Survey of Results Related to the Minimum Rank Problem

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The minimum rank problem for a simple graph (the minimum rank problem for short) asks us to determine the minimum rank among real symmetric matrices whose zero-nonzero pattern of off-diagonal entries is described by a given simple graph G; this problem has received considerable attention (see references) and is a main focus of this workshop. Since the maximum rank is always the order of G (e.g., use a diagonally dominant matrix), there is no interest in an analogous maximum rank problem.

The zero-nonzero pattern described by the graph has tremendous influence on minimum rank; for example, a matrix associated with a path on n vertices, i.e., a symmetric tridiagonal matrix with nonzero sub- and super-diagonal, has minimum rank n − 1, whereas the complete graph on n vertices has minimum rank 1.

The solution to the minimum rank problem is equivalent to the determination of the maximum multiplicity of an eigenvalue among the same family of matrices. Perhaps the earliest result on multiplicities of eigenvalues of such a family is Parter’s
1960 paper [P60] on trees, and most of the work on trees since then has relied on this work and Wiener’s 1984 paper [W84]. In 1996 Nylen initiated the study of minimum rank and gave a method for its computation for a tree, subsequently improved by others (see algorithms in Subsection 2.3). Only limited progress has been made on graphs that are not trees.

The known results for the minimum rank problem are summarized in Section 2 (Section 1 covers preliminary definitions). Subsequent sections survey related problems, including information about possible spectra of trees (Section 4) and minimum rank problems for other families of matrices related to a graph (Section 5). Each section is organized into Definitions, Results, and Examples. For results that are nontrivial, the original source is provided. For some results, a comment is provided; [H05] also provides general background. Many of the referenced papers are supplied in PDF format; in this case, the comment “PDF provided” appears in the bibliography item and the PDF file has the same file name as the citation, e.g., reference [BF04] is the file BF04.pdf.

Except when explicitly stated otherwise (primarily in Section 5), all matrices discussed are real and symmetric and all graphs are simple.

1 Preliminaries

This section contains basic background notation, definitions and results from linear algebra and graph theory. Suitable background references for linear algebra include [Z99] and [HJ85]; for graph theory, [W01] or [D00] (note that the latter uses the term “graph” to mean what we call a simple graph); or [HLA06] for everything discussed.

Definitions:

• A graph is a pair $G = (V, E)$, where $V$ is the set of vertices (usually $\{1, \ldots, n\}$ or a subset thereof) and $E$ is the set of edges. A graph allows multiple edges and/or loops. Every graph is finite (finite number of vertices and finite number of edges) and has nonempty vertex set.

• A simple graph allows neither multiple edges nor loops.

• The order of $G$, denoted $|G|$, is the number of vertices of $G$.

• A graph $G' = (V', E')$ is a subgraph of graph $G = (V, E)$ if $V' \subseteq V, E' \subseteq E$.

• The result of deleting a vertex $v$ of $G$ and its incident edges is denoted $G - v$. If $S \subset V$, $G - S$ is the result of deleting all vertices of $S$ and their incident edges from $G$. 
• The subgraph $G[S]$ of $G = (V, E)$ induced by $S \subseteq V$ is the subgraph with vertex set $S$ and edge set $\{\{i, j\} \in E \mid i, j \in S\}$.

• The union of $G_i = (V_i, E_i), i = 1, \ldots, h$ is $\bigcup_{i=1}^{h} G_i = (\bigcup_{i=1}^{h} V_i, \bigcup_{i=1}^{h} E_i)$.

• The intersection of $G_i = (V_i, E_i), i = 1, \ldots, h$ is $\bigcap_{i=1}^{h} G_i = (\bigcap_{i=1}^{h} V_i, \bigcap_{i=1}^{h} E_i)$ (provided the intersection of the vertices is nonempty).

• A graph $(V, E)$ is bipartite if the vertex set $V$ admits a partition into two parts, such that no edge of $E$ has both endpoints in one part (thus there are no loops).

• A simple graph $(V, E)$ is complete if $E$ consists of all sets of two distinct vertices from $V$.

• A bipartite simple graph $(V, E)$ with nonempty parts $V_1$ and $V_2$ is complete bipartite if $E$ consists of all unordered pairs from $V$ with one vertex in $V_1$ and one in $V_2$.

• The join $G \lor G'$ of two disjoint graphs $G = (V, E)$ and $G' = (V', E')$ is the union of $G \cup G'$ and the complete bipartite graph with with vertex set $V \cup V'$ and partition $\{V, V'\}$.

• A path in a graph is an alternating sequence $(v_{i_1}, e_{i_1}, \ldots, v_{i_{\ell-1}}, v_{i_\ell})$ of distinct vertices and edges such that $v_{i_j}$ and $v_{i_{j+1}}$ are endpoints of $e_{i_j}$ for $j = 1, \ldots, \ell$.

• A cycle in a graph is an alternating sequence $(v_{i_1}, e_{i_1}, \ldots, v_{i_{\ell-1}}, v_{i_{\ell}}, v_{i_1})$ of distinct vertices and edges such that $v_{i_j}$ and $v_{i_{j+1}}$ are endpoints of $e_{i_j}$ for $j = 1, \ldots, \ell$, and $v_{i_{\ell}}$ and $v_{i_1}$ are endpoints of $e_{i_{\ell}}$.

• A graph is acyclic if it does not contain a cycle.

• The length of a path or cycle is the number of edges.

• A graph is connected if there is a path from any vertex to any other vertex. Otherwise it is disconnected. (A graph of order one is connected whether or not it has a loop.)

• A tree is a connected acyclic simple graph.

• A (connected) component of a graph is a maximum connected (induced) subgraph.

• The following simple graphs are denoted by special symbols:
  
  – $P_n$: path on $n$ vertices,
- $C_n$: cycle on $n$ vertices,
- $K_n$: complete graph on $n$ vertices
- $K_{p,q}$: complete bipartite graph on $p$ and $q$ vertices
- $\overline{G}$: complement of $G$

- The **distance** between two vertices in a graph $G$ is the number of edges in a shortest path between them.

- The **diameter** of $G$, $\text{diam}(G)$, is the maximum distance between any two vertices of $G$.

- An induced subgraph $G'$ of a graph $G$ is a **clique** or **complete subgraph** if $G'$ has an edge between every pair of vertices of $G'$ (i.e., $G'$ is isomorphic to $K_n$ where $n = |G'|$).

- A set of subgraphs of $G$, each of which is a clique and such that every edge of $G$ is contained in at least one of these cliques, is called a **clique covering** of $G$.

- The **clique covering number** of $G$, denoted by $\theta(G)$, is the smallest number of cliques in a clique covering of $G$.

- An induced subgraph $G'$ of a graph $G$ is a **coclique** or **independent set of vertices** if $G'$ has no edges.

- The largest number $k$ for which a coclique with $k$ vertices exists is called the **vertex independence number** of $G$ and denoted by $\alpha(G)$.

- A **vertex coloring** of $G$ is a partition of the vertex set into cocliques. A **color class** is one of the cocliques in such a partition into color classes.

- If $G$ does not have loops, the **chromatic number** $\chi(G)$ of $G$ is the smallest number of color classes of any vertex coloring of $G$.

- A vertex $v$ of a connected graph $G$ is a **cut-vertex** if $G - v$ is disconnected. More generally, $v$ is a cut-vertex of a graph $G$ if $v$ is a cut-vertex of a component of $G$.

- A graph is **2-connected** if its order is at least 3 and it has no cut-vertex.

- A subset $S$ of vertices of a connected graph $G$ is a **cut-set** if $G - S$ is disconnected. More generally, $S$ is a cut-set of a graph $G$ if $S$ is contained in one component of $G$ and $S$ a cut-set of that component of $G$. 
• The dual of a plane embedding of a planar graph $G$ is obtained as follows: Place a new vertex in each face of the embedding; these are the vertices of the dual. Two dual vertices are adjacent if and only if the two faces of $G$ share an edge of $G$.

• A graph $G$ is chordal if it does not contain an induced cycle on four or more vertices.

• The open neighborhood of a vertex $w$, $N(w)$ is the set of all vertices adjacent to $w$.

• The closed neighborhood of a vertex $w$ is $N(w) \cup \{w\}$.

• A vertex $u$ is a duplicate of a vertex $v$ if $u$ and $v$ have the same closed neighborhoods.

• For a matrix $A \in F^{n \times n}$, the spectrum of $A$ is the multiset of the $n$ roots of the characteristic polynomial in the algebraic closure of $F$, and is denoted $\sigma(A)$.

• Let $S_n$ denote the set of real symmetric $n \times n$ matrices.

• For $B \in S_n$, let $\text{mult}_B(\lambda)$ denote the multiplicity of $\lambda$ as a root of the characteristic polynomial of $B$ (i.e., the algebraic multiplicity of $\lambda$ if $\lambda$ is an eigenvalue of $B$ and 0 otherwise).

• For $B \in S_n$, let $\text{gmult}_B(\lambda)$ denote the geometric multiplicity of $\lambda$ as an eigenvalue of $B$ (i.e., the dimension of $\ker(B - \lambda I)$).

• If $R \subseteq \{1, 2, \ldots, n\}$ and $B \in S_n$, then $B[R]$ denotes the principal submatrix of $B$ whose rows and columns are indexed by $R$, and $B(R)$ is the complementary principal submatrix obtained from $B$ by deleting the rows and columns indexed by $R$. In the special case when $R = \{v\}$ is a singleton, we let $B(v) = B(R)$.

• A real symmetric matrix $B$ is positive semidefinite if for all $x \in \mathbb{R}^n$, $x^T B x \geq 0$. More generally, a matrix $B \in \mathbb{C}^{n \times n}$ is positive semidefinite if for all $x \in \mathbb{C}^n$, $x^T B x \geq 0$ (a complex positive semidefinite matrix is necessarily Hermitian).

