Optimal Model Order Reduction for Finite Length Segments of LTI System Unit Sample Response

A. Megretski*

July 22, 2005

Abstract

The paper presents an algorithm for finding optimal reduced models of discrete time LTI systems described by finite length segments of their unit sample response. The optimality is understood in a way similar to that of the classical Hankel model reduction. The algorithm utilizes semidefinite programming and has slow complexity growth as the number of time domain samples increases. Application to model reduction of continuous time LTI systems of extremely large order is described.

Key Words: optimal model reduction, semidefinite programming, large scale linear systems.

1 Introduction

In this section, the basic problem formulation is sketched and discussed in the context of modern approaches to LTI model order reduction.

Spectral analysis of lightly damped oscillations

The following task appears in a number of engineering applications involving signal processing and system modeling: for a given finite sequence $h = h_{1:n} = \{h_t\}_{t=1}^n$ of noisy samples $h_t \in \mathbf{R}$ of a lightly dampled oscillation, find a good approximation by a sum of a small number of decaying sinusoids:

$$h_t \approx \sum A_i \rho_i^t \cos(\omega_i t + \phi_i) \quad \text{for } t \in \{1, 2, \dots, n\}.$$

Equivalently, one can search for a low order strictly proper stable transfer function

$$\hat{H}(z) = \frac{p(z)}{q(z)} = \sum_{t=1}^{\infty} \hat{h}_t z^{-t} \quad (|z| \geq 1)$$

of order $m \ll n$ such that the first n samples $\hat{h}_{1:n} = \{\hat{h}_t\}_{t=1}^n$ of the corresponding unit sample response provide a good approximation of $h_{1:n} = \{h_t\}_{t=1}^n$.

*ameg@mit.edu. This work was supported by AFOSR, and DARPA

Ad-hoc algorithms: an overview

A very straightforward approach of *moments matching* is based on finding the $2m$ free coefficients of p, q from the set of $2m$ equations $h_t = \hat{h}_t$ for $t = 1, 2, \dots, 2m$. These equations can be re-written as linear with respect to p, q.

A number of inexpensive methods of finding a “reasonable” q are based on minimization of

$$J(q) = \sum_{t=1}^{n-m} e_t^2,$$

where

$$H(z)q(z) - p(z) = \sum_{t=-m+1}^{n} e_t z^{-t},$$

subject to some normalization constraint imposed on p and q. For example, fixing a lead coefficient of q, as in $q(0) = 1$, yields a classical *least squares* algorithm. Alternatively, fixing the Euclidean norm of the vector of coefficients of q yields the standard singular value decomposition algorithm, frequently referred to as the *proper orthogonal decomposition*.

These approaches rarely yield a high quality approximation, and frequently deliver disastrous results (such as an unstable $\hat{H}$). One possible explanation is that $q(z)$ can be very small at some points on the unit circle, and hence

$$H(z) - \frac{p(z)}{q(z)} = \frac{H(z)q(z) - p(z)}{q(z)}$$

can be large even though $H(z)q(z) - p(z)$ is small.

Norm minimization algorithms

An ideal approach to finding a good m-th order approximation $\hat{H}(z) = p(z)/q(z)$ of $H(z)$ would be based on defining $\hat{H}$ as the argument of minimum of $\|h - \hat{h}_{1:n})\|$, where $\hat{h}_{1:n}$ denotes the sequence of first n terms in the expansion

$$\hat{H}(z) = \frac{p(z)}{q(z)} = \sum_{t=1}^{\infty} \hat{h}_t z^{-t} \quad (|z| \geq 1),$$

and $\|\cdot\|$ is a norm appropriately defined on the set of all sequences $\{\delta_t\}_{t=1}^n$ of n real numbers. Among the norms definitely worth minimizing are the Euclidean norm, the l^1 norm, the l^∞ norm, and many other.

Unfortunately, minimization of a norm of model reduction approximation error subject to an order constraint is usually a problem without an efficient solution. Most likely, this is due to the fact that the set of all impulse responses of m-th order transfer functions does not carry any decent convexity structures (or, more precisely, that such structures have not been discovered yet). To the best knowledge of the author, there is no norm $\|\cdot\|$ defined on the set of all sequences $\{\delta_t\}_{t=1}^n$ of n real numbers for which the task of minimizing $\|h - \hat{h}_{1:n}\|$ subject to an order constraint imposed on $\hat{H}$ has a polynomial time solution algorithm.

