MOTIVIC GALOIS GROUPS

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1. TANNAKIAN CATEGORIES ([26], CF. [7])

K field of characteristic 0, $\mathcal{A}$ rigid tensor K-linear *abelian* category, L extension of K.

Definition 1. An L-valued fibre functor is a tensor functor $\omega : \mathcal{A} \to Vec_L$ which is *faithful* and *exact*.

Definition 2. $\mathcal{A}$ is

- neutralised Tannakian if one is given a K-valued fibre functor
- neutral Tannakian if $\exists$ K-valued fibre functor
- Tannakian if $\exists$ L-valued fibre functor for some L.

Example 1. G affine K-group scheme, $\mathcal{A} = Rep_K(G)$, $\omega : \mathcal{A} \to Vec_K$ the forgetful functor.

$(\mathcal{A}, \omega)$ neutralised Tannakian category: $G_K := Aut^{\otimes}(\omega)$ is (canonically) the K-points of an affine K-group scheme $G(\omega)$.

Theorem 1 (Grothendieck-Saavedra [26]). *a) For $(\mathcal{A}, \omega)$ as in Example 1, $G(\mathcal{A}, \omega) = G$.*
b) In general ω enriches into a tensor equivalence of categories

$$\tilde{\omega} : \mathcal{A} \xrightarrow{\sim} Rep_K(G(\mathcal{A}, \omega)).$$

c) Dictionary (special case): $\mathcal{A}$ semi-simple $\iff$ G proreductive.

When $\mathcal{A}$ Tannakian but not neutralised, need replace $G(\mathcal{A}, \omega)$ by a *gerbe* (or a groupoid): Saavedra-Deligne [8].

Theorem 2 (Deligne [8]). *$\mathcal{A}$ rigid K-linear abelian. Equivalent conditions:*

- *$\mathcal{A}$ is Tannakian*
- $\forall M \in \mathcal{A}$, $\exists n > 0$: $\Lambda^n(M) = 0$.
- $\forall M \in \mathcal{A}$, $\dim_{rigid}(M) \in \mathbf{N}$.

2. Are motives Tannakian?

Ideally, would like $Mot_{\text{num}}(k, \mathbf{Q})$ Tannakian, fibre functors given by Weil cohomologies H. Two problems:

- $$\begin{tikzcd}
\mathrm{Mot}_H \arrow[r, hook, "H"] \arrow[d, two heads] & Vec_K^* \\
\mathrm{Mot}_{\text{num}} \arrow[ur, dashed, "???"'] &
\end{tikzcd}$$
- $\mathrm{Mot}_{\text{num}}(k, \mathbf{Q})$ is *never Tannakian* because $\dim_{rigid}(X) = \chi(X)$ may be negative (e.g. X curve of genus g: $\chi(X) = 2 - 2g$).

Second problem: matter of commutativity constraint – need modify it.

Yields Grothendieck's *standard conjectures* ([13], cf. [20]):

- (HN) $\sim_H = \sim_{\text{num}}$.
- (C) $\forall X$ the Künneth components of $H(\Delta_X)$ are algebraic.

Another conjecture (B) (skipped):

- (HN) $\Rightarrow$ (B) $\Rightarrow$ (C).
- (HN) $\iff$ (B) in characteristic 0.

Theorem 3 (Lieberman-Kleiman [19]). *Conjecture (B) holds for abelian varieties.*

Theorem 4 (Katz-Messing [18]). *Conjecture (C) is true if k finite.*

Corollary 1 (Jannsen [14]). *If k finite, a suitable modification $\widetilde{\mathrm{Mot}}_{\mathrm{num}}(k, \mathbf{Q})$ is (abstractly) Tannakian.*

Apart from this, wide open!

Definition 3. When $\widetilde{\mathrm{Mot}}_{\mathrm{num}}(k, \mathbf{Q})$ exists, the gerbe that classifies it is called the [pure] motivic Galois group $GMot_k$. H Weil cohomology with coefficients K: fibre of $GMot_k$ at H is proreductive K-group $GMot_{H,k}$.

More generally, $\mathcal{A}$ thick rigid subcategory of $\mathrm{Mot}_{\mathrm{num}}$, get an "induced" Galois group $GMot(\mathcal{A})$ of $\mathcal{A}$, quotient of the motivic Galois group. E.g. $\mathcal{A}$ thick rigid subcategory generated by $h(X)$: get the motivic Galois group of X $GMot_{H,k}(X)$(of finite type).

Examples 2.