• For $B \in S_n$, the graph of $B$, denoted $\mathcal{G}(B)$, is the simple graph with vertices $\{1, \ldots, n\}$ and edges $\{i, j\} \mid b_{ij} \neq 0$ and $i \neq j$. Note that the diagonal of $B$ is ignored in determining $\mathcal{G}(B)$. 
**Results:**

Let $B \in S_n, k \in \{1, \ldots, n\}, R \subseteq \{1, \ldots, n\}$.

1. All eigenvalues of $B$ are real and the algebraic multiplicity is the same as the geometric multiplicity.

2. $\text{mult}_B(0) + \text{rank}(B) = n$.
   (Since $\text{mult}_B(0) = \text{gmult}_B(0)$.)

3. $\text{rank}(B) - 2 \leq \text{rank}B(k) \leq \text{rank}(B)$.
   ($B(k)$ is a submatrix of $B$. $B(k)$ is obtained form $B$ by the deletion of 2 lines (one row, one column), and removing a line can decrease rank by at most 1.)

4. $\text{rank}B[R] \leq \text{rank}B$.

5. $B$ is positive semidefinite if and only if every eigenvalue of $B$ is nonnegative.
   See [Z99], [HJ85], or [HLA06] for many other equivalent conditions.

6. [Z99, HJ85] Interlacing Theorem Let $k \in \{1, \ldots, n\}$. If the eigenvalues of $B$ are $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$ and the eigenvalues of $B(k)$ are $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_{n-1}$, then $\beta_1 \leq \theta_1 \leq \beta_2 \leq \theta_2 \leq \cdots \leq \beta_{n-1} \leq \theta_{n-1} \leq \beta_n$.

7. If $\lambda \in \sigma(B)$, then $\text{mult}_B(\lambda) - 1 \leq \text{mult}_{B(k)}(\lambda) \leq \text{mult}_B(\lambda) + 1$.
   (Follows from Interlacing Theorem.)

8. $G(B(k)) = G - k$.
   (There is a slight abuse of notation here- technically we need to index the entries of $B(k)$ by $1, \ldots, k-1, k+1, \ldots, n$.)

9. Let $A \in \mathbb{C}^{n \times n}$ be positive semidefinite with rank $A = r < n$. Then $A$ can be factored as $A = B^*B$ with $B \in \mathbb{C}^{r \times n}$. If $A$ is a real matrix, then $B$ can be taken to be real and $A = B^TB$. Equivalently, there exist vectors $v_1, v_2, \ldots, v_n \in \mathbb{C}^r$ (or $\mathbb{R}^r$) such that $a_{ij} = v_i^*v_j$ (or $v_i^Tv_j$).

**Examples:**

1. For the matrix $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 3.1 & -1.5 & 2 \\ 0 & -1.5 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$, $G(B)$ is shown in Figure 1.
2 Minimum Rank of Symmetric Matrices described by a Simple Graph

2.1 Basics

All graphs in this subsection are simple.

Definitions:
Let $B \in S_n$ and let $G$ be a simple graph with vertices $\{1, \ldots, n\}$.

- The set of symmetric matrices of graph $G$ is
  \[ S(G) = \{ B \in S_n : G(B) = G \} \]

- The minimum rank of $G$ is
  \[ \text{mr}(G) = \min \{ \text{rank}(B) : B \in S(G) \} \]

- The maximum multiplicity of $G$ is
  \[ M(G) = \max \{ \text{mult}_B(\lambda) : B \in S(G), \lambda \in \mathbb{R} \} \]

Results:
Let $B \in S_n$ and let $G$ be a simple graph with vertices $\{1, \ldots, n\}$.

1. The adjacency matrix $A_G$, the Laplacian matrix $L_G$ and the signless Laplacian $|L_G|$ are all in $S(G)$, but they generally do not achieve the minimum rank.

2. If $B \in S(G)$ and $R \subseteq V(G), k \in V(G)$, then $B[R] \in S(G[R])$ and $B(k) \in S(G - k)$.
   
   (There is a slight abuse of notation here- technically we need to index the entries of $B[R]$ by $R$.)
3. If the connected components of $G$ are $G_1, \ldots, G_t$, then
\[ \text{mr}(G) = \sum_{i=1}^{t} \text{mr}(G_i). \]
(With a properly chosen ordering of vertices, a matrix in $\mathcal{S}(G)$ is block diagonal.)

4. If $G'$ is an induced subgraph of $G$ then $\text{mr}(G') \leq \text{mr}(G)$.
(Any matrix in $\mathcal{S}(G')$ can be embedded as a principal submatrix of a matrix in $\mathcal{S}(G)$.)

5. $M(G) + \text{mr}(G) = |G|$.
(Consider eigenvalue 0, translating by a multiple of the identity if necessary.)

6. $\text{mr}(G) \leq |G| - 1$.
(Consider the Laplacian, $L_G$.)

7. For the path on $n$ vertices, $\text{mr}(P_n) = n - 1$.
(For $B \in \mathcal{S}(P_n)$, delete row 1 and column $n$ to obtain an upper triangular matrix with nonzero diagonal.)

8. [F69] If $\text{mr}(G) = |G| - 1$, then $G = P_{|G|}$.

9. For a connected simple graph $G$, $\text{diam}(G) \leq \text{mr}(G)$.
(If $d = \text{diam}(G)$, then $G$ contains $P_{d+1}$ as an induced subgraph.)

10. For a connected simple graph $G$ of order $n > 1$, $\text{mr}(G) = 1$ if and only if $G = K_n$.
(If $\text{mr}(G) = 1$ then by Result 9, $\text{diam}(G) = 1$, i.e., $G = K_n$.)

11. For the cycle on $n$ vertices, $\text{mr}(C_n) = n - 2$.
($C_n$ is not a path and $P_{n-1}$ is an induced subgraph.)

Examples:

1. For the graph $G$ in Figure 1, $\text{mr}(G) = 2$, because $G$ has 4 vertices, $G \neq K_4$ and $G \neq P_4$. The matrix $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \in \mathcal{S}(G)$ and $A$ realizes minimum rank (and illustrates Result 1 in Subsection 2.4).
2.2 Special Vertices

The minimum rank of a tree can be easily computed through a variety of techniques (see Subsection 2.3), most involving the use of certain special vertices. This subsection surveys the definitions of the various types of special vertices that have been used.

All graphs in this subsection are simple.

Definitions:

Let $B \in S_n$ and let $G$ be a simple graph with vertices $\{1, \ldots, n\}$.

- A path $P$ in $G$ is a **pendent path** of vertex $v$ if $P$ is a component of $G - v$ and (in $G$) $P$ is connected to $v$ by one of its end-points.

- [JS02] A vertex of degree at least 3 is a **high degree vertex**; otherwise it is a **low degree vertex**. The set of high degree vertices of $G$ is denoted $H(G)$.

- A tree is a **generalized star** if it has at most one high degree vertex; if a generalized star has a high degree vertex, this vertex is called the **center**.

- A **pendent generalized star** of $G$ is a connected induced subgraph $S$ of $G$ such that:
  1. There is exactly one vertex $v$ of $S$ that is a high degree vertex of $G$.
  2. $G - v$ has $k + 1$ components and exactly $k$ of the components of $G - v$ are pendent paths of $v$.
  3. $S$ is induced by the vertices of the $k$ pendent paths and $v$.

  The high degree vertex is called the **center**.

- [N96] A vertex $v$ of $T$ is **appropriate** if $T - v$ has at least two components that are pendent paths.

- [WW01] A vertex $v$ of $G$ **typical** if $v$ has at least two low-degree neighbors.

- [BFH04] A vertex $v$ is **rank-strong** if $mr(G) = mr(G - v) + 2$.

Results:

1. A center of a pendent generalized star or generalized star is appropriate, but not conversely. See Example 1.

2. [WW01] An appropriate vertex is typical, but not conversely. See Example 2.

(For an appropriate vertex, the end-points of the 2 or more pendent paths are the required low degree neighbors.)
3. [BFH04] In a tree, a typical vertex is rank-strong, but not conversely. In a graph that is not a tree, a typical vertex need not be rank-strong. See Example 3.

4. A high degree vertex $v$ is typical if and only if $\deg_G(v) - \deg_{H(G)}(v) \geq 2$.
   (The low degree neighbors cause the degree difference and vice versa.)

Examples:

1. The converse of Result 1 is not true, as can be seen by considering a degree 2 vertex in a path on at least 3 vertices.

2. [WW01] The converse of Result 2 is not true. Let $T$ be the tree in Figure 2. Vertex $u$ is not an appropriate vertex but is a typical vertex.

   Figure 2: A tree with a typical vertex $u$ that is not appropriate

3. [BFH04] The converse of Result 3 is not true. Let $T$ be the tree in Figure 3. Vertex $v$ is not a typical vertex. However $v$ is a rank-strong vertex, since $mr(T) = 5$, $mr(T - v) = 3$ (these minimum ranks can be calculated using algorithms in the next subsection).

   Figure 3: A tree with a rank-strong vertex $v$ that is not typical

In the cycle $C_n$, every vertex is typical but none is rank-strong.
2.3 Trees

All graphs in this subsection are simple.

Definitions:
Let \( B \in S_n, 1 \leq k \leq n \), and let \( G \) be a simple graph.

- \( k \) is a Parter-Wiener (PW) vertex of \( B \) for eigenvalue \( \lambda \) if 
  \[ \text{mult}_{B(k)}(\lambda) = \text{mult}_B(\lambda) + 1. \]

- Vertex \( k \) is a strong PW vertex of \( B \) for \( \lambda \) if \( k \) is a PW vertex of \( B \) for \( \lambda \) and \( \lambda \) is an eigenvalue of at least three components of \( G(B) - k \).

What is here called a strong PW vertex has also been called a Parter vertex and a Wiener vertex in the literature, and the term weak has sometimes been applied to what is here called a PW vertex.

