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Graphs whose positive semi-definite matrices have nullity at most two[☆]

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Abstract

Let $G = (V, E)$ be a undirected graph containing n vertices, and let $\mathscr{M}_G$ be the set of all Hermitian $n \times n$ matrices $M = (m_{i,j})$ with $m_{i,j} \neq 0$ if i and j are connected by one edge of G, with $m_{i,j} \in \mathbb{C}$ if i and j are connected by at least two edges, with $m_{i,j} = 0$ if $i \neq j$, and i and j are not connected by an edge of G, and with $m_{i,i}$ for $i = 1, \ldots, n$ a real number. What is the largest nullity attained by any positive semi-definite matrix $M \in \mathscr{M}_G$?

In this paper we characterize, for $t = 1$ and 2, those graphs G for which the maximum nullity is not greater than t.

Keywords: Positive semi-definite; Nullity; Symmetric matrices; Graph structure

1. Introduction

Suppose M is a positive semi-definite Hermitian matrix of which some of the off-diagonal entries are prescribed to be zero, some of them are prescribed to be nonzero, and some of them have no prescribed restrictions, and on the diagonal the entries are prescribed to be real. What can we say about the nullity of this matrix without exactly knowing the entries?

Let us reformulate the problem as follows. Let $G = (V, E)$ be an undirected graph with $V = \{1, \ldots, n\}$ and which is allowed to have multiple edges but which does not have loops. Let $\mathscr{M}_G$ be the set of all $n \times n$ Hermitian matrices $M = (m_{i,j})$

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with $m_{i,j} \neq 0$ if i and j are connected by one edge, $m_{i,j} \in \mathbb{C}$ if i and j are connected by at least two edges, $m_{i,j} = 0$ if $i \neq j$ and i and j are not adjacent in G, and $m_{i,i} \in \mathbb{R}$ for $i = 1, \ldots, n$. So, if i and j are connected by at least two edges, then we allow $m_{i,j} = 0$. What is the largest nullity attained by any positive semi-definite matrix $M \in \mathscr{M}_G$? (Notice that $m_{i,i} \geqslant 0$ for any such matrix M.) Conversely, if we are given that number k such that the largest nullity attained by any positive semi-definite $M \in \mathscr{M}_G$ is k, can we infer the structure of the graph G?

In this paper we describe, for $k = 1$ and 2, the graphs G for which the largest nullity attained by any positive semi-definite matrix $M \in \mathscr{M}_G$ is k.

The outline of the paper is as follows. In Section 2 we give preliminaries. In Section 3 we recall in short the graph invariant $\nu(G)$, introduced by Colin de Verdière in [4], and some theorems about it. Then we give a kind of Courant nodal theorem for Hermitian matrices, and we see that this gives a way to show upper bounds for the nullity of matrices. In the final section we characterize, for $k = 1$ and 2, the graphs G for which the largest nullity of any positive semi-definite matrix $M \in \mathscr{M}_G$ is at most k.

2. Preliminaries

2.1. Graph theory

For graph theory we refer to [2,6]. In this paper all graphs are undirected and are allowed to have multiple edges but no loops. Let $G = (V, E)$ be a graph and let $X \subseteq V$ (sometimes we use the notation VG for V and EG for E). Then the subgraph of G induced by the vertices of X is denoted by $G[X]$. A set of edges of G forms a *parallel class* if every two edges form a cycle and if the subset is maximal with respect to this property. Two edges are *in parallel* if they belong to the same parallel class. The *degree* of a vertex v of G is the number of incident edges.

If v is a vertex of G, then $G - v$ is the graph obtained from G by deleting v and the edges incident with it. By *contracting* edge $e = vw$ in G we mean the operation of deleting e and identifying v with w. A graph H that can be obtained from G by a series of edge deletions, edge contractions and deletions of isolated vertices is called a *minor* of G. We say that G has an H-minor if G has a minor isomorphic to H.