(1) $\mathcal{A}$ = Artin motives (generated by $h(\operatorname{Spec} E)$, $[E : k] < \infty$): $GMot(\mathcal{A}) = G_k$.
(2) $\mathcal{A}$ = pure Tate motives (generated by L or $h(\mathbf{P}^1)$): $GMot(\mathcal{A}) = \mathbb{G}_m$.
(3) $\mathcal{A}$ = pure Artin-Tate motives (put these two together): $GMot(\mathcal{A}) = G_k \times \mathbb{G}_m$.
(4) E elliptic curve over $\mathbf{Q}$, $H = H_{Betti}$.
  - E not CM $\Rightarrow GMot_{H,\mathbf{Q}}(E) = GL_2$.
  - E CM $\Rightarrow GMot_{H,\mathbf{Q}}(E)$ = torus in GL_2 or its normaliser.

Example 3. Suppose Conjecture (HN) true.

- Characteristic 0: Betti cohomology yields (several) $\mathbf{Q}$-valued fibre functors, as long as $card(k) \leq card(\mathbf{C})$: $\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q})$ is neutral. Comparison isomorphisms $\Rightarrow$ isomorphisms between various motivic Galois groups.
- Characteristic p: $k \supseteq \mathbf{F}_{p^2}$ finite $\Rightarrow$ $\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q})$ is *not neutral*: if $K \subseteq \mathbf{R}$ or $K \subseteq \mathbf{Q}_p$, no K-valued fibre functor (Serre: endomorphisms of a supersingular elliptic curve = quaternion $\mathbf{Q}$-algebra nonsplit by $\mathbf{R}, \mathbf{Q}_p$).

3. CONNECTION WITH HODGE AND TATE CONJECTURES

3.1. Tate conjecture.

k finitely generated, $G_k := Gal(\bar{k}/k)$, $H = H_l$ ($l \neq \operatorname{char} k$): the $\otimes$-functor

$$H_l : \mathrm{Mot}_H \to Vec^*_{\mathbf{Q}_l}$$

enriches into a $\otimes$-functor

$$\hat{H}_l : \mathrm{Mot}_H \longrightarrow Rep^{cont}_{\mathbf{Q}_l}(G_k)^*.$$

Tate conjecture $\iff$ $\tilde{H}_l$ *fully faithful* (it is faithful by definition).

Proposition 1. *Tate conjecture $\Rightarrow$ Conjecture (B).*

Hence under Tate conjecture, Conjecture (C) holds and can modify commutativity constraint:

$$\tilde{H}_l : \widetilde{\mathrm{Mot}}_H \longrightarrow Rep^{cont}_{\mathbf{Q}_l}(G_k).$$

($Rep^{cont}_{\mathbf{Q}_l}(G_k)$, forgetful functor) neutralised Tannakian $\mathbf{Q}_l$-category with fundamental group Γ_k: for $V \in Rep^{cont}_{\mathbf{Q}_l}(G_k)$, $\Gamma_k(V) =$ Zariski closure of G_k in $GL(V)$.

Proposition 2 (folklore, cf. [27], [17])**.** *Assume Tate conjecture. Equivalent conditions:*

- *Conjecture (HN);*
- $Im\tilde{H}_l \subseteq Rep^{cont}_{\mathbf{Q}_l}(G_k)_{ss}$ *(full subcategory of semi-simple representations).*

Under these conditions, $\mathrm{Mot}_{\mathrm{num}}$ Tannakian, reduce to Γ_k^{ss} (for $Rep^{cont}_{\mathbf{Q}_l}(G_k)_{ss}$) proreductive and canonical epimorphism

$$\Gamma_k^{ss} \longrightarrow\!\!\!\!\rightarrow GMot_{H_l,k}.$$

In particular, $\forall X$, $GMot_{H_l,k}(X) =$ Zariski closure of G_k in $GL(H_l(X))$.

Delicate question: essential image of $\tilde{H}_l$? Conjectural answers for k finite (see below) and k number field (Fontaine-Mazur [11]).

3.2. Hodge conjecture.

$\sigma : k \hookrightarrow \mathbf{C}$, $H = H_\sigma$: this time enriches into $\otimes$-functor

$$\hat{H}_\sigma : \mathrm{Mot}_{H_\sigma} \to PHS^*_{\mathbf{Q}}$$

(graded pure Hodge structures over $\mathbf{Q}$). Hodge conjecture $\iff$ $\hat{H}_\sigma$ *fully faithful.*

Proposition 3. *Hodge conjecture $\Rightarrow$ Conjecture (B) $\iff$ Conjecture (HN).*

Hence, under Hodge conjecture, get modified fully faithful tensor functor

$$\tilde{H}_\sigma : \widetilde{\mathrm{Mot}}_{\mathrm{num}} \to PHS_{\mathbf{Q}}.$$

Latter category semi-simple neutralised Tannakian (via forgetful functor). If extend scalars to $\mathbf{R}$, fundamental group = Hodge torus $S = R_{\mathbf{C}/\mathbf{R}}\mathbb{G}_m$. Over $\mathbf{Q}$ it is the Mumford-Tate group MT: for $V \in PHS_{\mathbf{Q}}$, $MT(V) = \mathbf{Q}-$ Zariski closure of S in $GL(V)$.