- [JLD99] \( \Delta(G) = \max\{p - q : \text{there is a set of } q \text{ vertices whose deletion leaves } p \text{ paths}\} \) (Note that an isolated vertex is a path of order 1.)

- The path cover number of \( G \), \( P(G) \), is the minimum number of vertex disjoint paths occurring as induced* subgraphs of \( G \) that cover all the vertices of \( G \). *Note: some authors do not require the paths to be induced; this distinction is irrelevant for trees, but relevant for graphs.

Results:
Let \( T \) be a tree.

1. [N96] If \( T \) has at least 3 vertices, then \( T \) has an appropriate vertex.

2. Any tree is a generalized star or has a pendent generalized star.

   Proof: A path is a generalized star. If \( T \) is not a path, define \( T' \) to be the tree obtained from \( T \) by removing every path that is pendent from a high degree vertex. If \( |T'| = 1 \) then \( T \) is a generalized star; otherwise, any vertex \( v \) of degree 1 in \( T' \) together with its pendent paths (in \( T \)) forms a pendent generalized star (such a vertex must exist since \( T' \) is a tree).

3. Nylen [N96] gave the first method for computing minimum rank by successively deleting appropriate vertices.

4. [JLD99] \( M(T) = P(T) = \Delta(T) = n - \text{mr}(T) \).

5. [JS02] \( \Delta(T) \) and hence \( \text{mr}(T) \) can be computed by Algorithm 1.
Algorithm 1: Computation of mr(T) and Δ(T).
Input: A tree T on n vertices.
Output: mr(T), Δ(T), and a set Q of vertices of T whose deletion realizes Δ(T).
1. Set Q = ∅ and T′ = T.
2. While H(T′) ≠ ∅:
   a) Remove from T′ the set Q′ of all high degree typical vertices, i.e., the set of vertices v of H(T′) such that deg_T′(v) − deg_H(T′)(v) ≥ 2.
   b) Q = Q ∪ Q′.
3. Δ(T) = p − |Q| where p is the number of components (all of which are paths) in T − Q.
4. mr(T) = n − Δ(T).

Algorithm 2: Computation of mr(T) and Δ(T) for a tree T.
Input: A tree T on n vertices.
Output: mr(T), Δ(T), and a set Q of vertices of T whose deletion realizes Δ(T).
1. Set Q = ∅ and T′ = T.
2. While T′ has a pendent generalized star:
   a) Let Q′ be the the centers all pendent generalized stars of T′.
   b) Remove the vertices in Q′ from T′.
   c) Q = Q ∪ Q′.
3. If T′ is a generalized star with center v, Q = Q ∪ {v}.
4. Δ(T) = p − |Q| where p is the number of components in T − Q.
5. mr(T) = n − Δ(T).

6. Several variants of Algorithm 1, such as Algorithm 2, are also known; Algorithm 2 is sometimes more useful, e.g., another version of Algorithm 2 is generalized to Algorithm 3 in Section 5 (that version deletes high degree vertices whose deletion leaves at most one component containing a high degree vertex, resulting in the same set Q deleted in the same order as Algorithm 2).

7. [P60, W84, JLS03] Parter-Wiener Theorem If T is a tree, B ∈ S(T) and mult_B(λ) ≥ 2, then there is a strong PW vertex of B for λ.


9. A vertex v of tree T is a strong PW vertex for some eigenvalue λ of some matrix B ∈ S(T) if and only if v is a high degree vertex.
Proof: ⇒ is immediate from the definition of strong PW.
⇐: Let v be a high degree vertex, T_i be the components of T − v, and order the vertices of T so that v is a neighbor of the first vertex in T_i for all i. For each T_i, choose a singular matrix B_i ∈ S(T_i); in at least one of these cases, choose B_i is such a way that e_1 ∉ range(B_i) (where e_1 is the vector of correct length
having 1 in the first coordinate and 0s elsewhere). Embed the $B_i$ as principal submatrices of $B$ such that $B \in \mathcal{S}(T)$. Then $\text{rank}(B) = \text{rank}(B(1) + 2)$, so $v$ is a strong PW vertex of $B$ for eigenvalue 0.

**Examples:**

1. We use Algorithm 2 to compute $\Delta(T) = M(T)$ and $\text{mr}(T)$ for the tree $T$ shown in Figure 4. Step 2:

   - First iteration
     - $Q = Q' = \{1, 3, 5, 9\}$
   - Second iteration
     - $Q' = \{4, 7\}$
     - $Q = \{1, 3, 4, 5, 7, 9\}$

   Step 3: The tree $T - Q$ is shown in Figure 5. $\Delta(T) = M(T) = 19 - 6 = 13$.

   Step 4: $\text{mr}(T) = 32 - 13 = 19$.

**2.4 Graphs**

The limited progress on the minimum rank problem for graphs that are not trees has come mainly in two ways, by deleting cut-vertices, and characterizing graphs having relatively extreme minimal rank. Both of these approaches seem natural, since deletion of vertices was a powerful technique for trees, and since the graphs having minimum rank 1 and $n - 1$ have been described.
All graphs in this subsection are simple.

Definitions:
Let $G$ be a simple graph.

- [BFH04] The **rank-spread** of $G$ at vertex $v$ is defined as $r_v(G) = \mr(G) - \mr(G - v)$.

- A graph is **decomposable** if it can be expressed as a sequence of unions and joins of isolated vertices.

- A **linear 2-tree** is a 2-connected graph $G$ that can be embedded in the plane such that the graph obtained from the dual of $G$ after deleting the vertex corresponding to the infinite face is a path.

Results:
Let $G$ be a simple graph.

1. $\mr(G) \leq \theta(G)$.
   (This is a special case of the next result, since the minimum rank of a clique is 1.)

2. If $G = \bigcup_{i=1}^{h} G_i$ then $\mr(G) \leq \sum_{i=1}^{h} \mr(G_i)$.
   (A matrix $A \in \mathcal{S}(G)$ of rank at most $\sum_{i=1}^{h} \mr(G_i)$ can be obtained by choosing for each $i = 1, \ldots, h$ a matrix $A_i$ that realizes $\mr(G_i)$, embedding $A_i$ in a matrix $\tilde{A}_i$ of size $|G|$ and then letting $A = \sum_{i=1}^{h} \tilde{A}_i$.)

3. [N96] For any vertex $v$ of $G$, $0 \leq r_v(G) \leq 2$. 
4. [N96] Adding or removing an edge from a graph $G$ can change minimum rank by at most 1.

5. [BFH04], [H01] (cut-vertex reduction) If $G$ has a cut-vertex, the problem of computing the minimum rank of $G$ can be reduced to computing minimum ranks of certain subgraphs. Specifically, let $v$ be a cut-vertex of $G$. For $i = 1, \ldots, h$, let $W_i \subseteq V(G)$ be the vertices of the $i$th component of $G - v$ and let $G_i$ be the subgraph induced by $\{v\} \cup W_i$. Then

$$r_v(G) = \min \left\{ \sum_{i=1}^h r_v(G_i), \ 2 \right\}$$

and thus

$$\text{mr}(G) = \sum_{i=1}^h \text{mr}(G_i - v) + \min \left\{ \sum_{i=1}^h r_v(G_i), \ 2 \right\}.$$ 

6. The following corollary to Result 5 (with the notation of that result) has been observed by Wayne Barrett’s students: If $r_v(G_i) = 0$ for all but at most one of the $G_i$, then

$$\text{mr}(G) = \sum_{i=1}^h \text{mr}(G_i).$$

7. Hein van der Holst has cut-set reduction results for cut-sets of order 2.

8. [BF06] The minimum rank of a decomposable graph can be computed recursively from its pieces. In many cases this is true of the minimum rank of a join.

9. [BvdHL04] A connected simple graph $G$ has $\text{mr}(G) \leq 2$ if and only if $G$ does not contain as an induced subgraph any of $P_4$, $K_{3,3,3}$ (the complete tripartite graph), Dart, or ⋁ (all shown in Figure 6).

![Figure 6: Forbidden induced subgraphs for $\text{mr}(G) \leq 2$](image)
10. [BvdHL04] A simple graph $G$ has $mr(G) \leq 2$ if and only if the complement of $G$ is of the form

$$(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \lor K_r$$

for appropriate nonnegative integers $k, s_1, s_2, p_1, q_1, \ldots, p_k, q_k, r$ with $p_i+q_i > 0$ for all $i = 1, \ldots, k$. See also Result 5.1.18.

11. [HvdH06] If $G$ is 2-connected and $mr(G) = |G| - 2$, then $G$ is a linear 2-tree. See also Result 4 of Section 3.

12. For any simple graph $G$, $\Delta(G) \leq M(G)$.
   (Follows from the Interlacing Theorem.)

13. [BFH04] For any simple graph $G$, $\Delta(G) \leq P(G)$.

14. [BFH04] For graphs (as opposed to trees), $P$ and $M$ are non-comparable. See Examples 2 and 3.

15. The Parter-Wiener Theorem need not be true for graphs that are not trees; see Example 4.

Examples:

1. [H05] This example illustrates the use of Result 6. The graph $G$ shown in Figure 7 has cut-vertex 4 and the induced subgraphs $G[\{1, 2, 3, 4\}]$ and $G_1 = G[\{4, 5, 6, 7, 8, 9, 10, 11, 12\}]$ associated with the two components.

![Figure 7: A graph to which Result 6 can be applied to compute minimum rank](image)

By Results 9 and 2.1.7, $mr(G[\{1, 2, 3, 4\}]) = 2 = mr(G[\{1, 2, 3\}])$, so $r_4(G[\{1, 2, 3, 4\}]) = 0$. Thus

$$mr(G) = mr(G_1) + mr(G[\{1, 2, 3, 4\}]) = mr(G_1) + 2.$$ 

The graph $G_1$ has the cut-vertex 6 and induced subgraphs $G[\{4, 5, 6, 7, 8\}]$, $G[\{6, 9, 10\}]$, and $G[\{6, 11, 12\}]$. By Result 2.1.10,
mr(G[{6,9,10}]) = mr(G[{9,10}]) = 1, so $r_6(G[{6,9,10}]) = 0$, and similarly $r_6(G[{6,11,12}]) = 0$. Thus,

$$\text{mr}(G_1) = \text{mr}(G[{4,5,6,7,8}]) + \text{mr}(G[{6,9,10}]) + \text{mr}(G[{6,11,12}]) = 3 + 1 + 1 = 5.$$ 

Thus, \( \text{mr}(G) = \text{mr}(G_1) + 2 = 5 + 2 = 7 \).

