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Minimum energy on trees with k pendent vertices

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Abstract

The energy of G, denoted by $E(G)$, is defined as the sum of the absolute values of the eigenvalues of G. In this paper, we characterize the tree with minimal energy among the trees with k pendent vertices.

AMS classification: 05C35; 05C50

Keywords: Energy; Tree; Pendent vertex

1. Introduction

Let G be a graph on n vertices and $A(G)$ the adjacency matrix of G. Then the characteristic polynomial of $A(G)$, denoted by $\phi(G) = |xI - A(G)|$, is called the characteristic polynomial of G. The n roots of the equation $\phi(G) = 0$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, are called the eigenvalues of G. Since $A(G)$ is real and symmetric, all eigenvalues of G are real.

The *energy* of G, denoted by $E(G)$, is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

This concept was introduced by Gutman and is intensively studied in chemistry, since it can be used to approximate the total π-electron energy of a molecule (see, e.g., [10,11]). There are a lot

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of results on $E(G)$ (e.g., see, [1,3,5–10,12–23,25–27,29–34]). However, up to now, very little is known for graphs with extremal energy.

If G is a bipartite graph, then the characteristic polynomial of G can be written as (see [11]):

$$\phi(G) = |xI - A(G)| = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k b(G,k) x^{n-2k},$$

where n is the order of G. Note that $b(G,0) = 1$, $b(G,k) \geqslant 0$ for $1 \leqslant k \leqslant \lfloor n/2 \rfloor$. For the other k, we assume $b(G,k) = 0$, for convenience. The energy of bipartite graph G can be expressed as the Coulson integral formula (see [18]):

$$E(G) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dx}{x^2} \ln \left[1 + \sum_{k=1}^{\lfloor n/2 \rfloor} b(G,k) x^{2k} \right].$$

It is easy to see that $E(G)$ is a strictly monotonously increasing function of $b(G,k)$. This fact inspired Gutman to define a quasiordering to compare the energies for trees and further for a set of graphs.

Let G_1 and G_2 be two bipartite graphs of order n, whose characteristic polynomials are

$$\phi(G_1) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k b(G_1,k) x^{n-2k} \quad \text{and} \quad \phi(G_2) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k b(G_2,k) x^{n-2k}.$$

If $b(G_1,k) \geqslant b(G_2,k)$ holds for all $k \geqslant 0$, we call $G_1 \succeq G_2$ or $G_2 \preceq G_1$. If $G_1 \succeq G_2$ and there is a k such that $b(G_1,k) > b(G_2,k)$, we call $G_1 \succ G_2$. By the strict monotonicity of $E(G)$, if $G_1 \succeq G_2$, then $E(G_1) \geqslant E(G_2)$; if $G_1 \succ G_2$, then $E(G_1) > E(G_2)$.

This method has been successfully applied in the study of the extremal values of energies in some classes of graphs. Gutman [6] showed that the path has the maximal energy and the star has the minimal energy among the trees on n vertices. Zhang and Li characterized the trees with maximal energy [30] and minimal energy [29], respectively, among the trees with perfect matchings. Hou determined the graphs with minimal energy among all the trees with a given size of matching [19] and all unicyclic graphs [18], respectively. Zhang et al. determined the graphs with maximal energy [32] and minimal energy [31], respectively, among the hexagonal chains. Recently, Yan and Ye [26] characterized the tree with maximal energy among the trees with order n and at least $\left\lfloor \frac{n+2}{2} \right\rfloor$ pendent vertices. Lin et al. [22] determined the tree with maximal energy among the trees with order n and maximum degree Δ ($3 \leqslant \Delta \leqslant n-2$) and the tree with minimal energy among the trees with order n and maximum degree Δ $\left(\left\lceil \frac{n+1}{3} \right\rceil \leqslant \Delta \leqslant n-2\right)$.

Let $\mathscr{T}_{n,k}$ be the set of all trees with n vertices and k pendent vertices. In this paper, we show that $P_{n,k}$ (as shown in Fig. 1) is the tree with minimal energy in $\mathscr{T}_{n,k}$.