The graph K_n ($n \in \mathbb{N}$) denotes the complete (simple) graph on n vertices. The graph $K_{n,m}$ ($n, m \in \mathbb{N}$) denotes the simple graph in which the vertex set is partitioned into two classes, one of size n and one of size m, such that an edge connects two distinct vertices if and only if these vertices belong to different classes. The graph C_3^2 denotes the graph obtained from K_3 by replacing each edge by two edges in parallel. The graph T_3 denotes the graph obtained from K_3 by replacing each edge by a triangle. So C_3^2 is obtained from T_3 by contracting one edge incident to each vertex of degree two.

A graph is called *planar* if it can be embedded in the plane. It is called *outerplanar* if it can be embedded in the plane such that all vertices are incident to the infinite face. A graph is planar if and only if it has no K_5- or $K_{3,3}$-minor, and it is outerplanar if and only if it has no K_4- or $K_{2,3}$-minor.

A vertex v of a component C of a graph is a *cutvertex* if $C - v$ is disconnected. A *block* of a graph is a maximal connected subgraph of it without a cutvertex. A block of a graph that has at least three vertices is 2-connected. A block of a graph that has at most two vertices is either an isolated vertex or a parallel class with the ends of the edges.

Two graphs $G = (VG, EG)$ and $H = (VH, EH)$ are called 2-*isomorphic* if there is a bijection $\phi : EG \longrightarrow EH$ such that for each circuit C of G, $\phi(EC)$ forms the edge set of a circuit of H. (So the cycle matroid of G is isomorphic to the cycle matroid of H, see [12,14].)

A graph G is called a *suspended forest* if it is obtained from a forest T by adding a new vertex v and edges (allowing multiple edges) from this vertex v to some of the vertices of the forest. In the case that the forest T is a tree, we call G a *suspended tree*. If it is 2-connected, then the vertex v is adjacent to all vertices of degree one of the tree.

Any minor of a suspended forest is again a suspended forest. Hence, if G is 2-isomorphic to a suspended forest and G' is a minor of it, then also G' is 2-isomorphic to a suspended forest (if G has an isolated vertex then it is 2-isomorphic to the graph without that isolated vertex).

For any forest embedded in the plane, we have that each vertex of the forest belongs to the infinite face, and hence a suspended forest can be embedded in the plane. So we can talk about the dual of an embedding of the suspended forest in the plane (see [6, Chapter 4]).

Theorem 2.1. *A graph G is* 2*-isomorphic to a suspended forest if and only if the dual of an embedding of G is outerplanar*; *that is*, *if and only if G contains no K_4- or C_3^2-minor.*

Proof. It is clear that K_4 and C_3^2 are not 2-isomorphic to a suspended forest. So a graph 2-isomorphic to a suspended forest has no K_4- or C_3^2-minor.

Conversely, suppose G has no K_4- or C_3^2-minor. Then G is embeddable in the plane since K_4 is a minor of $K_{3,3}$ and of K_5. Since deletions and contractions of edges correspond to contractions and deletions, respectively, of edges in the dual of an embedding, the dual H of an embedding of G in the plane has no K_4- or $K_{2,3}$-minor, which means that H is outerplanar. Embed H in the plane such that all vertices of H are incident with the infinite face. Then the dual of H is 2-isomorphic to G. Disregarding the infinite face, the dual of H is a forest. Hence the dual of the embedding is a suspended forest. Hence G is 2-isomorphic to a suspended forest. □

For simple graphs we have:

Theorem 2.2. *A simple graph G is* 2*-isomorphic to a suspended forest if and only if it contains no K_4- or T_3-minor.*

2.2. Matrix theory

For matrix theory we refer to [11,13]. If $M = (m_{i,j})$ is a matrix, then we denote the conjugate transpose of M by M^*. A matrix M is *Hermitian* if $M^* = M$. The null-space of a matrix M is denoted by $\ker(M)$. The nullity of a matrix M is the dimension of $\ker(M)$; we denote this by $\mathrm{null}(M)$.

Let M be a Hermitian matrix. *Sylvester's Law of Inertia* tells us that A^*MA has the same number of negative, the same number of zero, and the same number of positive eigenvalues as M whenever A is a nonsingular matrix.