2. The wheel on 5 vertices, \( W_5 \), shown in Figure 8, has \( P(W_5) = 2 \) by inspection, and \( \text{mr}(W_5) = 2 \) (by Result 9), so \( M(W_5) = 3 > P(W_5) \).

3. The penta-sun, \( H_5 \), which is shown in Figure 8, has \( P(H_5) = 3 \) by inspection. By repeated application of Result 5, it is shown in [BFH04] that \( \text{mr}(H_5) = 8 \) and thus that \( P(H_5) > M(H_5) = 2 \).

4. The cycle has \( \text{mr}(C_n) = n - 2 \) so there is a matrix \( B \in \mathcal{S}(C_n) \) with \( \text{mult}_B(0) = 2 \) but there is no PW vertex, since the deletion of any vertex leaves \( P_{n-1} \).

3 Colin de Verdière-type Parameters

Recall that minimum rank is monotone on induced subgraphs (i.e., if \( G' \) is an induced subgraph of a graph \( G \) then \( \text{mr}(G') \leq \text{mr}(G) \), cf. Result 2.1.4, and this property can be useful in bounding \( \text{mr}(G) \) from below. Unfortunately, maximum multiplicity of an eigenvalue is not monotone for induced subgraphs (see Example 1 below). In 1990 Colin de Verdière ([CdV93] in English) introduced the graph parameter \( \mu \) equal to the maximum multiplicity of eigenvalue 0 among generalized Laplacian matrices having a given graph and also satisfying the Strong Arnold Hypothesis. The Colin de Verdière number \( \mu \) was the first of several related parameters that are both minor monotone and bound the maximum eigenvalue multiplicity from below. In this section we discuss several Colin de Verdière-type parameters and their use for computing the maximum eigenvalue multiplicity (or equivalently, the minimum rank) of a graph. These parameters are most useful when the graph has a large number of edges (since a matrix with many nonzero entries is more likely
to satisfy the Strong Arnold Hypothesis), and least useful for trees, where a convenient method already exists for evaluation of maximum multiplicity and minimum rank (see Algorithms 1 and 2 in Subsection 2.3). For our purposes, these parameters should be used only for connected graphs (each component should be analyzed separately).

Reference [vdHLS99] provides an excellent introduction to the parameter $\mu$ from a linear algebra perspective. More information about the parameter $\xi$ can be found in [BFH05xi]. All graphs in this section are simple.

Definitions:

Let $G$ be a simple graph.

- The symmetric matrix $L$ is a **generalized Laplacian matrix** of $G$ if $G(L) = G$ and all off-diagonal entries of $L$ are non-positive.

- A symmetric real matrix $M$ is said to satisfy the **Strong Arnold Hypothesis** provided there does not exist a nonzero symmetric matrix $X$ satisfying:
  - $MX = 0$.
  - $M \circ X = 0$.
  - $I \circ X = 0$.

  where $\circ$ denotes the Hadamard (entrywise) product and $I$ is the identity matrix.

- [CdV93] The **Colin de Verdière number** $\mu(G)$ is the maximum multiplicity of 0 as an eigenvalue among matrices $L$ that satisfy:
  - $L$ is a generalized Laplacian matrix of $G$.
  - $L$ has exactly one negative eigenvalue (of multiplicity 1).
  - $L$ satisfies the Strong Arnold Hypothesis.

- [CdV98] The parameter $\nu(G)$ (also denoted $\nu^R(G)$) is defined to be the maximum multiplicity of 0 as an eigenvalue among matrices $A$ that satisfy:
  - $A \in S(G)$.
  - $A$ is positive semidefinite.
  - $A$ satisfies the Strong Arnold Hypothesis.

- [BFH05xi] The parameter $\xi(G)$ is the maximum multiplicity of 0 as an eigenvalue among matrices $A$ that satisfy:
A ∈ S(G).
A satisfies the Strong Arnold Hypothesis.

- A contraction of \( G \) is obtained by identifying two adjacent vertices of \( G \), suppressing any loops or multiple edges that arise in this process.

- A minor of \( G \) arises by performing a series of deletions of edges, deletions of isolated vertices, and/or contraction of edges.

- A graph parameter \( \zeta \) is minor monotone if for any minor \( G' \) of \( G \), \( \zeta(G') \leq \zeta(G) \).

- A graph parameter \( \zeta \) is monotone on induced subgraphs if for any induced subgraph \( H \) of \( G \), \( \zeta(H) \leq \zeta(G) \).

Results:
Let \( G \) be a simple graph.

1. Obviously, a minor monotone graph parameter is monotone on subgraphs and thus on induced subgraphs.

2. [CdV93], [vdHLS99] The Strong Arnold Hypothesis is equivalent to the requirement that certain manifolds intersect transversally.

3. The parameter \( \xi \) is minor monotone [BFH05xi], as are \( \mu \) [CdV93], [vdHLS99], and \( \nu \) [CdV98].

4. [HvdH06] Let \( G \) be a 2-connected graph of order \( n \). The following are equivalent:
   (a) \( \xi(G) = 2 \).
   (b) \( M(G) = 2 \).
   (c) \( \text{mr}(G) = n - 2 \).
   (d) \( G \) has no \( K_{4,\ast} \), \( K_{2,3,\ast} \), or \( T_3 \)-minor (see Figure 9).
   (e) \( G \) is a linear 2-tree.

5. \( \mu(G) \leq \xi(G) \leq M(G) \) and \( \nu(G) \leq \xi(G) \leq M(G) \). An example in which both \( \mu(G) < \xi(G) \) and \( \nu(G) < \xi(G) \) is given in [BFH05xi]. Both Results 6 and 8 below provide immediate examples of graphs \( G \) having \( \xi(G) < M(G) \).

6. [BFH05xi] If \( G \) is the disjoint union of components \( G_i, i = 1, \ldots, k \) then \( \xi(G) = \max_{i=1}^k \xi(G_i) \), whereas \( M(G) = \sum_{i=1}^k M(G_i) \). Thus if \( \xi \) is used to study maximum multiplicity (or minimum rank), consideration should be restricted to connected graphs.
7. For a path $P_n$, $\mu(P_n) = \nu(P_n) = \xi(P_n) = 1$.

8. [BFH05xi]
   (a) If $T$ is a tree that is not a path, then $\xi(T) = 2$.
   (b) $\xi(K_n) = n - 1$
   (c) For $p \leq q$ and $q \geq 3$, $\xi(K_{p,q}) = p + 1$

9. If $K_p$ is a subgraph of $G$ then $M(G) \geq p - 1$.
   (Follows from 8, 3, 5.)

10. [CdV93, LS98, vdHLS99] $\mu$ is used to characterize planarity:
    (a) $\mu(G) \leq 1$ if and only if $G$ is a disjoint union of paths,
    (b) $\mu(G) \leq 2$ if and only if $G$ is outerplanar,
    (c) $\mu(G) \leq 3$ if and only if $G$ is planar,
    (d) $\mu(G) \leq 4$ if and only if $G$ is linklessly embeddable in $\mathbb{R}^3$.

11. See Subsection 5.2 for more information on $\nu$ and its applications to positive semidefinite minimum rank.

Examples:

1. Maximum eigenvalue multiplicity $M$ is not monotone on induced subgraphs of the graph $G$ shown in Figure 10. It is shown in [BFH04] (by using Fact 5 of Subsection 2.4) that $mr(G) = 4$, so $M(G) = 2$, but deletion of the vertex of degree three produces $K_{1,4}$ and $M(K_{1,4}) = 3$. The Strong Arnold Hypothesis is essential to minor-monotonicity: any matrix realizing $M(K_{1,4}) = 3$ does not satisfy the Strong Arnold Hypothesis.

2. This example show how we can use monotonicity to compute minimum rank of the graph $G$ shown in Figure 11(a).
As indicated in red in Figure 11(b), $P_4$ is an induced subgraph of $G$, so $3 = \mr(P_4) \leq \mr(G)$.

Figures 11(c),(d) show that $K_4$ is a minor of $G$ (delete red edges and vertex in (c) and then contract on red edge in (d)). So $3 = \xi(K_4) \leq \xi(G) \leq \M(G)$. Thus $\mr(G) \leq 6 - 3 = 3$, and so $\mr(G) = 3$.

4 Ordered Multiplicity Lists and The Inverse Eigenvalue Problem of a Graph

The minimum rank problem is a small part of the Inverse Eigenvalue Problem of a Graph, which asks what spectra are possible among all the symmetric matrices described by the graph. With the exception of results about maximum multiplicity of an eigenvalue, most of the progress on this problem has been limited to specific families of trees. Much of this work is based on determination of possible ordered list of multiplicities for the tree. It was once thought that the Inverse Eigenvalue problem of a tree was equivalent to the determination of the ordered multiplicity lists of the tree; however, in 2003 Barioli and Fallat [BF04] gave an example showing that this is not the case (see Result 2 below), thereby rendering the study of ordered multiplicity lists somewhat less significant.

All graphs in this section are simple.

Definitions:
Let $G$ be a simple graph.
• The **Inverse Eigenvalue Problem of a graph** (IEP-G) is to characterize the possible spectra of matrices in $S(G)$.

• A tree is a **double generalized star** if it can be constructed from two generalized stars by joining their centers by an edge.

