))=0$$

From property 2 on page 56, it follows that the limit of $\Omega$-quasigeodesics is a $\Omega$-quasigeodesic. Moreover, if for any initial data we can construct an $\Omega$-quasigeodesic and an $\Omega'$-quasigeodesic, then it is possible to construct an $\Omega\cup\Omega'$-quasigeodesic for any initial data; for $\Upsilon \Subset \Omega\cup\Omega'$, $\Upsilon$-quasigeodesic can be constructed by joining together pieces of $\Omega$ and $\Omega'$-quasigeodesics and $\Omega\cup\Omega'$-quasigeodesic can be constructed as a limit of $\Upsilon_n$-quasigeodesics as $\Upsilon_n\to\Omega\cup\Omega'$.

Let us denote by $\Omega$ the maximal open set such that for any initial data one can construct an $\Omega$-quasigeodesic. We have to show then that $\Omega = A$.

Let $\mathfrak{C} = A\backslash\Omega$, and let $p\in\mathfrak{C}$ be the point with almost maximal $\mathrm{vol}_{m-1}\Sigma_p$. We will arrive to a contradiction by constructing a $B_\varepsilon(p)\cup\Omega$-quasigeodesic for any initial data.

Choose a finite set of points $q_i$ as in A.3.6. Given $\varepsilon>0$, it is enough to construct an $\Omega$-quasigeodesic $\gamma_\varepsilon:[0,T)\to A$, for some fixed $T>0$ with the given initial data $x\in\bar{B}_\varepsilon(p)$, $\xi\in\Sigma_x$, such that the entropies $\mu_{\gamma_\varepsilon}((0,T))\to 0$ as $\varepsilon\to 0$.

The $\Omega$-quasigeodesic $\gamma_\varepsilon$ which we are going to construct will have the following property: one can present $[0,T)$ as a countable union of disjoint half-open intervals $[a_i,\bar{a}_i)$ such that

$$\text{if}\quad \frac{\gamma^+(a_i)}{|\gamma^+(a_i)|}\in\Sigma_{\gamma(a_i)}\mathfrak{C}\quad\text{then}\quad \mu_\gamma([a_i,\bar{a}_i))\leqslant\varepsilon\vartheta(a_i,\bar{a}_i)$$

and

$$\text{if}\quad \frac{\gamma^+(a_i)}{|\gamma^+(a_i)|}\notin\Sigma_{\gamma(a_i)}\mathfrak{C}\quad\text{then}\quad \mu_\gamma([a_i,\bar{a}_i))=0$$

Existence of $\gamma_\varepsilon$ is being proved the same way as in the $\delta$-strained case, with the use of one additional observation: if

$$\frac{\gamma^+(t_{\max})}{|\gamma^+(t_{\max})|}\notin\Sigma_{\gamma(a_i)}\mathfrak{C}$$

then any $\Omega$-quasigeodesic in this direction has zero entropy for a short time.

Then, just as in the $\delta$-strained case, applying inequality A.3.6 we get that $\mu_{\gamma_\varepsilon}(0,T)\to 0$ as $\varepsilon\to 0$. Therefore, passing to a partial limit $\gamma_\varepsilon\to\gamma$ gives a $B_\varepsilon(p)\cup\Omega$-quasigeodesic $\gamma:[0,T)\to A$ for any initial data in $B_\varepsilon(p)$. $\square$

## A.4 Quasigeodesics in extremal subsets.

The second part of theorem A.0.1 follows from the above construction, but we have to modify Milka's lemma A.3.2:

**A.4.1. Extremal Milka's lemma.** *Let $E\subset T_p$ be an extremal subset of a tangent cone then for any vector $v\in E$ there is a polar vector $v^*\in E$ such that $|v|=|v^*|$.*

*Proof.* Set $X = E \cap \Sigma_p$. If $\Sigma_\xi X \neq \varnothing$ then the proof is the same as for the standard Milka's lemma; it is enough to choose a direction in $\Sigma_\xi X$ and shoot a quasigedesic $\gamma$ of length $\pi$ in this direction such that $\gamma \subset X$ ($\gamma$ exists from the induction hypothesis).

If $X = \{\xi\}$ then from the extremality of $E$ we have $B_{\pi/2}(\xi) = \Sigma_p$. Therefore $\xi$ is polar to itself.

Otherwise, if $\Sigma_\xi X = \varnothing$ and $X$ contains at least two points, choose $\xi^*$ to be closest point in $X \backslash \xi$ from $\xi$. Since $X \subset \Sigma_p$ is extremal we have that for any $\eta \in \Sigma_p$ $\measuredangle_{\Sigma_p} \eta\xi^*\xi \leqslant \pi/2$ and since $\Sigma_\xi X = \varnothing$ we have $\measuredangle_{\Sigma_p} \eta\xi\xi^* \leqslant \pi/2$. Therefore, from triangle comparison we have

$$|\xi\eta|_{\Sigma_p} + |\eta\xi^*|_{\Sigma_p} = \measuredangle(\xi,\eta) + \measuredangle(\eta,\xi^*) \leqslant \pi$$

□

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