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A typical vertex of a tree

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Abstract

Let T denote a tree with at least three vertices. Observe that T contains a vertex which has at least two neighbors of degree one or two. A class of algorithms on trees related to the observation are discussed and characterized. One of the example is an algorithm to compute the minimum rank $m(T)$ of the symmetric matrices with prescribed graph T, which is easier to process than the algorithm previous found by Nylen [Linear Algebra Appl. 248 (1996) 303–316]. Two interpretations of the number $m(T)$ in terms of some combinatorial properties on trees are given. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction and results

Let T denote a tree with $n(T)$ vertices. We also use T as its vertex set. We refer the reader to [2, pp. 376–388] for the definition and the properties of trees. For a vertex subset $U \subseteq T$, let $T \setminus U$ denote the subgraph induced on the vertex subset $T \setminus U$ of T. Let p be a vertex of T, and let $T_p^1, \ldots, T_p^t$ denote the connected components of $T \setminus \{p\}$. Note that each T_p^i is a tree. Observe

$$n(T) = n(T_p^1) + \cdots + n(T_p^t) + 1. \tag{1}$$

Let P_n denote the simple path with n vertices. Line (1) can be viewed as a trivial algorithm on trees to compute $n(T)$ provided the initial condition $n(P_1) = 1$. The choice of a vertex p does not affect the value $n(T)$.

We shall give another algorithm on trees. We need a few definitions first. For an $n \times n$ symmetric matrix $A = [a_{ij}]$, we associate with it the graph $\Gamma(A)$ having n vertices

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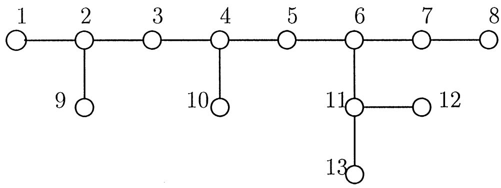

Fig. 1.

labeled $1, 2, \ldots, n$. For $i \neq j$, the unordered pair (i, j) will be an edge in $\Gamma(A)$ if and only if $a_{ij} \neq 0$. Given a graph G on n vertices, we define the number $m(G)$ by

$$m(G) := \min\{\text{rank } A \mid \Gamma(A) = G\}. \tag{2}$$

The study of $m(G)$ can be found in [3–5]. Observe

$$m(P_1) = 0, \qquad m(P_2) = 1. \tag{3}$$

A vertex p of T is called *appropriate* if at least two of the connected components in $T \backslash \{p\}$ are the simple paths (one or more vertices) which were connected to p through an endpoint. It is not difficult to see that every tree T with at least 3 vertices has an appropriate vertex, see [3, Lemma 3.1] for details. Provided the initial conditions in (3), Nylen [3] gives the algorithm

$$m(T) = m(T_p^1) + \cdots + m(T_p^t) + 2 \tag{4}$$

to compute $m(T)$, where $n(T) \geqslant 3$ and p is an appropriate vertex of T. The choice of p among the appropriate vertices of T does not affect the number $m(T)$ also.

Motivated by the above definition, we define a vertex p of T to be *typical* if p has at least two neighbors of degrees 1 or 2 in T. It is immediate from the definition that an appropriate vertex is a typical vertex. In Fig. 1, the vertices labeled 2, 4, 6, 11 are typical and only the vertices labeled 2, 11 are appropriate.

We shall prove in Theorem 1.7 that the condition p being appropriate in line (4) can be replaced by p being typical. We study a general class of algorithms on trees first. Fix three reals a, b, c. We assign a tree T with the real numbers $f(T)$ recursively by the following rules:

$$f(P_1) = a, \qquad f(P_2) = b, \tag{5}$$

$$f(T) = f(T_p^1) + \cdots + f(T_p^t) + c, \tag{6}$$

where p is a typical vertex of T. Note that $f(T)$ may not have a unique solution, since the choice of a typical vertex p may be different. For $a = 1$, $b = 2$, $c = 1$, $f = n$, (5)–(6) is the case of (1) with p typical. We list our results in this section and the proofs shall be in next section.