• A tree is a **double path** if it can be constructed from two paths $T_1$ and $T_2$ each of order $\geq 3$ by joining a vertex of degree two in $T_1$ to a vertex of degree two in $T_2$ by an edge.

• If the distinct eigenvalues of $B \in S_n$ are $\tilde{\beta}_1 < \cdots < \tilde{\beta}_q$ with multiplicities $m_1, \ldots, m_q$, then $(m_1, \ldots, m_q)$ is called the **ordered multiplicity list** of $B$.

• The number of distinct eigenvalues of $B \in S_n$ is denoted $q(B)$.

• The minimum number of distinct eigenvalues of $G$ is $q(G) = \min\{q(B) : B \in S(G)\}$.

• [KS06] The $(3, \ell)$-whirl ($\ell \geq 1$), is the tree on $n = 6\ell + 4$ vertices with 6 pendant paths $\alpha_i, \beta_i, \gamma_i$, each with $\ell$ vertices, for $i = 1, 2$ as illustrated in Figure 12.

![Figure 12: The $(3, \ell)$-whirl (picture courtesy of Kim and Shader)](image)

**Results:**
Let $T$ be a tree.
1. [BF05], [F69], [JLD02], [JLDS03a] The possible ordered multiplicity lists of matrices in \( S(T) \) have been determined for the following families of trees:
- paths
- double paths
- stars
- generalized stars
- double generalized stars

Furthermore, for \( T \) in any of these families, if there is a matrix \( B \in S(T) \) with distinct eigenvalues \( \tilde{\beta}_1 < \cdots < \tilde{\beta}_r \) having multiplicities \( m_1, \ldots, m_r \), then for any real numbers \( \gamma_1 < \cdots < \gamma_r \), there is a matrix in \( S(T) \) having eigenvalues \( \gamma_1, \ldots, \gamma_r \) with multiplicities \( m_1, \ldots, m_r \). Thus for any of these trees, determination of the possible ordered multiplicity lists of the graph is equivalent to the solution of the Inverse Eigenvalue Problem of the graph, and that problem has been solved.

2. [BF04] Sometimes there are restrictions on which real numbers can appear as the eigenvalues for an attainable ordered multiplicity list. That is, there exist trees for which the determination of the ordered multiplicity list of the tree is not equivalent to the solution of the Inverse Eigenvalue Problem of the tree.

For the tree \( T_{BF} \) shown in Figure 13, which is a \((3,1)\)-whirl, the spectrum of the adjacency matrix is \( \sigma(A_{T_{BF}}) = \{-\sqrt{5}, -\sqrt{2}, -\sqrt{2}, 0, 0, 0, \sqrt{2}, \sqrt{2}, \sqrt{5}\} \), so the ordered multiplicity list of \( A_{T_{BF}} \) is \((1, 2, 4, 2, 1)\). But the trace technique in [BF04] shows that if \( B \in S(T_{BF}) \) has the five distinct eigenvalues \( \tilde{\beta}_1 < \tilde{\beta}_2 < \tilde{\beta}_3 < \tilde{\beta}_4 < \tilde{\beta}_5 \) with multiplicities \( \text{mult}_B(\tilde{\beta}_1) = \text{mult}_B(\tilde{\beta}_5) = 1, \text{mult}_B(\tilde{\beta}_2) = \text{mult}_B(\tilde{\beta}_4) = 2, \text{mult}_B(\tilde{\beta}_3) = 4 \), then \( \tilde{\beta}_1 + \tilde{\beta}_5 = \tilde{\beta}_2 + \tilde{\beta}_4 \).

3. If the order of \( T \) is \( n \), \( B \in S(T) \), and the eigenvalues of \( B \) are \( \lambda_1 \leq \cdots \leq \lambda_n \), then \( \lambda_1 \) and \( \lambda_n \) are simple eigenvalues.

(Follows from the Interlacing Theorem and Parter-Wiener Theorem.)
4. If $T$ is a generalized star and $B \in \mathcal{S}(T)$, then there must be a simple eigenvalue between any two multiple eigenvalues of $B$.

   (Follows from the Interlacing Theorem and Parter-Wiener Theorem.)

5. $q(T) \geq \text{diam}(T) + 1$ [LDJ02]. Barioli and Fallat [BF04] gave the first example showing $q(t) > \text{diam}(T) + 1$ is possible (the example is what is now called the $(3,2)$-whirl). Kim and Shader [KS06] generalized this: If $W$ is the $(3, \ell)$-whirl with $\ell \geq 2$, then

   $$q(W) \geq \frac{9}{8} \text{diam}(W) + \frac{1}{2} = \text{diam}(W) + 1 + \frac{\ell - 1}{4} > \text{diam}(W) + 1.$$  

6. Kim and Shader [KS06] recently introduced the use of Smith Normal Form to study the IEP-G.

5 Minimum Rank Problems for Other Families of Matrices

In this section we survey other families of matrices described by a graph for which the minimum rank question has been studied. This section is divided into subsections, each covering a different family of matrices. Many basic facts are the same for all the types of minimum rank (although the value of minimum rank varies with the family).

5.1 Matrices over Other Fields

All graphs in this subsection are simple.

Definitions:
Let $G$ be a simple graph on $n$ vertices and let $F$ be a field.

- $\mathcal{S}^F(G) = \{ B : B$ is a symmetric $n \times n$ matrix over $F$ and $\mathcal{G}(B) = G \}$.

- $\text{mr}^F(G) = \min \{ \text{rank}(B) : B \in \mathcal{S}^F(G) \}$.

- $M^F(G) = \max \{ \text{mult}_B(\lambda) : B \in \mathcal{S}^F(G), \lambda \in \sigma(B) \}$.

- $gM^F(G) = \max \{ g\text{mult}_B(\lambda) : B \in \mathcal{S}^F(G), \lambda \in \sigma(B) \}$.

- $\mathcal{H}(G) = \{ B : B$ is a Hermitian $n \times n$ matrix over $\mathbb{C}$ and $\mathcal{G}(B) = G \}$.

- $\text{hmr}(G) = \min \{ \text{rank}(B) : B \in \mathcal{H}(G) \}$.  

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Note that $S(G) = S^R(G)$, $\text{mr}(G) = \text{mr}^R(G)$ and $M(G) = M^R(G) = gM^R(G)$.

**Results:**

Let $G$ be a simple graph, $F$ a field, and $B \in F^{n \times n}$.

1. [BvdHL04] $\text{mr}^F(G)$ may be larger or smaller than $\text{mr}(G)$, depending on $F$ and $G$ (see Example 1).

2. [CDHMP] For a tree $T$, minimum rank is independent of field, and so $\text{mr}^F(T) = |T| - \Delta(T)$ can be computed by Algorithm 1 or 2.

3. If $K$ is a field and $F \subset K$, then $\text{rank}(B)$ does not depend on whether $B$ is viewed as a matrix over $F$ or over $K$.
   (Use RREF.)

4. If $K$ is a field and $F \subset K$, then $\text{mr}^F(G) \geq \text{mr}^K(G)$ and strict inequality is possible (see Example 2).
   ($S^F(G) \subset S^K(G)$ and Result 3.)

5. $\text{rank}(B) - 2 \leq \text{rank}(B(k)) \leq \text{rank}(B)$.
   ($B(k)$ is obtained from $B$ by one row and one column deletion, each of which leaves rank unchanged or diminishes it by 1.)

6. $\text{mr}^F(G) - 2 \leq \text{mr}^F(G - v) \leq \text{mr}^F(G)$.
   (Follows from 5.)

7. If $B \in S^F(G)$ and $R \subseteq V(G), k \in V(G)$, then $B[R] \in S^F(G[R])$ and $B(k) \in S^F(G - k)$.

8. If the connected components of $G$ are $G_1, \ldots, G_t$, then $\text{mr}^F(G) = \sum_{i=1}^t \text{mr}^F(G_i)$.

9. $\text{gmult}_B(0) + \text{rank}(B) = n$.

10. $gM^F(G) + \text{mr}^F(G) = |G|$.

11. If $G'$ is an induced subgraph of $G$ then $\text{mr}^F(G') \leq \text{mr}^F(G)$.

12. $\text{mr}^F(G) \leq |G| - 1$.

13. If $F$ is infinite, then $\text{mr}^F(G) \leq \theta(G)$. This need not be true for finite fields (see Example 1). Result 2 in Subsection 2.4 is also true for infinite fields.

14. For a connected simple graph $G$ of order $n > 1$, $\text{mr}^F(G) = 1$ if and only if $G = K_n$. 

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15. For the path $P_n$ on $n$ vertices, $\text{mr}^F(P_n) = n - 1$.

16. [CDHMP] If $F \neq \mathbb{Z}_2$ and $\text{mr}^F(G) = |G| - 1$, then $G$ is a path.

17. [BvdHL04] The graphs $P_4$, Dart, $\star$ shown in Figure 6, and $P_3 \cup K_2$ and $3K_2$ all have minimum rank 3 over any field. The rank of $K_{3,3,3}$ (see Figure 6) depends on the field (see Example 1).

18. [BvdHL04] Let $F$ be an infinite field of characteristic $\neq 2$. The following are equivalent:
   
   (a) $\text{mr}^F(G) \leq 2$.
   
   (b) $\overline{G}$ is of the form
   $$(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \lor K_r$$
   
   for appropriate nonnegative integers $k, s_1, s_2, p_1, q_1, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$ for all $i = 1, \ldots, k$.
   
   (c) $G$ does not contain as an induced subgraph any of
   $P_4$, Dart, $\star$, $P_3 \cup K_2$, $3K_2$, or $K_{3,3,3}$.

   ($P_4$, Dart, $\star$, $K_{3,3,3}$ are shown in Figure 6.)

19. [BvdHL05] Let $F$ be a finite field of prime characteristic $p \neq 2$ having $p^t$ elements. The following are equivalent:
   
   (a) $\text{mr}^F(G) \leq 2$.
   