In order to state our results, we introduce some notation and terminology. Other undefined notation may refer to [2]. If $W \subseteq V(G)$, we denote by $G - W$ the subgraph of G obtained by

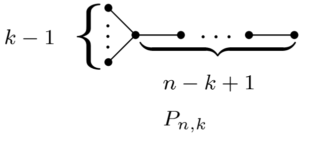

Fig. 1.

deleting the vertices of W and the edges incident with them. Similarly, if $E' \subseteq E(G)$, we denote by $G - E'$ the subgraph of G obtained by deleting the edges of E'. If $W = \{v\}$ and $E' = \{xy\}$, we write $G - v$ and $G - xy$ instead of $G - \{v\}$ and $G - \{xy\}$, respectively. If a graph G has components $G_1, G_2, \ldots, G_t$, then G is denoted by $\bigcup_{i=1}^{t} G_i$. We denote by P_n and S_n the path and the star on vertices, respectively. We denote by $m(G, k)$ the number of k-matchings of G.

2. Lemmas and results

Lemma 2.1 [4,24]. *Let G be a forest and v be a vertex of G. Then the characteristic polynomial of G satisfies*

$$\phi(G) = x\phi(G - v) - \sum_{u} \phi(G - \{u, v\}),$$

where the summation extends over all vertices adjacent to v.

In particular, if v is a pendent vertex of a forest G and u is the unique vertex adjacent to v, then $\phi(G) = x\phi(G - v) - \phi(G - \{u, v\})$.

Lemma 2.2 [4,24]. *Let G be a forest and $e = uv$ be an edge of G. The characteristic polynomial of G satisfies*

$$\phi(G) = \phi(G - e) - \phi(G - \{u, v\}).$$

Lemma 2.3 [4]. *If $G_1, G_2, \ldots, G_t$ are the components of a forest G, we have*

$$\phi(G) = \prod_{i=1}^{t} \phi(G_i).$$

Lemma 2.4 [29]. *Let G and G' be two forests of order n with characteristic polynomials*

$$\phi(G) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_k x^{n-2k} \quad \text{and} \quad \phi(G') = \sum_{k=0}^{\lfloor n/2 \rfloor} a'_k x^{n-2k},$$

respectively, then $G \succeq G'$ if and only if $a_0 - a'_0 = 0$ and $(-1)^k (a_k - a'_k) \geqslant 0$ for $k = 1, 2, \ldots, \lfloor n/2 \rfloor$; $G \succ G'$ if and only if $G \succeq G'$ and there is a k $(1 \leqslant k \leqslant \lfloor n/2 \rfloor)$ such that $(-1)^k (a_k - a'_k) > 0$.

Note that if G and G' are two forests, $(-1)^k a_k = m(G, k)$, $(-1)^k a'_k = m(G', k)$ and $m(G, 0) = m(G', 0) = 1$. Thus the above lemma can also be written as the following.

Let G and G' be two forests of order n. $G \succeq G'$ if and only if $m(G, k) \geqslant m(G', k)$ for $k = 1, 2, \ldots, \lfloor n/2 \rfloor$; $G \succ G'$ if and only if $G \succeq G'$ and there is a k $(1 \leqslant k \leqslant \lfloor n/2 \rfloor)$ such that $m(G, k) > m(G', k)$.

Lemma 2.5 [26]. *Let G be a forest of order n $(n > 1)$ and G' be a spanning subgraph (respectively a proper spanning subgraph) of G. Then $G \succeq G'$ (respectively $G \succ G'$).*

Lemma 2.6 [26]. *Let T and T' be two trees of order n. Suppose that uv(respectively $u'v'$) is a pendent edge of T (respectively T') and u(respectively u') is a pendent vertex of T (respectively T'). Let $T_1 = T - u$, $T_2 = T - \{u, v\}$, $T'_1 = T' - u'$ and $T'_2 = T' - \{u', v'\}$. If $T_1 \succeq T'_1$ and $T_2 \succ T'_2$, or $T_1 \succ T'_1$ and $T_2 \succeq T'_2$, then $T \succ T'$.*

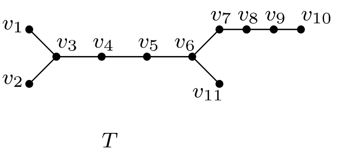

Fig. 2.