Hankel Model Reduction

In the case when $n = \infty$, i.e. a complete unit sample response $h = \{h_t\}_{t=1}^\infty$ is available, the classical *Hankel model reduction* gives an example of an efficient norm minimization algorithm. The *Hankel norm*

$\|\delta\|_H$ is defined on the set of exponentially decaying sequences δ_t as the largest singular number of the associated *Hankel matrix*

$$\Gamma = \Gamma_h = \begin{bmatrix} h_1 & h_2 & h_3 & \dots \\ h_2 & h_3 & & \\ h_3 & & & \\ \vdots & & & \ddots \end{bmatrix},$$

or, equivalently, as the infimum of the H-Infinity norm $\|H - V\|_\infty$, where

$$\|\Delta\|_\infty = \max_{|z|=1} |\Delta(z)|,$$

and V ranges over the set of all anti-stable proper transfer functions

$$V(z) = \sum_{t=-\infty}^{0} v_t z^{-t} \quad (|z| \leq 1).$$

It turns out that, for an h defined as a unit sample response of a stable rational transfer function, minimization of the Hankel norm of model reduction error can be performed efficiently by a polynomial time algorithm. In applications, the resulting Hankel norm optimal reduced models tend to have very high quality (this observation is also supported, to some extent, by rigorous mathematical analysis). Moreover, the minimal Hankel norm model reduction error serves as a lower bound for the minimal H-Infinity norm model reduction error, which, due to the classical small gain theorem, is one of the most adequate measures of model reduction quality.

Unfortunately, the algorithms for Hankel model order reduction, based on the famous Adamyan-Arov-Krein theorem, do not generalize easily to the case of model reduction of finite seqments of unit sample responses. When the order of H is large (tens of thousands or more states), the computational complexity of Hankel model reduction becomes prohibitively high.

Maximal real part model reduction

The author has recently proposed an alternative to Hankel norm model reduction, based on using the *maximal real part norm* $\|\cdot\|_{mrp}$ in place of the Hankel norm:

$$\|\Delta\|_{mrp} = \|\mathrm{Re}(\Delta)\|_\infty = \max_{|z|=1} |\mathrm{Re}(\Delta(z))|.$$

It turns out that the problem of minimizing the maximal real part norm of model reduction error can also be solved in polynomial time. When applied to a complete model of H, the new model reduction algorithm has higher complexity (still, growing polynomially) compared to Hankel model reduction. However, the real benefit of the new approach is that it can be also applied to the case when H is defined by a finite set of its samples $H(z_i)$ at points on the unit circle, and the functional to be minimized is

$$\|D\|_{smrp} = \max_i |\mathrm{Re}(\Delta(z_i))|.$$

Thus, for all practical purposes, complexity of the new algorithm grows very slowly with the order of the original model H: its most expensive operation is calculation of the samples of $H(z)$.

Even more encouraging, a simple generalization of the maximal real part model reduction approach yields an efficient optimization algorithm for the problem

$$\|H - \hat{H} - V\|_\infty \to \min,$$

where $\hat{H}, V$ are constrained by

$$\hat{H}(z) = \frac{p(z)}{q(z)}, \quad V(z) = \frac{r(z)}{z^m q(1/z)},$$

and q is a Schur polynomial of order m. This optimization is guaranteed to provide a lower bound for the minimal H-Infinity norm model reduction error which is at least as good (and, in most examples, strictly better) than the one delivered by the Hankel model order reduction algorithm. This optimization can also be performed when the information available about H is limited to a finite set of frequency domain samples $H(z_i)$, $|z_i| = 1$, in which case the functional to be minimized is

$$\max_i |H(z_i) - \hat{H}(z_i) - V(z_i)|.$$

Note that, in most situations of practical interest, the frequency samples $H(z_i)$ cannot be defined in terms of a finite segment $h_{1:n}$ of the unit sample response $h = h_t$ of H (this can be done easily when the values $|h_t|$ are small enough for $t > n$, but that case is not of interest in this paper). This makes it difficult to apply directly the ideas of maximal real part model reduction to systems defined by a finite segment of their sample response.