   (b) $\overline{G}$ is of one of the forms:
   $$(K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \lor K_r$$
   
   for appropriate nonnegative integers $k, p_1, q_1, \ldots, p_k, q_k, r$, with $p_i + q_i > 0$ for all $i = 1, \ldots, k$ and $k \leq \frac{p^t+1}{2}$;
   
   or
   $$(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \lor K_r$$
   
   for appropriate nonnegative integers $k, s_1, s_2, p_1, q_1, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, for all $i = 1, \ldots, k$ and $k \leq \frac{p^t-1}{2}$.
   
   (c) $G$ does not contain as an induced subgraph any of
   $P_4$, Dart, $\star$, $P_3 \cup K_2$, $3K_2$, $K_{3,3,3}$, $(m+2)K_2 \cup K_1$, $K_2 \cup 2K_1 \cup mP_3$, or $K_1 \cup (m+1)P_3$.

   where $m = \frac{p^t-1}{2}$.  

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20. [BvdHL04] Let $F$ be an infinite field of characteristic 2. The following are equivalent:

(a) $\text{mr}^F(G) \leq 2$.

(b) $\overline{G}$ is of one of the forms

$$(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}) \vee K_r$$

$$(K_{s_1} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$$

for appropriate nonnegative integers $k, s_1, s_2, p_1, q_1, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$ for all $i = 1, \ldots, k$.

(c) $G$ does not contain as an induced subgraph any of $P_4, \text{Dart}, \kappa, P_3 \cup K_2, 3K_2$, or $P_3 \cup 2K_3$.

21. [BvdHL05] gives characterizations of minimum rank 2 as the graph complement and using forbidden induced subgraphs for finite fields of characteristic 2, including $\mathbb{Z}_2$.

22. Wayne Barrett has pointed out that the proof of Result 5 of Section 2.4, which provides a formula for minimum rank in terms of subgraphs for a graph with cut-vertex, remains valid for arbitrary fields. Hence its simpler corollary, Result 2.4.6, is also valid for all fields.

23. [Z99, HJ85] The Interlacing Theorem (Result 6 in Section 1) applies to Hermitian matrices.

24. [LDJ02] If $A \in \mathbb{C}^{n \times n}$ is Hermitian and $G(A) = T$ is a tree, then there exists a unitary diagonal matrix $U$ such that $B = U^*AU$ is real, symmetric, and all off-diagonal entries are nonnegative.

(The same technique used to symmetrize a sign-symmetric real matrix whose graph is a tree works.)

25. Thus for a tree $T$, $\text{hmr}(T) = \text{mr}(T)$ and the Parter-Wiener Theorem and its consequences (including Result 3 in Section 4 that the first and last eigenvalue are simple) are valid for Hermitian matrices.

(follows immediately from the preceding Result.)

26. [BvdHL04] The graphs $P_4$, Dart, $\kappa$ shown in Figure 6, and $P_3 \cup K_2$ and $3K_2$ all have Hermitian minimum rank 3, but $\text{hmr}(K_{3,3,3}) = 2$ (see Example 2).

27. [BvdHL04] The following are equivalent:
(a) $hmr(G) \leq 2$.

(b) $\overline{G}$ is of the form

$$(K_{s_1} \cup \cdots \cup K_{s_t} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$$

for appropriate nonnegative integers $r, t, k, s_1, \ldots, s_t, p_1, q_1, \ldots, p_k, q_k$ with $p_i + q_i > 0$ for all $i = 1, \ldots, k$.

(c) $G$ does not contain as an induced subgraph any of $P_4$, Dart, $\kappa$, $P_3 \cup K_2$, or $3K_2$.

28. Reference [JS04] contains results about PW vertices of a Hermitian $A$, even when $G(A)$ is not a tree, based on the structure of eigenvectors.

Examples:

1. $mr^z_2(K_{3,3,3}) = 2$ by choosing all diagonal elements 0, and $mr(K_{3,3,3}) = 3$ by Result 18.

For the graph $G$ shown in Figure 14, $mr(G) = 2$ by considering $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

$(G$ is the 2-clique-sum of two copies of $K_4$). $mr^z_2(G) = 3$ by computation with all possible diagonal elements or by [BvdHL04].

![Figure 14: A graph $G$ with $mr^z_2(G) > mr(G)$.

2. $mr^z_3(3K_2 \cup K_1) = 3$ by Result 19 and $mr^F(3K_2 \cup K_1) = 2$ where $F$ is an infinite field of characteristic 3 by Result 18. $3K_2 \cup K_1$ is shown in Figure 15.
Figure 15: A graph for which minimum rank over a subfield is strictly greater.

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & i & i & i \\
0 & 0 & 0 & 1 & 1 & 1 & i & i & i \\
0 & 0 & 0 & 1 & 1 & 1 & i & i & i \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

3. \( \text{rank} \)

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
-i & -i & -i & 1 & 1 & 1 & 0 & 0 & 0 \\
-i & -i & -i & 1 & 1 & 1 & 0 & 0 & 0 \\
-i & -i & -i & 1 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

\[= 2\]

5.2 Positive Semidefinite Matrices of a Graph

All graphs in this subsection are simple.

Definitions:
Let \( G \) be a simple graph of order \( n \).

- \( S^+(G) = \{ B \in S(G) : B \text{ is positive semidefinite} \} \).
- \( H^+(G) = \{ B \in H(G) : B \text{ is positive semidefinite} \} \).
- \( \text{mr}^+(G) = \min \{ \text{rank}(B) : B \in S^+(G) \} \).
- \( \text{hmr}^+(G) = \min \{ \text{rank}(B) : B \in H^+(G) \} \).
- \( M^+_A(G) = \max \{ \text{mult}_B(\lambda) : B \in S^+(G), \lambda \in \sigma(B) \} \). (It is necessary to distinguish between eigenvalues because translation by a negative is no longer possible.)
- \([CdV98]\) The parameter \( \nu^R(G) \) (also denoted \( \nu(G) \)) is defined to be the maximum multiplicity of 0 as an eigenvalue among matrices \( A \) that satisfy:
  - \( A \in S^+(G) \).
  - \( A \) satisfies the Strong Arnold Hypothesis (see Section 3).
• A Hermitian matrix $M$ is said to satisfy the **complex Strong Arnold Hypothesis** provided there does not exist a nonzero Hermitian matrix $X$ satisfying:

  - $MX = 0$.
  - $M \circ X = 0$.
  - $I \circ X = 0$.

  where $\circ$ denotes the Hadamard (entrywise) product and $I$ is the identity matrix.

• [CdV98] The parameter $\nu^c(G)$ is defined to be the maximum multiplicity of 0 as an eigenvalue among matrices $A$ that satisfy:

  - $A \in \mathcal{H}^+(G)$.
  - $A$ satisfies the complex Strong Arnold Hypothesis.

• [Netal] A **vector representation** of $G$ is a set $\tilde{W} = \{w_1, \ldots, w_n\} \subset \mathbb{C}^m$, where none of the $w_i$ is the zero vector and for $i \neq j$, $\langle w_i, w_j \rangle \neq 0$ if vertices $i$ and $j$ are connected and $\langle w_i, w_j \rangle = 0$ if $i$ and $j$ are not connected.

• The **rank** of vector representation $\tilde{W} = \{w_1, \ldots, w_n\} \subset \mathbb{C}^m$ is $\dim \text{Span}(W)$.

• [Netal] For any graph $G$, define the **minimum vector rank** of $G$ to be the minimum rank over all vector representations of $G$, and denote it by $\text{mv}(G)$.

**Results:**

Let $G$ be a simple graph.

1. $\mathcal{S}^+(G) \subseteq \mathcal{S}(G)$ and $\mathcal{S}^+(G) \subseteq \mathcal{H}^+(G)$.

2. $\text{mr}^+(G) \geq \text{mr}(G)$, and strict inequality is possible (see Example 1).

3. [Netal] $\text{mr}^+(G) \geq \text{hmr}^+(G) \geq \text{hmr}(G)$. $\text{hmr}^+(G) > \text{hmr}(G)$ is possible but it is not known whether $\text{mr}^+(G) > \text{hmr}^+(G)$ is possible.

4. For any $G$, $\text{mr}^+(G) \leq |G| - c$, where $c$ is the number of connected components of $G$.

   (The Laplacian $L_G$ of $G$ has an eigenvalue 0 with multiplicity $c$ and is a positive semidefinite matrix in $\mathcal{S}(G)$.)

5. $\text{mr}^+(P_n) = n - 1$ but the converse is false (see Example 1).

6. [Netal] $\text{hmr}^+(P_n) = n - 1$ but the converse is false (cf. Result 13).
7. For a connected graph \( G \) of order \( n > 1 \), \( \text{mr}^+(G) = 1 \) if and only if \( G = K_n \).

8. If \( G' \) is an induced subgraph of \( G \) then \( \text{mr}^+(G') \leq \text{mr}^+(G) \).

9. If \( B \in S^+(G) \) and \( R \subseteq V(G), k \in V(G) \), then \( B[R] \in S^+(G[R]) \) and \( B(k) \in S^+(G - k) \).

   (A principal submatrix of a positive semidefinite matrix is positive semidefinite.)

10. If \( B \in H^+(G) \) and \( R \subseteq V(G), k \in V(G) \), then \( B[R] \in H^+(G[R]) \) and \( B(k) \in H^+(G - k) \).

11. If the connected components of \( G \) are \( G_1, \ldots, G_t \), then \( \text{mr}^+(G) = \sum_{i=1}^t \text{mr}^+(G_i) \).

12. If the connected components of \( G \) are \( G_1, \ldots, G_t \), then \( h\text{mr}^+(G) = \sum_{i=1}^t h\text{mr}^+(G_i) \).

13. [Netal] \( h\text{mr}^+(G) = |G| - 1 \) if and only if \( G \) is a tree.

14. \( \text{mr}^+(G) = |G| - 1 \) if and only if \( G \) is a tree.