Lemma 2.7 [29]. *Let* $\varphi(x) = \sum_{k=0}^{\lfloor m/2 \rfloor} \varphi_k x^{m-2k}$ *and* $\varphi'(x) = \sum_{k=0}^{\lfloor m'/2 \rfloor} \varphi'_k x^{m'-2k}$, *where* $\varphi_0 = \varphi'_0 = 0$, $(-1)^k \varphi_k \geqslant 0$ $(1 \leqslant k \leqslant \lfloor m/2 \rfloor)$ *and* $(-1)^k \varphi'_k \geqslant 0$ $(1 \leqslant k \leqslant \lfloor m'/2 \rfloor)$. *If*

$$\Pi(x) = \sum_{k=0}^{\lfloor (m+m')/2 \rfloor} \pi_k x^{m+m'-2k} = \varphi(x)\varphi'(x),$$

then we have $\pi_0 = 0$ *and* $(-1)^k \pi_k \geqslant 0$ $(1 \leqslant k \leqslant \lfloor (m+m')/2 \rfloor)$.

Let $P = v_0 v_1 \cdots v_k$ $(k \geqslant 1)$ be a path of a tree T. If $d_T(v_0) \geqslant 3$, $d_T(v_k) \geqslant 3$ and $d_T(v_i) = 2$ $(0 < i < k)$, we call P an internal path of T. If $d_T(v_0) \geqslant 3$, $d_T(v_k) = 1$ and $d_T(v_i) = 2$ $(0 < i < k)$, we call P a pendent path of T with root v_0 and particularly when $k = 1$, we call P a pendent edge. Let $s(T)$ be the number of vertices in T with degree more than 2 and $p(T)$ the number of pendent paths in T with length more than 1. For example, we consider the tree T as shown in Fig. 2. $v_3v_4v_5v_6$ is an internal path of T, while $v_6v_7v_8v_9v_{10}$, v_6v_{11}, v_3v_1 and v_3v_2 are all pendent paths of T; $s(T) = 2$ and $p(T) = 1$.

If $T \in \mathscr{T}_{n,k}$ $(3 \leqslant k \leqslant n-2)$, $T \ncong P_{n,k}$ and $p(T) \neq 0$, then T can be seen as the tree as shown in Fig. 3, where P_s $(s \geqslant 3)$ is the pendent path of T with s vertices and root u, T_1 and T_2 are two subtrees of T with vertices v and u as roots, respectively, and $T_1, T_2 \ncong P_1$. If T' is obtained from T by replacing P_s with a pendent edge and replacing the edge uv with a path P_s, we say that T' is obtained from T by **Operation I** (as shown in Fig. 3). It is easy to see that $T' \in \mathscr{T}_{n,k}$.

Lemma 2.8 [28]. *If* T' *is obtained from* T *by Operation I, then*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m(T', k) < \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m(T, k).$$

Now we show that Operation I makes the energy of a tree decrease strictly. In the following proof, we shall use the same notations as above.

Lemma 2.9. *If* T' *is obtained from* T *by Operation I, then* $E(T') < E(T)$.

Proof. Let A_l and B_l are the trees as shown in Fig. 4.

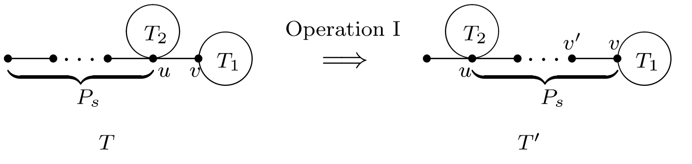

Fig. 3.