Let $n > 0$ be any integer. By I_n we denote the $n \times n$ identity matrix, and by 0_n we denote the $n \times n$ zero matrix. If M_1 is an $n_1 \times n_1$ matrix and M_2 is an $n_2 \times n_2$ matrix, then $M_1 \oplus M_2$ denotes the $(n_1 + n_2) \times (n_1 + n_2)$ matrix whose upper left $n_1 \times n_1$ submatrix is M_1, whose lower right $n_2 \times n_2$ submatrix is M_1, and whose upper right $n_1 \times n_2$ and lower left $n_2 \times n_1$ matrices are equal to zero.

Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$. Let $M = (m_{i,j})$ be an Hermitian $n \times n$ matrix. If V_1 and V_2 are subsets of V, then by $M_{V_1 \times V_2}$ we denote the submatrix of M consisting of all entries $m_{i,j}$ with $i \in V_1$ and $j \in V_2$.

3. Discrete magnetic Schrödinger operators

Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$. Recall that $\mathscr{M}_G$ is the set of all Hermitian $n \times n$ matrices $M = (m_{i,j})$ with

- $m_{i,j} \neq 0$ if i and j are connected by exactly one edge,
- $m_{i,j} \in \mathbb{C}$ if i and j are connected by at least two edges,
- $m_{i,j} = 0$ if $i \neq j$ and i and j are not adjacent in G,
- $m_{i,i} \in \mathbb{R}$.

These matrices were called discrete Schrödinger operators with a magnetic field in [4] for the case that G has no multiple edges and that G is connected, because they can be viewed as discrete analogues of the continuous version of Schrödinger operators with a magnetic field added. Discrete Schrödinger operators with magnetic fields are obtained if one makes the continuous version discrete using the methods of finite elements.

In the definition of $\mathscr{M}_G$ we have put the condition $m_{i,j} \in \mathbb{C}$ if i and j are connected by at least two edges. We shall see in Lemma 3.7 that, by subdividing all but one of the edges in every parallel class, we can reduce the case in which the graphs can have parallel edges to the case in which the graphs are simple.

For any graph $G = (V, E)$, let $\tau(G)$ be the largest nullity attained by any positive semi-definite matrix $M \in \mathscr{M}_G$. It is possible that $\tau(G') > \tau(G)$ if G' is a minor of G. An example is formed by the graph G' which consists of two isolated vertices.

Then G' is a minor of the graph G which consists of one edge. But $\tau(G') = 2 > \tau(G) = 1$ (see Theorems 4.1 and 4.3).

In [4], Colin de Verdière introduced the graph parameter $\nu(G)$, which looks like the invariant $\tau(G)$. For any connected simple graph the parameter $\nu(G)$ is defined as the largest nullity attained by any positive semi-definite matrix $M \in \mathscr{M}_G$ fulfilling the Strong Arnol'd Property. A matrix $M \in \mathscr{M}_G$ of nullity k is said to fulfill the Strong Arnol'd Property if the linear span of the tangent space of $\mathscr{M}_G$ at M and the tangent of the manifold of all Hermitian $n \times n$ matrices of nullity k is equal to the linear space of all Hermitian $n \times n$ matrices. This can be translated to the following criterion: a matrix $M \in \mathscr{M}_G$ fulfills the Strong Arnol'd Property if and only if for every Hermitian $n \times n$ matrix A there exists a Hermitian $n \times n$ matrix $B = (b_{i,j})$ with $b_{i,j} = 0$ if i and j are nonadjacent in G, such that for every $x \in \ker(M)$, $x^*Ax = x^*Bx$.

It turns out that with the Strong Arnol'd Property, the graph parameter is monotone under taking minors, that is:

Theorem 3.1 [4]. *Let G be a connected simple graph and let G' be a connected minor of G which is simple. Then $\nu(G') \leqslant \nu(G)$.*

The invariant $\nu(G)$ can be defined for graphs that have multiple edges and that are disconnected, for which we refer to [9]. For more information on $\nu(G)$ we refer to [4,10].