Lemma 1.1. *Suppose the algorithm in* (5)–(6) *generates a unique solution* $f(T)$ *for each tree* T. *Then* $3a - 2b + c = 0$.

We shall prove the converse of Lemma 1.1 in Theorem 1.4. In fact, if $3a-2b+c=0$ then we can express $f(T)$ into a linear combination of $n(T)$ and the number $s(T)$ defined below. For a vertex subset $U \subseteq T$, let $c_T(U)$ denote the number of connected components in the subgraph $T \setminus U$. The *separating number* of a tree T is the number

$$s(T) := \max\{c_T(U) - |U| \mid U \subseteq T\}. \tag{7}$$

U is a *separating* set of T if $c_T(U) - |U| = s(T)$. Note that if U is a separating set of $T, T \setminus U$ is a union of simple paths. Observe

$$s(P_1) = 1, \qquad s(P_2) = 1. \tag{8}$$

Theorem 1.2 gives an algorithm to construct a separating set, and to determine the separating number of a tree.

Theorem 1.2. *Let T be a tree with at least 3 vertices and p be a typical vertex of T. Let $T_p^1, \ldots, T_p^t$ be the connected components of $T \setminus \{p\}$. Let U be a subset of vertices of T containing p. Then U is a separating set of T if and only if for each i $(1 \leqslant i \leqslant t)$, $U \cap T_p^i$ is a separating set of T_p^i. Furthermore,*

$$s(T) = s(T_p^1) + \cdots + s(T_p^t) - 1. \tag{9}$$

Note that (8)–(9) is the case $a=1$, $b=1$, $c=-1$ and $f=s$ of (5)–(6). It follows from (8)–(9) that $s(P_n)=1$. Corollary 1.3 improves the algorithm in Theorem 1.2.

Corollary 1.3. *Let U be a subset of the typical vertices of T satisfying the following $(*)$ condition of T:*

$(*)$ *Each vertex of U with degree 2 in T is not adjacent to other vertices in U.*

Let $T_U^1, \ldots, T_U^l$ be the connected components of $T \setminus U$. Suppose S_j is a separating set of T_U^j $(1 \leqslant j \leqslant l)$. Then,

$$U \cup \left(\bigcup_{1 \leqslant j \leqslant l} S_j \right)$$

is a separating set of T. Furthermore,

$$s(T) = s(T_U^1) + \cdots + s(T_U^l) - |U|. \tag{10}$$

The following theorem shows that $n(T)$ and $s(T)$ span all the functions defined on trees satisfying (5)–(6).

Theorem 1.4. *Suppose $3a-2b+c=0$. Then $f(T)$ are numbers generated from (5)–(6) for trees T if and only if*

$$f(T) = \frac{a+c}{2} n(T) + \frac{a-c}{2} s(T) \tag{11}$$

for trees T. In particular, $f(T)$ has a unique solution for each tree T.

For graph theoretical interest, we give another interpretation of $s(T)$ in Corollary 1.6. Let $e(T)$ denote the number of edges in T. Note that $e(T)=n(T)-1$. A subset F of the edge set $E(T)$ of T *dissolves* the tree T if the subgraph $T\backslash F$ obtained from T by deleting all edges in F is a disjoint union of simple paths. Set

$$s^*(T):=\min\{|F|\,|\,F\subseteq E(T) \text{ dissolves } T\}. \tag{12}$$

An edge subset F is a *separating* edge set of T if F dissolves T and $|F|=s^*(T)$. Observe $s^*(P_n)=0$.