In this paper, a specific norm $\|\delta\|_{n,\infty}$ is defined on the set of all real sequences $\{\delta_t\}_{t=0}^n$ of length n, so that, for a given sequence $h = h_{0:n}$, the taks of minimizing

$$E(\hat{h}, v) = \|h - \hat{h}_{0:n} - v_{0:n}\|_{n,\infty},$$

where

$$\frac{p(z)}{q(z)} = \sum_{t=0}^{\infty} \hat{h}_t z^{-t} \quad (|z| \geq 1),$$

$$\frac{r(z)}{z^m q(1/z)} = \sum_{t=0}^{\infty} v_{n+1-t} z^t \quad (|z| \leq 1),$$

q, r are real polynomials of a given degree $m \ll n$, p is a real polynomial of degree $m-1$, and q is a Schur polynomial, has a polynomial time solution.

Continuous time systems

In a typical application of LTI model reduction in fluid dynamics or electromagnetics, the task is formulated as that of finding a good low order approximation of a given stable strictly proper transfer function $G(s)$ defined in an "implicit" state space format

$$G(s) = C(sE - A)^{-1}B,$$

where A, B, C, E are given real matrices, E is not necessarily invertible, and the dimension N of the generalized state vector (i.e. the dimesion of vector B) is very large (at least 10^4, but more likely 10^5 or 10^6 state variables).

It was recently proposed to use Fourier series expansions

$$H(z) = \sum_{t=0}^{\infty} h_t z^{-t},$$

where the stable proper discrete-time transfer function H is defined by

$$H\left(\frac{\omega_0 - s}{\omega_0 + s}\right) = \frac{s + \omega_0}{2\omega_0} G(s),$$

and $\omega_0 > 0$ is an appropriately selected "base frequency". It turns out that calculation of first n Fourier coefficients h_t reduces to solving n linear equations $(\omega_0 E - A)x = y$ with respect to x, which, in many applications, can be done quite efficiently. The model reduction algorithm described in this paper can be used to extract a low order approximate model of H (and hence of G) from the first $n+1$ coefficients h_t.

2 Main Results

In this section, an optimal model reduction setup is defined, and a polynomial time algorithm is described for its solution.

The basic model reduction problem

Let $\mathcal{L}_n$ denote the set of all sequences $y = \{y_t\}_{t=0}^n$ of real variables. For $y \in \mathcal{L}_n$ define its Fourier transform $\tilde{y} = \{\tilde{y}_i\}_{i=0}^n$ by

$$\tilde{y}_i = \sum_{t=0}^{n} y_t e^{\frac{2\pi i t j}{n+1}}.$$

Define the norm $\|\cdot\|_{n,\infty}$ on $\mathcal{L}_n$ by

$$\|y\|_{n,\infty} = \max_i |\tilde{y}_i|.$$

The optimal model reduction problem under consideration in this paper is defined as follows: *given a sequence $h \in \mathcal{L}_n$ and a positive integer $m \ll n$, find polynomials $p, q \in \mathbf{R}[z]$ of order not larger than m, and a polynomial $r \in \mathbf{R}[z]$ of order less than m, which minimize*

$$E(p, q, r) = \|h - \hat{h}_{0:n} - v_{0:n}\|_{n,\infty} \to \min \tag{1}$$

subject to q being a normalized Schur polynomial:

$$q(z) \neq 0 \quad \text{for} \quad |z| \geq 1, \quad q(1) = 1, \tag{2}$$

where

$$\frac{p(z)}{q(z)} = \sum_{t=0}^{\infty} \hat{h}_t z^{-t} \quad (|z| \geq 1), \tag{3}$$

$$\frac{r(z)}{z^m q(1/z)} = \sum_{t=0}^{\infty} v_{n+1-t} z^t \quad (|z| \leq 1). \tag{4}$$

Here h represents a finite segment of unit sample response of a stable discrete time LTI SISO system, to be approximated by a lower order stable system,

$$\hat{H}(z) = \frac{p(z)}{q(z)}$$

is the resulting reduced model, and v is the unstable component of the approximation, similar to the one used in Hankel and maximal real part norm model reduction algorithms.