Clearly, $\tau(G) \geqslant \nu(G)$ for every graph G. The following proposition shows that for any connected simple graph G which is not a tree, there is a positive semi-definite matrix $M \in \mathscr{M}_G$ with $\mathrm{null}(M) \geqslant 2$, and that for any connected simple graph G which is not 2-isomorphic to a suspended tree, there is a positive semi-definite matrix $M \in \mathscr{M}_G$ with $\mathrm{null}(M) \geqslant 3$.

Proposition 3.2 [4]. $\nu(K_3) = 2$, $\nu(K_4) = 3$, *and* $\nu(T_3) = 3$.

Let $M \in \mathscr{M}_G$ be a Hermitian matrix and let $x \in \ker(M)$. Then the *support* of x is defined as $\{i \mid x_i \neq 0\}$; the notation for the support of x is $\mathrm{supp}(x)$. The vector x has *minimal support* if there is no nonzero $y \in \ker(M)$ with $\mathrm{supp}(y)$ a proper subset of $\mathrm{supp}(x)$.

The next theorem allows us to prove some upper bounds for the nullity of any positive semi-definite matrix $M \in \mathscr{M}_G$. It can be seen as a discrete version of the Courant nodal theorem; see [1,3,5] for the continuous version and see [7–9] for graphical versions.

Theorem 3.3. *Let $M \in \mathscr{M}_G$ be positive semi-definite. If $x \in \ker(M)$ has minimal support, then $G[\mathrm{supp}(x)]$ is connected.*

Proof. Suppose that $G[\mathrm{supp}(x)]$ is disconnected. Let $C_1, \dots, C_t$ be the components of $G[\mathrm{supp}(x)]$. Let $x(C_l)$, for $l = 1, \dots, t$, be the vector with $x(C_l)_j = x_j$

if $j \in C_l$ and $x(C_l)_j = 0$ otherwise. Since the support of $Mx(C_l)$ is a subset of $V \setminus \mathrm{supp}(x(C_l))$, $x(C_l)^*Mx(C_l) = 0$, which implies that $x(C_l) \in \ker(M)$ because M is positive semi-definite. This gives a contradiction, as $\mathrm{supp}(x(C_l))$ is a proper subset of $\mathrm{supp}(x)$. □

The following theorem is also proved in [4]. Here we give a different proof.

Theorem 3.4. *If G is a tree then* $\mathrm{null}(M) \leqslant 1$ *for every positive semi-definite* $M \in \mathscr{M}_G$.

Proof. Let G be a tree, and suppose to the contrary that there exists a positive semi-definite matrix $M \in \mathscr{M}_G$ with $\mathrm{null}(M) > 1$. Let $x \in \ker(M)$ be nonzero, with $x_v = 0$ for some vertex v of G, which exists because $\mathrm{null}(M) > 1$. We assume that x has minimal support; so $G[\mathrm{supp}(x)]$ is connected. Then $\mathrm{supp}(x)$ is not the whole set of vertices of G, and since G is connected, $\mathrm{supp}(x)$ must be adjacent to a vertex w with $x_w = 0$. But if w is adjacent to at least one vertex u_1 with $x_{u_1} \neq 0$ then it must adjacent to another vertex u_2 with $x_{u_2} \neq 0$, since $Mx = 0$. Since $\mathrm{supp}(x)$ induces a connected subgraph of G, it follows that G has a circuit—a contradiction. □

Theorem 3.5. *Let $G = (V, E)$ be a 2-connected graph which is 2-isomorphic to a suspended tree. Then* $\mathrm{null}(M) \leqslant 2$ *for every positive semi-definite matrix* $M \in \mathscr{M}_G$.

Proof. Let H be a suspended tree such that G is 2-isomorphic to H. Since H is a suspended tree, there is a vertex v such that $H - v$ is a tree; let t be a vertex of degree one of this tree. Let e be an edge connecting t and v.