Theorem 1.5. *Let T be a tree with at least 3 vertices and p be a typical vertex of degree t. Let $e_1,\dots,e_t$ denote the edges incident on p, and $T_p^1,\dots,T_p^t$ the connected components of $T\backslash\{p\}$. Assume each of e_{t-1},e_t is incident on a vertex different from p of degree at most 2 in T. Suppose F_i is a separating edge set of T_p^i $(1\leqslant i\leqslant t)$. Then*

$$\{e_1,\dots,e_{t-2}\}\cup\bigcup_{1\leqslant i\leqslant t}F_i$$

is a separating edge set of T. Furthermore,

$$s^*(T)=s^*(T_p^1)+\cdots+s^*(T_p^t)+t-2. \tag{13}$$

Equivalently, $g(T):=e(T)-s^*(T)$ satisfies

$$g(T)=g(T_p^1)+\cdots+g(T_p^t)+2. \tag{14}$$

Corollary 1.6.

$$s(T)=s^*(T)+1. \tag{15}$$

Theorem 1.7. *Let T be a tree with at least 3 vertices and p be a typical vertex of degree t. Let $T_p^1,\dots,T_p^t$ be the connected components of $T\backslash\{p\}$. Then*

$$m(T)=m(T_p^1)+\cdots+m(T_p^t)+2, \tag{16}$$

where $m(T)$ is defined in (2).

Following the above lines, we reprove the following corollary which was proved by Johnson and Duarte [1].

Corollary 1.8. $m(T)=e(T)-s^*(T)=n(T)-s(T)$.

To end this section, we show how to compute $m(T)$ for the tree T in Fig. 1. The best algorithm is Corollary 1.3. We set $U=\{2,4,6,11\}$ which of course satisfies $(*)$ condition of Corollary 1.3. Since $T\backslash U$ contains 8 simple paths, the separating number $s(T)=8-4=4$ by (10). Now $m(T)=13-4=9$ by Corollary 1.8.

2. Proofs of results

Proof of Lemma 1.1. Suppose the algorithm in (5)–(6) generates a unique solution $f(T)$ for each tree T. Considering the simple path P_3 of three vertices, the middle vertex is typical, so $f(P_3)=2a+c$ by (5)–(6). For the simple path P_5 of five vertices, there are essentially two different ways to choose a typical vertex. According to these two ways,

$$\begin{aligned} f(P_5) &= f(P_2)+f(P_2)+c \\ &= 2b+c \end{aligned}$$

and

$$\begin{aligned} f(P_5) &= f(P_1)+f(P_3)+c \\ &= a+(2a+c)+c. \end{aligned}$$

Hence $3a-2b+c=0$. $\square$

Proof of Theorem 1.2. We find an upper bound of $s(T)$ first. Let V denote a vertex subset of T. We shall prove

$$c_T(V)-|V| \leqslant s(T_p^1)+\cdots+s(T_p^t)-1. \tag{17}$$

Set $V_i=V\cap T_p^i$ $(1\leqslant i\leqslant t)$. Suppose $p\in V$. Then

$$|V|=1+\sum_{i=1}^{t}|V_i| \tag{18}$$

and the components in $T\backslash V$ are exactly those in $T_p^i\backslash V_i$ $(1\leqslant i\leqslant t)$. Hence,

$$\begin{aligned} c_T(V)-|V| &= \sum_{i=1}^{t} c_{T_p^i}(V_i)-\left(1+\sum_{i=1}^{t}|V_i|\right) \\ &= \sum_{i=1}^{t}(c_{T_p^i}(V_i)-|V_i|)-1 \\ &\leqslant s(T_p^1)+\cdots+s(T_p^t)-1. \end{aligned} \tag{19}$$

Suppose $p\notin V$. Then

$$|V|=\sum_{i=1}^{t}|V_i|. \tag{20}$$

Let u denote the number of neighbors of p in $T\backslash V$. Each of the u vertices is in a connected component of $T_p^i\backslash V_i$ which contains it, and p merges these u components into a single connected component of $T\backslash V$. Then

$$c_T(V)=1-u+\sum_{i=1}^{t}c_{T_p^i}(V_i). \tag{21}$$

Let v denote the number of neighbors of p in V which have degrees 1 or 2 in T. Since each of these v vertices has degree 0 or 1 in the subgraph T_p^i which contains it, and by the fact, a separating set contains no endpoints, we have the corresponding V_i is not a separating set of T_p^i. Hence there are at least v indices i such that

$$c_{T_p^i}(V_i) - |V_i| + 1 \leqslant s(T_p^i).$$

Then

$$v + \sum_{i=1}^{t} (c_{T_p^i}(V_i) - |V_i|) \leqslant \sum_{i=1}^{t} s(T_p^i). \tag{22}$$