Hodge conjecture $\iff$ $\forall X$, $GMot_{k,H_\sigma}(X) = MT(X) \subseteq GL(H_\sigma(X))$.

Sometimes gives proof of Hodge conjecture (for powers of X, X abelian variety)!

4. UNCONDITIONAL MOTIVIC GALOIS GROUPS

Want an unconditional theory of motives (not assuming the unproven standard conjectures)

4.1. First approach (Deligne, André).

Both are in characteristic 0.

- **Deligne** [10]**:** replace motives by systems of compatible realisations: motives for absolute Hodge cycles (systems of cohomology classes corresponding to each other by comparison isomorphisms). Gives semi-simple Tannakian category.
  Hodge conjecture ⇒ absolute Hodge cycles are algebraic so same category.
- **André** [3]: only adjoin to algebraic cycles the inverses of the Lefschetz operators: motives for cycles. Gives semi-simple Tannakian category.
  Conjecture (B) ⇒ motivated cycles are algebraic so same category.
  (Hodge conjecture ⇒ Conjecture (B) so cheaper approach!)

A abelian variety over number field:

Theorem 5 (Deligne [9])**.** *Every Hodge cycle on A is absolutely Hodge.*

Corollary 2. *Tate conjecture $\Rightarrow$ Hodge conjecture on A.*

Better:

Theorem 6 (André [3])**.** *Every Hodge cycle on A is motivated.*

Corollary 3. *Conjecture (B) for abelian fibrations on curves $\Rightarrow$ Hodge conjecture on A.*

Tannakian arguments:

Theorem 7 (Milne [23])**.** *Hodge conjecture for complex CM abelian varieties $\Rightarrow$ Tate conjecture for all abelian varieties over a finite field.*

Theorem 8 (André [4])**.** *A abelian variety over a finite field: every Tate cycle is motivated.*

4.2. Second approach (André-K): tensor sections.

$\mathcal{A}$ pseudo-abelian $\mathbf{Q}$-linear category, $\mathcal{R}$ Kelly radical of $\mathcal{A}$ (like Jacobson radical of rings): smallest ideal such that $\mathcal{A}/\mathcal{R}$ semi-simple.

If $\mathcal{A}$ tensor category, $\mathcal{R}$ may or may not be stable under $\otimes$. True e.g. if $\mathcal{A}$ Tannakian.

Theorem 9 (André-K [6]). *Suppose that* $\mathcal{R}$ *is* $\otimes$*-ideal,* $\mathcal{A}(\mathbf{1},\mathbf{1}) = \mathbf{Q}$ *and* $\mathcal{R}(M,M)$ *nilpotent ideal of* $\mathcal{A}(M,M)$ *for all* M*. Then the projection functor*

$$\mathcal{A} \to \mathcal{A}/\mathcal{R}$$

has tensor sections, and any two are tensor-conjugate.

Application:
H classical Weil cohomology,

$$\begin{aligned}\mathcal{A} &= \mathrm{Mot}_H^{\pm}(k, \mathbf{Q}) \\ &:= \{M \in \mathrm{Mot}_H(k, \mathbf{Q}) \mid \text{ sum of even Künneth projectors of } M \text{ algebraic}\}.\end{aligned}$$

Then $\mathcal{A}$ satisfies assumptions of Theorem 9: in characteristic 0 by comparison isomorphisms, in characteristic p by Weil conjectures.

Theorem 10 (André-K [5]). *a)* $\mathrm{Mot}_{\mathrm{num}}^{\pm} := Im(\mathrm{Mot}_H^{\pm} \to \mathrm{Mot}_{\mathrm{num}})$ *independent of* H*.*
b) Can modify commutativity constraints in $\mathrm{Mot}_H^{\pm}$ *and* $\mathrm{Mot}_{\mathrm{num}}^{\pm}$*, yielding* $\widetilde{\mathrm{Mot}}_H^{\pm}$ *and* $\widetilde{\mathrm{Mot}}_{\mathrm{num}}^{\pm}$*.*
c) Projection functor $\widetilde{\mathrm{Mot}}_H^{\pm} \to \widetilde{\mathrm{Mot}}_{\mathrm{num}}^{\pm}$ *has tensor sections* σ*; any two are tensor-conjugate.*

$$\begin{tikzcd}
\mathrm{Mot}_H \arrow[r, hook, "H"] \arrow[d] & Vec_K^* \\
\mathrm{Mot}_{\mathrm{num}} \arrow[ur, dashed, "???"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{\mathrm{Mot}}_H^{\pm} \arrow[r, hook, "H"] \arrow[d, shift right] & Vec_K \\
\widetilde{\mathrm{Mot}}_{\mathrm{num}}^{\pm} \arrow[u, shift right, "\sigma"'] \arrow[ur, "H\circ\sigma"'] &
\end{tikzcd}$$

Variant with

$$\mathrm{Mot}_H^*(k, \mathbf{Q}) := \{M \in \mathrm{Mot}_H(k, \mathbf{Q}) \mid \text{ all Künneth projectors of } M \text{ algebraic}\}.$$

5. DESCRIPTION OF MOTIVIC GALOIS GROUPS

Assume all conjectures (standard, Hodge, Tate).