Since G is 2-isomorphic to $H = (W, F)$, there is a bijection $\phi : F \longrightarrow E$ such that for each circuit C of G, $\phi^{-1}(E(C))$ forms the edge set of a circuit of H. Let G' be obtained from G by adding an edge parallel to $\phi(e)$. Then G' is 2-isomorphic to a suspended tree and $M \in \mathscr{M}_{G'}$. Let u and w be the ends of $\phi(e)$.

Suppose to the contrary that $\mathrm{null}(M) > 2$. Then there is a nonzero $x \in \ker(M)$ with $x_u = x_w = 0$; we assume that x has minimal support. Let Z be the set of vertices of $VG' \setminus \mathrm{supp}(x)$ which are connected to $\mathrm{supp}(x)$. Then, since $Mx = 0$, each vertex $z \in Z$ is connected by at least two edges to $\mathrm{supp}(x)$.

Since G' is 2-connected, there are two paths Q_1, Q_2 from u, w to $\mathrm{supp}(x)$ which are, except possibly for their ends in $\mathrm{supp}(x)$, vertex-disjoint. Let q_1 and q_2 be the ends of Q_1 and Q_2 in $\mathrm{supp}(x)$, respectively (so possibly $q_1 = q_2$). We may assume that the paths Q_1, Q_2 have, except for q_1, q_2, no other vertices in common with $\mathrm{supp}(x)$. For $i = 1, 2$, let p_i be the vertex of Q_i adjacent to q_i; then $p_1, p_2 \in Z$ and $p_1 \neq p_2$. Since $G'[\mathrm{supp}(x)]$ is connected, there is a spanning tree T for $G'[\mathrm{supp}(x)]$. The vertices p_1 and p_2 are connected each by at least two edges to this tree. Delete each edge of $G'[\mathrm{supp}(x)]$ not in this tree, and contract each edge in this tree. Since p_1 and p_2 were connected each by at least two edges to $\mathrm{supp}(x)$, we have that p_1 and p_2 are connected each by at least two edges to the new vertex that results from contracting T. Now contract each edge in the subpaths of Q_1 and Q_2 between u and p_1, and between w and p_2. Since there is an edge parallel to e, we get that the resulting graph has a subgraph isomorphic to C_3^2, and hence G' has a minor isomorphic to C_3^2. But a graph 2-isomorphic to a suspended tree cannot have a C_3^2-minor, a contradiction, and hence $\mathrm{null}(M) \leqslant 2$. □

The following two lemmata will be needed in the next section.

Lemma 3.6. *Let $G = (V, E)$ be a graph and let $M = (m_{i,j}) \in \mathscr{M}_G$ be a positive semi-definite matrix. Let v be a vertex of G of degree one. Let $G' = (V', E')$ be the graph obtained from G by deleting v. Then there is a positive semi-definite matrix $M' \in \mathscr{M}_{G'}$ with* $\mathrm{null}(M') = \mathrm{null}(M)$.

Proof. We first show that $m_{v,v} > 0$. Suppose to the contrary that $m_{v,v} = 0$. Then the vector x with $x_i = 0$ for $i \neq v$ and with $x_v = 1$, belongs to $\ker(M)$ as $x^*Mx = 0$ and M is positive semi-definite. Let w be the vertex of G to which v is adjacent. Then w is connected to $\mathrm{supp}(x)$ by one edge, as v has degree one, which gives a contradiction. Hence $m_{v,v} > 0$.

Let $L = VG \setminus \{v, w\}$ and let n_2 be the number of vertices in L. We may write

$$M = \begin{pmatrix} m_{v,v} & m_{v,w} & 0 \\ m_{w,v} & m_{w,w} & M_{\{w\}\times L} \\ 0 & M_{L\times\{w\}} & M_{L\times L} \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 1 & -m_{v,v}^{-1}m_{v,w} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n_2} \end{pmatrix}.$$