Note that

$$u + v \geqslant 2, \tag{23}$$

since p is typical. Then by (20)–(23),

$$\begin{aligned} c_T(V) - |V| &= 1 - u + \sum_{i=1}^{t} c_{T_p^i}(V_i) - \sum_{i=1}^{t} |V_i| \\ &= 1 - u + \sum_{i=1}^{t} (c_{T_p^i}(V_i) - |V_i|) \\ &\leqslant s(T_p^1) + \cdots + s(T_p^t) + 1 - u - v \\ &\leqslant s(T_p^1) + \cdots + s(T_p^t) - 1. \end{aligned} \tag{24}$$

This proves (17). To prove Theorem 1.2, set $V = U$ in (17). Then $p \in V$. Suppose $V_i = V \cap T_p^i$ is a separating set of T_p^i for all i. Then equality holds in (19). Hence for the vertex set $V, c_T(V) - |V|$ attains its maximum in (17). We conclude V is separating set of T, and (9) holds. To prove the other direction, suppose V is a separating set of T. Then equality holds in (17) and (19). This forces

$$c_{T_p^i}(V_i) - |V_i| = s(T_p^i) \quad (1 \leqslant i \leqslant t),$$

where $V_i = V \cap T_p^i$. Hence for each i $(1 \leqslant i \leqslant t)$, $V \cap T_p^i$ is a separating set of T_p^i. This proves the theorem. □

Proof of Corollary 1.3. We prove the corollary by induction on the cardinality of U. This is clear if U is empty. Assume U is not empty. Pick $p \in U$. Let $T_p^1, \ldots, T_p^t$ denote the connected components of $T \backslash \{p\}$. Fix an integer i $(1 \leqslant i \leqslant t)$. Observe that T_p^i contains those T_U^j it intersects. First we prove that

$$(U \cap T_p^i) \cup \left(\bigcup_{S_j \subseteq T_p^i} S_j \right) \tag{25}$$

is a separating set of T_p^i, and

$$s(T_p^i) = \sum_{T_U^j \subseteq T_p^i} s(T_U^j) - |U \cap T_p^i|. \tag{26}$$

Eqs. (25)–(26) follow from induction, if we prove $U \cap T_p^i$ contains typical vertices of T_p^i satisfying $(*)$ condition of T_p^i. Let x denote the neighbor of p in T_p^i. Note that for vertices in T_p^i, the degrees in T and the degrees in T_p^i are the same except the vertex x whose degrees are decreased by 1. Hence, we only need to show that if $x \in U$ then x is also typical in T_p^i, and furthermore, if x has degree 2 in T_p^i then x is not adjacent to other vertices in $U \cap T_p^i$. Suppose $x \in U$. Then p has degree at least 3, since U satisfies the $(*)$ condition of T. Hence x is also typical in T_p^i by the definition of typical. Furthermore, suppose x has degree 2 in T_p^i. By the definition of typical again, the two neighbors of x in T_p^i have degrees 1 or 2 in T, and then are not contained in U since U satisfies the $(*)$ condition of T. This proves (25)–(26). By applying Theorem 1.2 to (25)–(26),

$$\{p\} \cup \bigcup_{1 \leqslant i \leqslant t} \left((U \cap T_p^i) \cup \left(\bigcup_{S_j \subseteq T_p^i} S_j \right) \right) = U \cup \left(\bigcup_{1 \leqslant j \leqslant l} S_j \right)$$

is a separating set of T, and

$$\begin{aligned} s(T) &= s(T_p^1) + \cdots + s(T_p^t) - 1 \\ &= \sum_{1 \leqslant i \leqslant t} \left(\sum_{T_U^j \subseteq T_p^i} s(T_U^j) - |U \cap T_p^i| \right) - 1 \\ &= s(T_U^1) + \cdots + s(T_U^l) - |U|. \end{aligned}$$