   (If \( T \) is a tree, then \( \text{mr}^+(T) = |T| - 1 \) since the least (and greatest) eigenvalue of \( T \) must be simple cf. Result 3 in Section 4.
   Let \( \text{mr}^+(G) = |G| - 1 \). By Result 17 below, \( \nu^R(G) \leq |G| - \text{mr}^+(G) = 1 \). \( G \) is connected by Result 4. So then by Result 19 below, \( G \) is a tree.)

15. For any \( \lambda > 0 \), \( M^+_\lambda(G) = M(G) \). Also, \( M^+_0(G) \leq M(G) \) and strict inequality is possible (see Example 1).

   (Let \( B \in S(G) \) have \( \text{mult}_B(\mu) = M(G) \). Let \( \tau \) be the least eigenvalue of \( B \).
   Let \( A = \frac{\lambda}{\mu - \tau + 1}(B + (1 - \tau)I) \). Then \( \text{mult}_A(\lambda) = M(G) \) and \( A \) is positive definite, since the least eigenvalue of \( B + (1 - \tau)I \) is 1.)

16. \( M^+_0(G) + \text{mr}^+(G) = |G| \).

17. \( \nu^R(G) \leq M^+_0(G) \) and so \( \text{mr}^+(G) \leq |G| - \nu^R(G) \) and \( \nu^R(G) \leq |G| - \text{mr}^+(G) \).

18. \( h\text{mr}^+(G) \leq |G| - \nu^C(G) \) and \( \nu^C(G) \leq |G| - h\text{mr}^+(G) \).

19. Let \( G \) be connected. \( \nu^R(G) = 1 \) if and only if \( G \) is a tree.

20. Let \( G \) be connected. \( \nu^C(G) = 1 \) if and only if \( G \) is a tree.

21. [vdH02] has characterizations of graphs \( G \) such that \( \nu^C(G) \leq 2 \) and \( \nu^C(G) \leq 3 \).
22. [Netal] If $G$ has a cut-vertex, the problem of computing the minimum rank of $G$ can be reduced to computing minimum ranks of certain subgraphs. Specifically, let $v$ be a cut-vertex of $G$. For $i = 1, \ldots, h$, let $W_i \subseteq V(G)$ be the vertices of the $i$th component of $G - v$ and let $G_i$ be the subgraph induced by $\{v\} \cup W_i$. Then
\[ \text{hmr}^+(G) = \sum_{i=1}^{h} \text{hmr}^+(G_i). \]

23. $\text{mr}^+(G) \leq \theta(G)$.

   (This is a special case of the next result, since the positive semidefinite minimum rank of a clique is 1.)

24. If $G = \bigcup_{i=1}^{h} G_i$ then $\text{mr}^+(G) \leq \sum_{i=1}^{h} \text{mr}^+(G_i)$.

   (A positive semidefinite matrix $A \in S(G)$ of rank at most $\sum_{i=1}^{h} \text{mr}(G_i)$ can be obtained by choosing for each $i = 1, \ldots, h$ a positive semidefinite matrix $A_i$ that realizes $\text{mr}(G_i)$, embedding $A_i$ in a matrix $\tilde{A}_i$ of size $|G|$ and then letting $A = \sum_{i=1}^{h} \tilde{A}_i$.)

25. [Netal] $\text{hmr}^+(G) \leq \theta(G)$.

   (This is a special case of the next result, since $\text{hmr}^+(K_n) = 1$.)

26. If $G = \bigcup_{i=1}^{h} G_i$ then $\text{hmr}^+(G) \leq \sum_{i=1}^{h} \text{hmr}^+(G_i)$.

   (Same proof as for mr$^+$ works.)

27. If $G$ is connected, then $\alpha(G) \leq \text{hmr}^+(G)$.

   (Let $R \subseteq V(G)$ be an independent set such that $|R| = \alpha(G)$. Let $B \in \mathcal{H}^+(G)$ such that $\text{rank}(B) = \text{hmr}^+(G)$. Since $G$ is connected and $B$ is positive semidefinite, $B[R]$ is a diagonal matrix with nonzero diagonal, so $\text{hmr}^+(G) = \text{rank}(B) \geq \text{rank}(B[R]) = |R| = \alpha(G)$.)

28. [Netal] If $G$ is a chordal graph, then $\text{hmr}^+(G) = \theta(G)$.

29. If $G$ is a chordal graph, then $\text{mr}^+(G) = \theta(G)$.

   (Follows from 3, 23, and 28.)

30. [Netal] $\text{hmr}^+(C_n) = n - 2$ for $n \geq 3$.

31. [Netal] Adding or removing an edge from a graph $G$ can change $\text{hmr}^+$ by at most 1.
32. [Netal] \( \text{mvr}(G) = hmr^+(G) + \ell \), where \( \ell \) is the number of isolated vertices of \( G \).

(If \( \hat{W} = \{w_1, \ldots, w_n\} \subset \mathbb{C}^m \) is a vector representation, let \( W \in \mathbb{C}^{r \times n} \) be the matrix whose columns are \( w_1, \ldots, w_n \). For each isolated vertex \( v \), if the \( i \)th coordinate of \( w_v \neq 0 \), remove the \( i \)th coordinates of all \( w_i \) and use these modified vectors as the columns of a \((m - \ell) \times n\) matrix \( \hat{W} \). Then \( A = \hat{W}^* \hat{W} \in H^+(G) \) and \( \text{rank}(A) \leq m - \ell \).

If \( A \in \mathbb{C}^{n \times n} \) realizes \( hmr^+(G) \) has rank \( r \), then there exists \( W \in \mathbb{C}^{r \times n} \) such that \( W^* W = A \) (cf. Result 9 of Section 1). If there are no isolated vertices then each column of \( W \) is nonzero and the columns of \( W \) form a vector representation. If there are \( \ell \) isolated vertices, a vector representation of rank \( r + \ell \) can be obtained by extending the dimension (extending the columns of \( W \) with zeros and using the additional standard basis vectors for the isolated vertices).

33. [Netal] For graphs \( G_1, G_2 \), \( \text{mvr}(G_1 \lor G_2) = \max \{ \text{mvr}(G_1), \text{mvr}(G_2) \} \).

34. [Netal] If \( u \) is a duplicate of a vertex \( v \) in a graph \( G \) with three or more vertices, \( hmr^+(G - u) = hmr^+(G) \).

35. [Netal] Let \( G \) be a bipartite graph with independent sets \( X, Y \) such that \( X \cup Y = V(G) \). Let \( |X| = m \geq |Y| = n \), and suppose \( |\bigcap_{v \in Y} N(v)| \geq n \). Then \( hmr^+(G) = m \). Thus the minimum Hermitian positive semidefinite rank of a complete bipartite simple graph is the cardinality of the larger independent set.

36. [Netal] The \( hmr^+ \) problem for simple graphs may be reduced to the bipartite case.

37. [Netal] If \( A \) is a minimum rank positive semidefinite matrix with simple graph \( G \) then every row (and column) in \( A \) is linearly dependent on the other rows (other columns).

38. [BvdHL04] The following are equivalent:

(a) \( \text{mr}^+(G) \leq 2 \).

(b) \( hmr^+(G) \leq 2 \).

(c) \( G \) is of the form \( (K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \lor K_r \)

for appropriate nonnegative integers \( k, p_1, q_1, \ldots, p_k, q_k, r \) with \( p_i + q_i > 0 \) for all \( i = 1, \ldots, k \).
(d) $G$ does not contain as an induced subgraph any of $P_4$, $K_{1,3}$, $P_3 \cup K_2$, $3K_2$.

39. Hein van der Holst reports that Lovász has proved there are an infinite number of forbidden induced subgraphs for $mr^+(G) \leq 3$. The proof uses Ramanujan graphs and the Lovász theta function.

40. [vdH03] has characterizations of graphs $G$ having $hmr^+(G) \geq |G| - 1$ and $hmr^+(G) \geq |G| - 2$.

41. Note that the Parter-Wiener Theorem applies to positive semidefinite matrices described by a tree and a principal submatrix remains positive semidefinite.

42. See also Subsection 5.3 for connections with the vertex independence number and $\eta$.

Examples:

1. Let $L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$ be the Laplacian of $K_{1,3}$; $\text{rank}(L) = 3$. $mr^+(K_{1,3}) = 3$, since no diagonal element can be 0 in a positive semidefinite matrix of a connected graph, so for any $B \in S^+(K_{1,3})$, $B(1)$ is a $3 \times 3$ diagonal submatrix with nonzero diagonal entries, so $mr^+(K_{1,3}) \geq 3$. Note $mr(K_{1,3}) = 2$. Also, $M_0^+(K_{1,3}) = 1 < 2 = M_1^+(K_{1,3})$ (since $\text{mult}_L(1) = 2$ and $M(K_{1,3}) = 2$).

5.3 A Family of Matrices and the Graph Parameter $\eta$

All graphs in this subsection are simple.

Definitions:
Let $G$ be a simple graph.

- $S^F_\eta(G) = \{B \in F^{n \times n} : B$ is a symmetric matrix over $F$, if $ij \notin E(G)$, then $b_{ij} = 0$ and $\forall i, b_{ii} \neq 0\}$.

(No requirement that the entry corresponding to an edge be nonzero.)

- $\eta(G) = \min \{\text{rank}(B) : B \in S^F_\eta, F$ any field\}$

For more information about $\eta$, see Chapter 28 of [HLA06].

Results:
Let $G$ be a simple graph.
1. $\eta(G)$ is not comparable to $\text{mr}(G)$ (see examples).

2. If the connected components of $G$ are $G_1, \ldots, G_t$, then $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$.

3. If $G'$ is an induced subgraph of $G$ then $\eta(G') \leq \eta(G)$.

4. If $B \in S^F_\eta(G)$ and $R \subseteq V(G)$, $k \in V(G)$, then $B[R] \in S^F_\eta(G[R])$ and $B(k) \in S^F_\eta(G - k)$.