By Sylvester's Law of Inertia,

$$A^*MA = \begin{pmatrix} m_{v,v} & 0 & 0 \\ 0 & m_{w,w} - m_{w,v}m_{v,v}^{-1}m_{v,w} & M_{\{w\}\times L} \\ 0 & M_{L\times\{w\}} & M_{L\times L} \end{pmatrix}$$

is positive semi-definite and has the same nullity as the matrix M. As $m_{v,v} > 0$, the matrix

$$M' := \begin{pmatrix} m_{w,w} - m_{w,v}m_{v,v}^{-1}m_{v,w} & M_{\{w\}\times L} \\ M_{L\times\{w\}} & M_{L\times L} \end{pmatrix}$$

is positive semi-definite and has the same nullity as M. It is clear that $M' \in \mathscr{M}_{G'}$. □

Lemma 3.7. *Let G be a connected graph and let v be a vertex of degree two with two distinct neighbors in G which are connected by an edge. Let G' be obtained from G by deleting v (and its incident edges) and by connecting the neighbors of v by* *an additional edge. Then for each positive semi-definite matrix* $M = (m_{i,j}) \in \mathscr{M}_G$ *there exists a positive semi-definite matrix* $M' \in \mathscr{M}_{G'}$ *with* $\mathrm{null}(M') = \mathrm{null}(M)$.

Proof. Since v is a vertex of degree two with two neighbors, $m_{v,v} > 0$, for the same reason as at the beginning of the proof of previous lemma. Let the two neighbors of v be s_1 and s_2. Let $L := VG \setminus \{v, s_1, s_2\}$ and let n_2 denote the number of vertices in L. Write

$$M = \begin{pmatrix} m_{v,v} & m_{v,s_1} & m_{v,s_2} & 0 \\ m_{s_1,v} & m_{s_1,s_1} & m_{s_1,s_2} & M_{\{s_1\}\times L} \\ m_{s_2,v} & m_{s_2,s_1} & m_{s_2,s_2} & M_{\{s_2\}\times L} \\ 0 & M_{L\times\{s_1\}} & M_{L\times\{s_2\}} & M_{L\times L} \end{pmatrix},$$

and let

$$A = \begin{pmatrix} 1 & -m_{v,v}^{-1}m_{v,s_1} & -m_{v,v}^{-1}m_{v,s_2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{pmatrix}.$$

Let

$$M' := \begin{pmatrix} m_{s_1,s_1} - m_{s_1,v}m_{v,v}^{-1}m_{v,s_1} & m_{s_1,s_2} - m_{s_1,v}m_{v,v}^{-1}m_{v,s_2} & M_{\{s_1\}\times L} \\ m_{s_2,s_1} - m_{s_2,v}m_{v,v}^{-1}m_{v,s_1} & m_{s_2,s_2} - m_{s_2,v}m_{v,v}^{-1}m_{v,s_2} & M_{\{s_2\}\times L} \\ M_{L\times\{s_1\}} & M_{L\times\{s_2\}} & M_{L\times L} \end{pmatrix}.$$

Then $M' \in \mathscr{M}_{G'}$. Since

$$A^*MA = \begin{pmatrix} m_{v,v} & 0 \\ 0 & M' \end{pmatrix},$$

we have, by Sylvester's Law of Inertia, that M' is positive semi-definite and that $\mathrm{null}(M') = \mathrm{null}(M)$. □

4. Matrices with nullity bounded by t for $t = 1, 2$

In this section we characterize, for $t = 1, 2$, the graphs G with $\mathrm{null}(M) \leqslant t$ for each positive semi-definite matrix $M \in \mathscr{M}_G$.

Theorem 4.1. $\mathrm{null}(M) \leqslant 1$ *for every positive semi-definite matrix* $M \in \mathscr{M}_G$ *if and only if* G *is a tree.*

Proof. If G is a tree, then, by Theorem 3.4, $\mathrm{null}(M) \leqslant 1$ for every positive semi-definite matrix $M \in \mathscr{M}_G$.