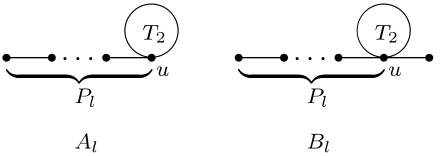

Fig. 4.

By Lemmas 2.1–2.3, we have

$$\begin{aligned}\phi(T)&=\phi(T-uv)-\phi(T-\{u,v\})\\&=\phi(T_1)\cdot\phi(A_s)-\phi(T_1-v)\cdot\phi(T_2-u)\cdot\phi(P_{s-1})\\&=x\cdot\phi(T_1)\cdot\phi(A_{s-1})-\phi(T_1)\cdot\phi(A_{s-2})-\phi(T_1-v)\cdot\phi(T_2-u)\cdot\phi(P_{s-1}),\\\phi(T')&=\phi(T'-v'v)-\phi(T'-\{v',v\})\\&=\phi(T_1)\cdot\phi(B_{s-1})-\phi(T_1-v)\cdot\phi(B_{s-2})\\&=x\cdot\phi(T_1)\cdot\phi(A_{s-1})-\phi(T_1)\cdot\phi(T_2-u)\cdot\phi(P_{s-2})-x\cdot\phi(T_1-v)\cdot\phi(A_{s-2})\\&\quad+\phi(T_1-v)\cdot\phi(T_2-u)\cdot\phi(P_{s-3}).\end{aligned}$$

Since $T_1, T_2 \ncong P_1$, $\phi(P_0)=1$, $\phi(P_1)=x$ and $\phi(P_n)=x\phi(P_{n-1})-\phi(P_{n-2})$ for $n \geqslant 2$, we have

$$\begin{aligned}&\phi(T)-\phi(T')\\&\quad=(\phi(T_1)-x\phi(T_1-v))(\phi(T_2-u)\cdot\phi(P_{s-2})-\phi(A_{s-2}))\\&\quad=-\left(\sum_{vv_i\in E(T_1)}\phi(T_1-\{v,v_i\})\right)\cdot\left(\sum_{uu_i\in E(T_2)}\phi(T_2-\{u,u_i\})\right)\cdot\phi(P_{s-3}).\end{aligned}$$

Let $\phi(T)-\phi(T')=\varphi(x)\cdot\varphi'(x)\cdot\varphi''(x)$, where

$$\begin{aligned}\varphi(x)&=-\sum_{vv_i\in E(T_1)}\phi(T_1-\{v,v_i\})=\sum_{k=0}^{\lfloor n_1/2\rfloor}\varphi_k x^{n_1-2k},\\\varphi'(x)&=-\sum_{uu_i\in E(T_2)}\phi(T_2-\{u,u_i\})=\sum_{j=0}^{\lfloor n_2/2\rfloor}\varphi'_j x^{n_2-2j},\\\varphi''(x)&=-\phi(P_{s-3})=\sum_{l=0}^{\lfloor (s-1)/2\rfloor}\varphi''_l x^{s-1-2l},\end{aligned}$$

and n_1, n_2 are the order of T_1, T_2, respectively. Since $T_1-\{v,v_i\}$, $T_2-\{u,u_i\}$ and P_{s-3} ($s \geqslant 3$) are forests, we have $\varphi_0=\varphi'_0=\varphi''_0=0$, $(-1)^k\varphi_k \geqslant 0$, $(-1)^j\varphi'_j \geqslant 0$ and $(-1)^l\varphi''_l \geqslant 0$ for any $1 \leqslant k \leqslant \lfloor n_1/2\rfloor$, $1 \leqslant j \leqslant \lfloor n_2/2\rfloor$ and $1 \leqslant l \leqslant \lfloor (s-1)/2\rfloor$. By Lemmas 2.7 and 2.4, we have $T' \preceq T$. Note that

$$\phi(T)=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k m(T,k)x^{n-2k} \quad\text{and}\quad \phi(T')=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k m(T',k)x^{n-2k},$$

we have $m(T', k) \leqslant m(T', k)$ for any k ($1 \leqslant k \leqslant \lfloor n/2 \rfloor$). By Lemma 2.8,

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m(T', k) < \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m(T, k).$$

Hence $m(T', k) \leqslant m(T', k)$ for any k ($1 \leqslant k \leqslant \lfloor n/2 \rfloor$) and there is a k ($1 \leqslant k \leqslant \lfloor n/2 \rfloor$) such that $m(T', k) < m(T', k)$. By Lemma 2.4, we have $T' \prec T$, and then $E(T') < E(T)$. □

From lemma 2.9, we immediately get the following result.