This proves the corollary. □

Proof of Theorem 1.4. First, assume $f(T)$ are numbers generated from (5)–(6). We prove by induction on the number $n(T)$. Note that $n(P_1) = 1$, $n(P_2) = 2$, $s(P_1) = s(P_2) = 1$, $f(P_1) = a$, $f(P_2) = b$. Hence (11) can be checked directly if $n(T) \leqslant 2$. Assume $n(T) \geqslant 3$. Pick a typical vertex p in T. By (6), induction, (1) and (9), we obtain

$$\begin{aligned} f(T) &= f(T_p^1) + \cdots + f(T_p^t) + c \\ &= \frac{a+c}{2} \sum_{i=1}^{t} n(T_p^i) + \frac{a-c}{2} \sum_{i=1}^{t} s(T_p^i) + c \\ &= \frac{a+c}{2} \left(\sum_{i=1}^{t} n(T_p^i) + 1 \right) + \frac{a-c}{2} \left(\sum_{i=1}^{t} s(T_p^i) - 1 \right) \\ &= \frac{a+c}{2} n(T) + \frac{a-c}{2} s(T). \end{aligned} \tag{27}$$

This proves the necessary condition (11). $f(T)$ has a unique solution, since $n(T), s(T)$ in (11) are well-defined functions. For the other direction, we assume (11) holds. (5) can be check directly. Reversing above four equalities in (27), we obtain $f(T)$ satisfies (6). This proves the theorem. □

Proof of Theorem 1.5. We give a lower bound of $s^*(T)$ first. Suppose $F' \subseteq E(T)$ dissolves T. We shall prove

$$|F'| \geqslant s^*(T_p^1) + \cdots + s^*(T_p^t) + t - 2. \tag{28}$$

Set $F_i' = F' \cap E(T_p^i)$ $(1 \leqslant i \leqslant t)$. Since the vertex p has degree t in T, and $T \backslash F'$ are simple paths, F' contains at least $t-2$ edges incident on p. Hence

$$|F'| \geqslant |F_1'| + \cdots + |F_t'| + t - 2. \tag{29}$$

Observe that F_i' dissolves T_p^i. Hence,

$$|F_i'| \geqslant s^*(T_p^i) \quad (1 \leqslant i \leqslant t). \tag{30}$$

Eq. (28) follows from (29)–(30). To prove the theorem, set

$$F' = \{e_1, \ldots, e_{t-2}\} \cup \left(\bigcup_{1 \leqslant i \leqslant t} F_i \right).$$

Hence $F_i' = F_i$. Observe F' dissolves T, and equalities hold in (29)–(30). Hence equality holds in (28). This proves that (13) holds and F' is a separating edge set of T. To prove (14), observe

$$\begin{aligned} g(T) &= e(T) - s^*(T) \\ &= e(T) - s^*(T_p^1) - \cdots - s^*(T_p^t) - t + 2 \\ &= \sum_{1 \leqslant i \leqslant t} (e(T_p^i) - s^*(T_p^i)) + 2 \\ &= \sum_{1 \leqslant i \leqslant t} g(T_p^i) + 2. \qquad \square \end{aligned}$$

Proof of Corollary 1.6. With the notation of Theorem 1.5, observe $g(P_n) = e(P_n) - s^*(P_n) = n - 1$, especially $g(P_1) = 0$ $g(P_2) = 1$. Hence (14) is the case $f = g$, $a = 0$, $b = 1$, and $c = 2$ in (5)–(6). We obtain $e(T) - s^*(T) = n(T) - s(T)$ by (11). Then $s(T) = s^*(T) + 1$, since $n(T) - e(T) = 1$. $\square$

Proof of Theorem 1.7. $m(T)$ is the unique solution of the algorithm in (3)–(4). However (3)–(4) is a special case of (5)–(6) with p appropriate, $a = 0, b = 1$ and $c = 2$. Since $3a - 2b + c = 0$, the algorithm in (5)–(6) with p typical has the unique solution $m(T)$ by Theorem 1.4. $\square$

Proof of Corollary 1.8. The result follows by applying (3), (16) to (11) using (15). $\square$

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