If G is not a tree, then either G is disconnected, or G is connected and G has a circuit. Let us first assume that G is disconnected. Let C_i, $i = 1, \ldots, k$ be the components of G. Let $M_{V(C_i)\times V(C_i)} \in \mathscr{M}_{C_i}$, for $i = 1, \ldots, k$, be positive semi-definite, with $\mathrm{null}(M_{V(C_i)\times V(C_i)}) \geqslant 1$, which exists because by choosing the right diagonal entries, we may assume that the smallest eigenvalue of $M_{V(C_i)\times V(C_i)}$ is equal to zero. Let $M := M_{V(C_1)\times V(C_1)} \oplus M_{V(C_2)\times V(C_2)} \oplus \cdots \oplus M_{V(C_k)\times V(C_k)}$. Then $\mathrm{null}(M) \geqslant 2$. Hence we may assume that G is connected and that G has a circuit. Let G' be the graph obtained from G by subdividing for every parallel class all but one edge. Then G' is a simple graph. Since it has a circuit, it has a K_3-minor. By Proposition 3.2, $\nu(K_3) = 2$, and hence $\nu(G') \geqslant 2$, which implies that there exists a positive semi-definite matrix $M' \in \mathscr{M}'_G$ with $\mathrm{null}(M') \geqslant 2$. By Lemma 3.7 there exists a positive semi-definite matrix $M \in \mathscr{M}_G$ with $\mathrm{null}(M) \geqslant 2$. □

In the proof of the characterization of the graphs G with $\mathrm{null}(M) \leqslant 2$ for each positive semi-definite matrix $M \in \mathscr{M}_G$, we need the following lemma:

Lemma 4.2. *Let G_1 and G_2 be graphs, and let G be a graph obtained from G_1 and G_2 by identifying a vertex G_1 with a vertex of G_2. If there are positive semi-definite matrices $N \in \mathscr{M}_{G_1}$ and $L \in \mathscr{M}_{G_2}$ with* $\mathrm{null}(N) \geqslant 2$ *and* $\mathrm{null}(L) \geqslant 2$, *then there is a positive semi-definite matrix $M \in \mathscr{M}_G$ with* $\mathrm{null}(M) \geqslant 3$.

Proof. Let $N' := N \oplus 0_{|VG_2|-1}$, let $L' := 0_{|VG_1|-1} \oplus L$, and let $M := N' + L'$. Then $\mathrm{rank}(M) \leqslant \mathrm{rank}(N') + \mathrm{rank}(L') \leqslant |VG_1| + |VG_2| - 4$, and hence $\mathrm{null}(M) = |VG_1| + |VG_2| - 1 - \mathrm{rank}(M) \geqslant 3$. □

Theorem 4.3. *Let G be a graph. Then* $\mathrm{null}(M) \leqslant 2$ *for every positive semi-definite matrix $M \in \mathscr{M}_G$ if and only if G is 2-isomorphic to a suspended forest and one of the following holds*:

(1) *G has two components, each of which is a tree, or*
(2) *G is connected and there is at most one block containing a circuit*

Proof. Let G be a graph such that $\mathrm{null}(M) \leqslant 2$ for every positive semi-definite $M \in \mathscr{M}_G$. Then G is 2-isomorphic to a suspended forest. For if not, let G' be the graph obtained from G by subdividing for every parallel class all but one edge. Then G' is a simple graph. Since it is not 2-isomorphic to a suspended forest, it has a T_3- or K_4-minor. By Proposition 3.2, $\nu(K_4) = 3$ and $\nu(T_3) = 3$, and hence $\nu(G') \geqslant 3$, which implies that there exists a positive semi-definite matrix $M' \in \mathscr{M}'_G$ with $\mathrm{null}(M') \geqslant 3$. By Lemma 3.7 there exists a positive semi-definite matrix $M \in \mathscr{M}_G$ with $\mathrm{null}(M) \geqslant 3$.

First suppose that G is disconnected with components $C_1, \ldots, C_l$. Then taking for each component C_i a positive semi-definite matrix $M_{C_i} \in \mathscr{M}_{C_i}$ with maximal nullity, shows that there are at most two components and that each component is a tree. We can therefore assume that G is connected.