Lemma 2.10. *Let* $T \in \mathscr{T}_{n,k}$ ($3 \leqslant k \leqslant n-2$), $T \ncong P_{n,k}$ *and* $p(T) \neq 0$.

(1) *If* $s(T) = 1$, *we can finally get a tree* T' *by Operation I with* $E(T') < E(T)$ *and* $p(T') = 1$; *it is easy to see that* $T' \cong P_{n,k}$.
(2) *If* $s(T) \geqslant 2$, *we can finally get a tree* T' *by Operation I with* $E(T') < E(T)$, $s(T') = s(T)$ *and* $p(T') = 0$.

If $T \in \mathscr{T}_{n,k}$ ($3 \leqslant k \leqslant n-2$), $T \ncong P_{n,k}$ and $p(T) = 0$, then we always can find two pendent vertices u_1 and v_1 of T such that $d(u_1, v_1) = \max\{d(u, v) : u, v \in V(T)\}$. Let $u_1u, v_1v \in E(T)$, then $N_T(u) = \{u_1, u_2, \ldots, u_s, w\}$ ($s \geqslant 2$), $N_T(v) = \{v_1, v_2, \ldots, v_t, w'\}$ ($t \geqslant 2$), where $u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_t$ are pendent vertices of T, $d_T(w) \geqslant 2$ and $d_T(w') \geqslant 2$. Note that $w = w'$, when $d(u_1, v_1) = 3$. If $T' = T - \{uu_2, \ldots, uu_s\} + \{vu_2, \ldots, vu_s\}$ or $T' = T - \{vv_2, \ldots, vv_t\} + \{uv_2, \ldots, uv_t\}$, we say that T' is obtained from T by **Operation II**. It is easy to see that $T' \in \mathscr{T}_{n,k}$, $p(T') = 1$ and $s(T') = s(T) - 1$.

Now we show that Operation II makes the energy of a tree decrease strictly. In the following proof, we shall use the same notations as above.

Lemma 2.11. *If* T' *is obtained from* T *by Operation II, then* $E(T') < E(T)$.

Proof. Let u_1, v_1 be the pendent vertices such that $d(u_1, v_1) = \max\{d(u, v) : u, v \in V(T)\}$. If $d(u_1, v_1) \geqslant 4$, without loss of generality, we suppose $u_1u, v_1v, uw \in E(T)$ and $d_T(w) \geqslant 2$. Then $w \neq v$. Without loss of generality, we suppose $T' = T - \{uu_2, \ldots, uu_s\} + \{vu_2, \ldots, vu_s\}$. Then T and T' can be seen as the trees shown in Fig. 5.

Denote $A = \sum_{v'v \in E(T_1)} \phi(T_1 - \{v', v\})$ and $B = \sum_{v'v \in E(T_1 - w)} \phi(T_1 - \{v', v, w\})$. By Lemmas 2.1–2.3, we have

$$\begin{aligned}\phi(T) &= \phi(T - uw) - \phi(T - \{u, w\}) \\ &= x^{s+t-2}[(x^2 - s)(x^2 - t)\phi(T_1 - v) - x(x^2 - s)A \\ &\qquad - x(x^2 - t)\phi(T_1 - \{w, v\}) + x^2 B],\end{aligned}$$