5. $\eta(G) = 1$ if and only if $G = K_n$.

6. If $G$ is connected, then $S^+(G) \subseteq S^\eta(G)$ and $\eta(G) \leq \text{mr}^+(G)$

   (No diagonal element can be 0 in a positive semidefinite matrix of a connected graph.)

7. [HLA06, Fact 28.5.9] $\alpha(G) \leq \eta(G) \leq \chi(G)$.

   (For the first inequality, let $R \subseteq V(G)$ be an independent set such that $|R| = \alpha(G)$. Let $B \in S^F_\eta(G)$ such that $\text{rank}(B) = \eta(G)$. Then $B[R]$ is a diagonal matrix with nonzero diagonal, so $\eta(G) = \text{rank}(B) \geq \text{rank}(B[R]) = |R| = \alpha(G)$.)

Examples:

1. Consider $K_{1,3}$ with vertex 1 being the unique high degree vertex. Then any matrix $A$ in the set over which the minimum rank is taken when evaluating $\eta$ has rank($A(1)$) = 3, and over $\mathbb{R}$, rank($L$) = 3 for the Laplacian $L$ of $K_{1,3}$, so $\eta(K_{1,3}) = 3$. Note $\text{mr}(K_{1,3}) = 2$.

2. By considering $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, we see $\eta(P_4) = 2$. Note $\text{mr}(P_4) = 3$. Also, $\theta(P_4) = 2$.

5.4 Graphs with Loops

Warning: This subsection is based on [DHHHW06] but the notation differs.

Definitions:

Let $G$ be a graph without multiple edges that may have loops.

- $S^f(G) = \{B \in S_n : b_{ij} \neq 0 \text{ if and only if } ij \in E(G)\}$

  Note that the diagonal zero-nonzero pattern is restricted by $G$. 

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• \( \text{mr}^\ell(G) = \min\{\text{rank}(B) : B \in S^\ell(G)\} \).

• \( M^\ell_\lambda(G) = \max\{\text{mult}_B(\lambda) : B \in S^\ell(G), \lambda \in \sigma(B)\} \). (It is necessary to distinguish between zero from nonzero eigenvalues because translation is no longer possible.)

• **G allows singularity** if there exists \( B \in S^\ell(G) \) that is singular.

• \( \hat{G} \) denotes the simple graph obtained from \( G \) by removing all loops from \( G \).

• The graph \( T \) is a **loop-tree** if \( \hat{T} \) is a (simple) tree.

• If \( G_1 \) is a subgraph of \( G \) and \( H \) is a subset of the vertices of \( G \), \( G_1 \) is H-free if \( G_1 \) has no vertex in \( H \).

• For a loop-tree \( T \), define it its **matrix of indeterminates** \( X_T \) as follows:
  For \( i \leq j, i, j \in V(T), ij \in E(T) \), let \( x_{ij} \) be independent indeterminates and \( (X_T)_{ij} = x_{ij} \) and \( (X_T)_{ji} = x_{ij} \), and let the entries that do not correspond to edges be 0.

• [DHHHW06] For \( Q \subseteq V(T) \), define \( c_\lambda(Q) \) to be the number of components of \( T(Q) \) that allow eigenvalue \( \lambda \) and
  \[
  C_\lambda(T) = \max\{c_\lambda(Q) - |Q| : Q \subseteq V(T)\},
  \]

**Results:**
Let \( G \) be a graph without multiple edges that may have loops and let \( T \) be a loop-tree.

1. \( S^\ell(G) \subseteq S(G) \) and \( \text{mr}^\ell(G) \geq \text{mr}(G) \).

2. \( \text{mr}^\ell(G) \) is usually different from \( \text{mr}(G) \).

3. If the connected components of \( G \) are \( G_1, \ldots, G_t \), then \( \text{mr}^\ell(G) = \sum_{i=1}^t \text{mr}^\ell(G_i) \).

4. \( M_0(G) + \text{mr}^\ell(G) = |G| \).

5. If \( G' \) is an induced subgraph of \( G \) then \( \text{mr}^\ell(G') \leq \text{mr}^\ell(G) \).

6. If \( B \in S^\ell(G) \), then \( B(k) \in S^\ell(G - k) \).

7. For a connected graph \( G \) of order \( n > 1 \), \( \text{mr}^\ell(G) = 1 \) if and only if \( G \) is the complete graph with a loop at every vertex.

8. [DHHHW06] \( T \) allows singularity if and only if \( \det X_T \) is identically zero or \( \det X_T \) has at least two distinct nonzero terms.

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9. [DHHHW06] For any loop-tree, $M_\lambda(T) = C_\lambda(T)$, and $C_\lambda(T)$ and $mr^\ell(T)$ can be computed by Algorithm 3 (which generalizes Algorithm 2).

<table>
<thead>
<tr>
<th>Algorithm 3: Computation of $mr^\ell(T)$ and $c_0(T)$ (from the outside working in).</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> A loop-tree $T$ on $n$ vertices.</td>
</tr>
<tr>
<td><strong>Output:</strong> $mr(T)$ and a set $Q$ of vertices of $T$ whose deletion realizes $C_0(T)$.</td>
</tr>
<tr>
<td><strong>Initialize:</strong> $H$ is the set of all high degree vertices of $\hat{T}$, $Q = \emptyset$, and $i = 1$. While $H \neq \emptyset$:</td>
</tr>
<tr>
<td>1. Set $\hat{T}_i$ the unique component of $\hat{T} - Q$ that contains an $H$-vertex.</td>
</tr>
<tr>
<td>2. Set $S_i = {B[V(\hat{T}_i)] : B \in S^\ell(T)}$.</td>
</tr>
<tr>
<td>3. Set $Q_i = \emptyset$.</td>
</tr>
<tr>
<td>4. Set $W_i = {w \in H: \text{all but possibly one component of } \hat{T}_i - w \text{ is } H\text{-free}}$.</td>
</tr>
<tr>
<td>5. For each vertex $w \in W_i$, if there are at least two $H$-free components of $T_i - w$ that allow singularity, then $Q_i = Q_i \cup {w}$. (Test for singularity using $\det X_{R_j}$ where $R_j$ is the component.)</td>
</tr>
<tr>
<td>6. $Q = Q \cup Q_i$.</td>
</tr>
<tr>
<td>7. Remove all the vertices of $W_i$ from $H$.</td>
</tr>
<tr>
<td>8. For each $v \in H$, if $\deg_{\hat{T} - Q} v \leq 2$, remove $v$ from $H$.</td>
</tr>
<tr>
<td>9. $i = i + 1$.</td>
</tr>
<tr>
<td>$C_0(T) = p -</td>
</tr>
</tbody>
</table>

10. For computing the minimum rank of a loop-tree $T$, it does not matter whether the matrices associated with $T$ are symmetric. (If $G(B)$ is a tree then the signs of the entries can be made symmetric by multiplication by a diagonal $\pm 1$ matrix. This does not change the rank. A sign symmetric matrix whose graph is a tree is similar by a diagonal similarity to a symmetric matrix with the same graph.)

11. A variant of Algorithm 3 can be used to compute minimum rank of tree sign patterns. For more information, see [DHHHW06].

**Examples:**

1. [DHHHW06] We apply Algorithm 3 to compute the minimum rank of the tree $T$ shown in Figure 16. Initially, $Q = \emptyset$, $i = 1$ and $H = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is the set of high degree vertices. For the first iteration of Algorithm 3, $T_1 = T$, and $W_1 = \{1, 3, 6, 7\}$. Deletion of vertex 1 leaves two $H$-free components both of which require non-singularity, so $1 \notin Q_1$. 

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Deletion of vertex 3 leaves four $H$-free components, three of which, $T[13, 14, 15], T[16]$, and $T[17, 18, 19, 20]$ allow singularity, as can be verified by computation of the determinant. Thus $3 \in Q_1$.

Deletion of vertex 6 leaves two $H$-free components, both of which allow singularity, so $6 \in Q_1$.

Deletion of vertex 7 leaves two $H$-free components, both of which require nonsingularity, so $7 \notin Q_1$.

Now $Q = Q_1 = \{3, 6\}$, $H = \{2, 4, 5, 8\}$ and the forest $T - Q_1$ is shown in...
Figure 17 (the only labels shown are for vertices currently in $H$).

For the second iteration of Algorithm 3, $T_2$ is the component that contains $2, 4, 5, 8$, and $W_2 = \{2, 5, 8\}$:

$T_2 - 2$ has two $H$-free components, both of which allow singularity. The fact that the component that contains vertex 1 (look at Figure 16 in order to see that label) allows singularity can be verified by computation of the determinant. Thus $2 \in Q_2$.

$T_2 - 5$ has five $H$-free components, three of which allow singularity, so $5 \in Q_2$.

$T_2 - 8$ has two $H$-free components, both of which allow singularity, so $8 \in Q_2$.

Thus $Q = \{2, 3, 5, 6, 8\}$ and $T - Q$ is shown in Figure 18. There is no third iteration since the only vertex remaining in $H$ after the removal of $W_2$, i.e. 4, no longer has high degree, and so is removed from $H$ also.

Since $T - Q$ has twelve components which allow singularity, $M_0(T) = C_0(T) = 12 - 5 = 7$. Thus $\text{mr}^f(T) = 35 - 7 = 28$.

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[Netal] This citation covers the following four papers. Some of these papers will be available at the workshop.


Matthew Booth, Phil Hackney, Benjamin Harris, Charles R. Johnson, Margaret Lay, Lon H. Mitchell, Sivaram K. Narayan, Amanda Pascoe, and Brian D. Sutton. The minimum semidefinite rank of a graph.

Phil Hackney, Benjamin Harris, Margaret Lay, Lon H. Mitchell, Sivaram K. Narayan, and Amanda Pascoe. Linearly independent vertices and minimum semidefinite rank.