5.1. In general:

Short exact sequence

$$1 \longrightarrow GMot_{\bar{k}} \longrightarrow GMot_k \longrightarrow G_k \longrightarrow 1$$

Last morphism: G_k corresponds to motives of 0-dimensional varieties (Artin motives). The group $GMot_{\bar{k}}$ is connected, hence $= GMot_k^0$.

If $k \subseteq k'$, $GMot_{k'}^0 \twoheadrightarrow GMot_k^0$ (but not iso unless k'/k algebraic: otherwise, "more" elliptic curves over k' than over k).

Conjecture (C) $\Rightarrow$ weight grading on $\mathrm{Mot}_{\mathrm{num}}$ $\iff$ central homomorphism

$$w : \mathbb{G}_m \longrightarrow GMot_k.$$

On the other hand, Lefschetz motive gives homomorphism

$$t : GMot_k \longrightarrow \mathbb{G}_m$$

and $t \circ w = 2$ (-2 with Grothendieck's conventions).

5.2. Over a finite field:

Theorem 11 (cf. [22]). *a)* $\mathrm{Mot}_{\mathrm{num}}$ *generated by Artin motives and motives of abelian varieties.*
b) Essential image of $\tilde{H}_l$*: l-adic representations of* G_k *whose eigenvalues are Weil numbers.*

Uses Honda's theorem [16]: every Weil orbit corresponds to an abelian variety.

Corollary 4. $GMot_k^0$ = *group of multiplicative type determined by action of* $G_{\mathbf{Q}}$ *on Weil numbers.*

Even though $\widetilde{\mathrm{Mot}}_{\mathrm{num}}$ not neutral, $GMot_k^0$ abelian so situation not so bad!

5.3. Over a number field:

$S := (GMot_k^0)^{ab}$: the Serre protorus: describe its character group $X(S)$:

$$\mathbf{Q}^{cm} = \bigcup \{E \mid E \text{ CM number field}\}$$

Complex conjugation c central in $Gal(\mathbf{Q}^{cm}/\mathbf{Q})$ (largest Galois subfield of $\bar{\mathbf{Q}}$ with this property).

Definition 4. $f : Gal(\mathbf{Q}^{cm}/\mathbf{Q}) \to \mathbf{Z}$ CM type if $f(s) + f(cs)$ independent of s. $G_{\mathbf{Q}}$ acts on CM types by $\tau f(s) = f(\tau s)$.

Theorem 12 ([24]). *$X(S) = \mathbf{Z}[CM\ types]$.*

Can also describe the centre C of $GMot_k^0$ (pro-isogenous to S), etc.: cf. [25].

6. MIXED (TATE) MOTIVES

Expect Tannakian category of mixed motives

$$\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q}) \subset \mathrm{MMot}(k, \mathbf{Q})$$

with socle $\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q})$, classifying non smooth projective varieties. Corresponding motivic Galois group extension of $GMot_k$ by a pro-unipotent group (or gerbe).

Constructions of MMot:

- Conjecturally, heart of “motivic t-structure” on DM (Deligne, Beilinson: cf. Hanamura [15]).
- In characteristic 0: explicit category constructed by Nori.
- Over a finite field: Tate conjecture ⇒ $\mathrm{Mot}_{\mathrm{num}} = \mathrm{MMot}$ (cf. [22]).
- Can settle for subcategory: mixed Tate motives TMMot_k. Exists unconditionally if k number field (cf. Levine’s talk and [21]).

Goncharov [12]**:** $\mathrm{TMMot}_{\mathbf{Z}}$ (mixed Tate motives over $\mathbf{Z}$) defined as full subcategory of $\mathrm{TMMot}_{\mathbf{Q}}$ by non-ramification conditions.

Γ the motivic Galois group corresponding to $\mathrm{TMMot}_{\mathbf{Z}}$: Proreductive quotient of Γ is $\mathbb{G}_m$ (see above).

Theorem 13 (Goncharov [12])**.** *Action of* $\mathbb{G}_m$ *on prounipotent kernel* U *yields a grading on* $Lie(U)$*: for this grading,* $Lie(U)$ *is free with one generator in every odd degree* ≤ -3*.*

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