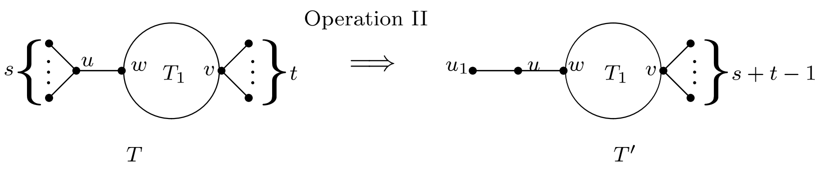

Fig. 5.

$$\begin{aligned}\phi(T')&=\phi(T''-uw)-\phi(T''-\{u,w\})\\&=x^{s+t-2}[(x^2-s-t+1)(x^2-1)\phi(T_1-v)-x(x^2-1)A\\&\qquad-x(x^2-s-t+1)\phi(T_1-\{w,v\})+x^2B].\end{aligned}$$

It is easy to see that

$$\begin{aligned}\phi(T)-\phi(T')&=(s-1)x^{s+t-2}[(t-1)\cdot\phi(T_1-v)-x\phi(T_1-\{w,v\})+xA]\\&=(s-1)x^{s+t-2}[(t-1)\cdot\phi(T_1-v)-\phi((T_1-\{w,v\})\cup\{w\})+xA].\end{aligned}$$

Let

$$\begin{aligned}(t-1)\cdot\phi(T_1-v)-\phi((T_1-\{w,v\})\cup\{w\})&=\sum_{k=0}^{\lfloor(n_1-1)/2\rfloor}\varphi_k x^{n_1-1-2k},\\xA&=\sum_{k=0}^{\lfloor(n_1-1)/2\rfloor}\varphi'_k x^{n_1-1-2k},\end{aligned}$$

where n_1 is the order of T_1. Thus

$$\begin{aligned}\phi(T)-\phi(T')&=\sum_{l=0}^{\lfloor n/2\rfloor}(a_l-a'_l)x^{n-2l}\\&=\sum_{k=0}^{\lfloor(n_1-1)/2\rfloor}(s-1)(\varphi_k+\varphi'_k)x^{n_1+s+t-3-2k}\\&=\sum_{k=0}^{\lfloor(n_1-1)/2\rfloor}(s-1)(\varphi_k+\varphi'_k)x^{n-2(2+k)}.\end{aligned}$$

Note that, for $2\leqslant l\leqslant\lfloor(n_1-1)/2\rfloor+2$,

$$\begin{aligned}(-1)^l(a_l-a'_l)&=(-1)^{k+2}(a_{k+2}-a'_{k+2})\\&=(-1)^{k+2}(s-1)(\varphi_k+\varphi'_k)=(-1)^k(s-1)(\varphi_k+\varphi'_k)\end{aligned}$$

$(0\leqslant k\leqslant\lfloor(n_1-1)/2\rfloor)$, and for the other l, $a_l-a'_l=0$.

Since $(T_1-\{w,v\})\cup\{w\}$ is a proper spanning subgraph of T_1-v and $t\geqslant 2$, by Lemmas 2.5 and 2.4, we have $(-1)^k\varphi_k\geqslant 0$ for $0\leqslant k\leqslant\lfloor(n_1-1)/2\rfloor$ and there is a k $(1\leqslant k\leqslant\lfloor(n_1-1)/2\rfloor)$ such that $(-1)^k\varphi_k>0$. Recall $A=\sum_{v'v\in E(T_1)}\phi(T_1-\{v',v\})$. We have $(-1)^k\varphi'_k\geqslant 0$ for $0\leqslant k\leqslant\lfloor(n_1-1)/2\rfloor$. Thus, $a_0-a'_0=0$, $(-1)^l(a_l-a'_l)\geqslant 0$ for $1\leqslant l\leqslant\lfloor n/2\rfloor$ and there is a l $(1\leqslant l\leqslant\lfloor n/2\rfloor)$ such that $(-1)^l(a_l-a'_l)>0$. By Lemma 2.4, we have $T\succ T'$, and then $E(T)>E(T')$.

