Axiomatic higher torsion

and answers to questions

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Questions (for me)

- Can you explain higher Franz-Reidemeister (FR) torsion in a way that everyone can understand it?
  -NSF referee
- What does analytic torsion have to do with the moduli space?
- Why do the Pontrjagin classes not satisfy the axioms?
- What is even torsion and odd torsion?
- How do you extend the higher FR-torsion to the mapping class group?
- Can you get the other classes (cohomology classes in B that are pushed down from E?)
- Where do $\zeta(2k+1)$ and the polylogarithms come from?
- documentclass{prosper}

Questions for audience

Take the Torelli group

$$\mathcal{I}^1_{g,n} = ker(M^1_{g,n} \to Sp(2g, \mathbb{Z}))$$

of homologically framed surfaces Σ with one (fixed) boundary component and n marked points.
Homological framing = fixed basis for $H_*(\Sigma)$

- Are $\psi_i^2 \in H^4(\mathcal{I}^1_{g,n}; \mathbb{Q})$ nonzero?
- Are $\kappa_{2k} \in H^{4k}(\mathcal{I}^1_{g,n}; \mathbb{Q})$ nonzero?

Summary of axiomatic higher torsion

- Def: A higher torsion invariant is a real characteristic class of unipotent smooth bundles satisfying two axioms: additivity and transfer.
- Every higher torsion invariant has even and odd parts:

$$\tau = \tau^+ + \tau^-$$

- The even part τ^+ is a scalar multiple of the (generalized) Miller-Morita-Mumford (MMM) class κ_{2k}
- The odd part τ^- is a scalar multiple of the odd part of the higher FR-torsion
- Conjecture (S. Goette): Nonequivariant higher analytic torsion classes are odd torsion invariants.

Axioms

- additivity If $E = E_1 \cup E_2$ then

$$\tau(E) = \frac{1}{2}\tau(DE_1) + \frac{1}{2}\tau(DE_2)$$

- transfer If $S^n \to D \to E$ is an oriented linear sphere bundle. Then the torsion $\tau_E(D), \tau_B(D)$ of D as a bundle over E, B resp. are related by:

$$\tau_B(D) = \chi(S^n)\tau(E) + tr_B^E(\tau_E(D))$$

$$tr_B^E(x) = p_*(x \cup e(E))$$

where $e(E) \in H^n(E)$ is the Euler class of the vertical tangent bundle.

Calculation of higher torsion

- Parameters s_1, s_2 are given by

$$\tau(S^n(\lambda)) = 2s_n ch_{4k}(\lambda) \in H^{4k}(B;\mathbb{R})$$

where $S^n(\lambda)$ is an S^1 or S^2 bundle associated to a complex line bundle λ over B.

- Theorem: Let $f : E \to \mathbb{R}$ be a fiberwise Morse function with distinct critical values. Let ξ_j, η_j be the negative and positive eigenspace bundles associated to the jth critical point. then

$$\tau(E) = \sum_j (-1)^i (s_1 + s_2) ch_{4k}(\eta_j) + (-1)^i (s_2 - s_1) ch_{4k}(\xi_j)$$

where i is the index of the critical point.

Miller-Morita-Mumford classes

- The MMM classes, for closed fiber F, are given by

$$\kappa_{2k}(E) = tr_B^E((2k)! ch_{4k}(T^v E))$$

where $ch_{4k}(T^v E) = \frac{1}{2} ch_{4k}(T^v E \otimes \mathbb{C})$.

- This is a higher torsion invariant with

$$s_1 = 0$$

$$s_2 = (2k)!$$

- Thm: Every even torsion invariant is a scalar mutiple of κ_{2k}.

Higher Franz-Reidemeister torsion

- Let $K(Z) = \mathbb{Z} \times BGL(\infty, \mathbb{Z})^+$. Then

$$H^*(\Omega K(Z); \mathbb{R}) \cong \mathbb{R}[b_2, b_4, \cdots]$$

  b_i: Borel regulator (continuous cohomology) classes

- $E \to B$ unipotent get: $C : B \to \Omega K(Z)$.

$$\tau_{2k}^{FR}(E) = C^*(b_{2k})$$

- This is a higher torsion invariant with

$$s_i = \frac{1}{2}(-1)^{k+i}\zeta(2k+1)$$

- Thm: Every odd higher torsion invariant is a scalar multiple of the odd part of τ_{2k}^{FR}.

Bismut-Lott analytic torsion

- Nonequivariant analytic torsion is a $4k$ form on B which is closed when $\pi_1 B$ acts trivially on $H_*(F;\mathbb{R})$
- Analytic torsion classes satisfy the transfer axiom (X. Ma)
- Additivity, an easy property of higher FR torsion, is unknown.
- Parameters are $s_2 = 0$,

$$s_1 = (-1)^k (2\pi)^{-2k} \frac{(4k+1)!}{2^{4k}(2k)!} \zeta(2k+1)$$

- Conjecture (which S. Goette now claims is true): τ_{2k}^{BL} is a scalar multiple of the odd part of τ_{2k}^{FR}.

Mapping class group

- Axiomatic higher torsion is only defined on (the classifying space $B\mathcal{I}_g$ of) the Torelli group.
- Higher FR torsion is defined on the mapping class group $\mathcal{M}_g$:

$$\tau_{2k}^{FR}(\mathcal{M}_g) = \frac{(-1)^k \zeta(2k+1)}{2(2k)!}\kappa_{2k} \in H^{4k}(\mathcal{M}_g;\mathbb{R})$$

- Equivariant higher FR-torsion gives the odd kappa's:

$$\tau_k^{FR}(\mathcal{M}_g;\zeta) = \frac{1}{2}m^k L_{k+1}(\zeta)\frac{1}{k!}\kappa_k \in H^{2k}(\mathcal{M}_g;\mathbb{R})$$

$$\zeta^m = 1,\ L_{k+1}(\zeta) = \mathcal{R}\left(\frac{1}{i^k}\sum \zeta^n/n^{k+1}\right)$$

Outer automorphism group

- F_n: free group on n letters.
- $Out(F_n) = Aut(F_n)/Inn(F_n)$: outer automorphism group.
- $BOut(F_n)$ is the moduli space of graphs $G \simeq \vee_n S^1$.
- $IOut(F_n)$ is defined to be the kernel:

$$1 \to IOut(F_n) \to Out(F_n) \to GL(n, \mathbb{Z}) \to 1$$

- Axiomatic torsion is define on $IOut(F_n)$:

$$\tau_{2k}(IOut(F_n)) \in H^{4k}(IOut(F_n); \mathbb{R})$$

- $\phi_j : \mathcal{M}_{g,n} \to Out(F_{2g})$, $\phi_j : \mathcal{I}_{g,n} \to IOut(F_{2g})$

$$\phi_j^*(\tau_{2k}) = \frac{-2s_1}{(2k)!}\left(\kappa_{2k} - \psi_j^{2k}\right) \in H^{4k}(\mathcal{I}_{g,n}; \mathbb{R}).$$