Suppose to the contrary that there are at least two blocks containing a circuit. Choose in G two of these blocks C_1 and C_2 such that the distance d between C_1 and

C_2 is as short as possible. Let P be a path connecting C_1 to C_2 of length d (so we allow P to have only one vertex). Let s be any vertex on P. Then s is a cutvertex, and there are distinct components D_1 and D_2 among the components of $G - s$, such that C_i is a subgraph of $H_i := G[V(D_i) \cup \{s\}]$, for $i = 1, 2$. By Theorem 4.1, we can find for $i = 1, 2$ positive semi-definite matrices $N_i \in \mathscr{M}_{H_i}$ with $\text{null}(N_i) \geqslant 2$. By Lemma 4.2, there exists a positive semi-definite matrix $M \in \mathscr{M}_G$ with $\text{null}(M) \geqslant 3$. This is a contradiction, and hence G is connected and has at most one block containing a circuit.

Conversely, suppose that G is 2-isomorphic to a suspended forest and that one of the following holds:

(1) G has two components, each of which is a tree, or
(2) G is connected and there is at most one block containing a circuit.

Let us first assume that G has two components and that each component is a tree. Since $\text{null}(M) \leqslant 1$ for each positive semi-definite matrix $M \in \mathscr{M}_T$ if T a tree, $\text{null}(M) \leqslant 2$ for each positive semi-definite matrix $M \in \mathscr{M}_G$.

So we may assume that G is connected and that it has at most one block containing a circuit. If G has no block containing a circuit then G is a tree. By Theorem 3.4, $\text{null}(M) \leqslant 1$. Hence, we can assume that G has a block containing a circuit; let this block be C. We can view G as C with several trees attached to it; that is, G is obtained from C and a collection of trees by identifying a vertex of each tree in this collection with a vertex of C. Hence we can always find a vertex of degree one, unless $C = G$. By repeatedly applying Lemma 3.6, $\text{null}(M)$ is at most the largest nullity of any positive semi-definite $N \in \mathscr{M}_C$. If C has two vertices, then, clearly, the largest nullity attained by any positive semi-definite $N \in \mathscr{M}_C$ is two. Hence we can assume that C has at least three vertices, so C is 2-connected. By Theorem 3.5, the largest nullity attained by any positive semi-definite $N \in \mathscr{M}_C$ is two. Hence $\text{null}(M) \leqslant 2$. □

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References

[1] G. Besson, Sur la multiplicité de la première valeur propre de surfaces riemanniennes, Ann. Inst. Fourier 30 (1) (1980) 109–128.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, 1976.
[3] S.Y. Cheng, Eigenfunctions and nodal sets, Comment. Math. Helv. 51 (1976) 43–55.

[4] Y. Colin de Verdière, Multiplicities of eigenvalues and tree-width of graphs, J. Combin. Theory Ser. B 74 (1998) 121–146.
[5] R. Courant, D. Hilbert, Methods of Mathematical Physics, vol. 1, Interscience, New York, 1953.
[6] R. Diestel, Graph Theory, second ed., Springer-Verlag, New York, 2000.
[7] M. Fiedler, A property of eigenvector of nonnegative symmetric matrices and its applications to graph theory, Czech. Math. J. 25 (1975) 619–633.
[8] J. Friedman, Some geometric aspects of graphs and their eigenfunctions, Duke Math. J. 69 (1993) 487–525.
[9] H. van der Holst, Topological and Spectral Graph Characterizations, Ph.D. thesis, University of Amsterdam, 1996.
[10] A. Kotlov, Spectral characterization of tree-width-two graphs, Combinatorica 20 (1) (2000) 147–152.
[11] P. Lancaster, M. Tismenetsky, The Theory of Matrices, second ed., with Applications, Academic Press, Orlando, 1985.
[12] J.G. Oxley, Matroid Theory, Oxford University Press, 1992.
[13] G. Strang, Linear Algebra and its Applications, second ed., Academic Press, Orlando, 1980.
[14] K. Truemper, Matroid Decomposition, Academic Press, San Diego, 1992.