If $d(u_1,v_1)=3$, we have

$$\begin{aligned}\phi(T)-\phi(T')&=(s-1)(t-1)x^{s+t-2}\\&=(s-1)(t-1)x^{n-4}.\end{aligned}$$

Noting that $s,t\geqslant 2$, by Lemma 2.4, $T'\prec T$, and then $E(T')<E(T)$. □

Theorem 2.1. *Let $T \in \mathscr{T}_{n,k}$. Then $E(T) \geqslant E(P_{n,k})$, and the equality holds if and only if $T \cong P_{n,k}$.*

Proof. Since $\mathscr{T}_{n,2} = \{P_n\}$ and $P_n \cong P_{n,2}$, $\mathscr{T}_{n,n-1} = \{S_n\}$ and $S_n \cong P_{n,n-1}$, we may assume $3 \leqslant k \leqslant n-2$ and it is sufficient to show that $E(T) > E(P_{n,k})$ for any $T \in \mathscr{T}_{n,k}$ and $T \ncong P_{n,k}$.

For $T \in \mathscr{T}_{n,k}$ $(3 \leqslant k \leqslant n-2)$ and $T \ncong P_{n,k}$, we know $1 \leqslant s(T) \leqslant n-k$, we shall show $E(T) > E(P_{n,k})$ by induction on $s(T)$. When $s(T) = 1$, note $T \ncong S_n, P_n, P_{n,k}$, by Lemma 2.10(1), we have $E(T) > E(P_{n,k})$. Suppose the result holds for any tree T' with $s(T') = s-1$. Let $s(T) = s \geqslant 2$. If $p(T) \neq 0$, by Lemma 2.10(2), we can get a tree $T_1 \in \mathscr{T}_{n,k}$ such that $p(T_1) = 0$, $s(T_1) = s$ and $E(T) > E(T_1)$. By Lemma 2.11, we can get a tree $T_2 \in \mathscr{T}_{n,k}$ from T_1 such that $p(T_2) = 1, s(T_2) = s-1$ and $E(T_1) > E(T_2)$. Hence $E(T) > E(T_1) > E(T_2)$. By the induction hypothesis, we have

$$E(T) > E(T_1) > E(T_2) > E(P_{n,k}).$$

Therefore, if $T \in \mathscr{T}_{n,k}$, then $E(T) \geqslant E(P_{n,k})$, and the equality holds if and only if $T \cong P_{n,k}$. □

Lemma 2.12. *For $3 \leqslant k \leqslant n-1$, we have $E(P_{n,k}) < E(P_{n,k-1})$.*

Proof. By Lemma 2.1, we have

$$\begin{aligned}\phi(P_{n,k}) &= x\phi(P_{n-1,k-1}) - x^{k-2}\phi(P_{n-k}),\\ &= x\phi(P_{n-1,k-1}) - \phi(P_{n-k} \cup (k-2)K_1),\\ \phi(P_{n,k-1}) &= x\phi(P_{n-1,k-1}) - \phi(P_{n-2,k-1}).\end{aligned}$$

Since $P_{n-k} \cup (k-2)K_1$ is a proper spanning subgraph of $P_{n-2,k-1}$, by Lemma 2.5, we have $P_{n-k} \cup (k-2)K_1 \prec P_{n-2,k-1}$. By Lemma 2.6, we have $P_{n,k} \prec P_{n,k-1}$. Thus for $3 \leqslant k \leqslant n-1$, we have $E(P_{n,k}) < E(P_{n,k-1})$. □

By Theorem 2.1 and Lemma 2.12, we immediately get the following results.

Corollary 2.1 [6]. *Let T be a tree with n vertices. Then*

(1) $E(T) \geqslant 2\sqrt{n-1}$, *the equality holds if and only if $T \cong S_n$.*
(2) *If* $T \ncong S_n$, $E(T) \geqslant \sqrt{2n-2+2\sqrt{n^2-6n+13}} + \sqrt{2n-2-2\sqrt{n^2-6n+13}}$, *the equality holds if and only if $T \cong P_{n,n-2}$.*

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