Reduction to semidefinite programming

An efficient algorithm for solving problem (1)-(4) is based on re-parameterization of the decision variables p, q, r. In the following, a function $f = f(t)$ of a real argument t is called a *trigonometric polynomial* of degree m if

$$f(t) = f_0 + \sum_{k=1}^{m} \{f_k \cos(kt) + g_k \sin(kt)\}$$

for some constants f_k, g_k. When all g_k are equal to zero, the trigonometric polynomial is *even*. Alternatively, when all f_k are equal to zero, the trigonometric polynomial is *odd*.

Let

$$t_i = \frac{2\pi i}{n+1}, \;\; z_i = e^{jt_i} \text{ for } i = 0, 1, \dots, n.$$

Consider the task of minimizing

$$\hat{E}(a, b, c) = \max_{i \in \{0,1,\dots,n\}} \left| \frac{b(t_i) + jc(t_i)}{a(t_i)} - \tilde{h}(z_i) \right| \to \min, \tag{5}$$

where a, b, c are trigonometric polynomials of degree not larger than m, such that a and b are even, c is odd, and

$$a(t) > 0 \;\; \forall\, t \in \mathbf{R}, \quad a(0) = 1. \tag{6}$$

Let $\mathcal{X}^m_{pqr}$ denote the set of all admissible decision parameters in problem (1)-(4), i.e. the set of all 3-typles (p, q, r) of polynomials, where p, q have degree not larger than m, r has degree less than m, and condition (2) is satisfied. Let $\mathcal{X}^m_{abc}$ denote the set of all admissible decision parameters in problem (5),(6), i.e. the set of all 3-typles (a, b, c) of trigonometric polynomials of degree not exceeding m, where a and b are even, c is odd, and condition (6) is satisfied.

The main technical result of this paper is given by the following statement.

Theorem 1 *Problem (1)-(4) is equivalent to problem (5),(6), in the sense that there exists a one-to-one correspondence* $\tau_m : \mathcal{X}^m_{abc} \mapsto \mathcal{X}^m_{pqr}$*, such that* $E(p, q, r) = \hat{E}(a, b, c)$ *whenever* $\tau_m(a, b, c) = (p, q, r)$*.*

Since $\hat{E}(a, b, c)$ can be defined as the minimal $\gamma \geq 0$ such that

$$|b(t_i) + jc(t_i) - \tilde{h}(z_i)a(t_i)| \leq \gamma a(t_i) \;\; \forall\, i, \tag{7}$$

and conditions (6),(7) are convex with respect to the coefficients of a, b, c, minimization of $\hat{E}(a, b, c)$ can be accomplished via a convex optimization. Moreover, condition $a(t) > 0$ is equivalent to existence of a positive definite matrix $M = M' > 0$ such that

$$a(t) = \begin{bmatrix} 1 \\ \cos(t) \\ \vdots \\ \cos(mt) \end{bmatrix}' M \begin{bmatrix} 1 \\ \cos(t) \\ \vdots \\ \cos(mt) \end{bmatrix}. \tag{8}$$

Hence, minimization of $\hat{E}(a, b, c)$ is equivalent to solving a semidefinite program with respect to M and the coefficients of b, c.

The proof of Theorem 1 is constructive, in the sense that the transformation τ is explicitly defined (as shown in the proof below), and can be performed efficiently using standard linear algebra manipulations.

Proof of Theorem 1

The proof follows the standard line of arguments for minimization of maximal real part model reduction error, except for the following additional observation.

Theorem 2 *Let $q \in \mathbf{R}[z]$ be a Schur polynomial of degree m. Then for every integer $n > 0$ there exists a nonsingular linear operator T_n acting on the set of all polynomials $p \in \mathbf{R}[z]$ of degree not larger than m, such that*

$$\frac{(Lp)(z_i)}{q(z_i)} = \sum_{t=0}^{n} f_t z_i^{-t} = f^{[n]}(z)$$

for all p and all

$$z_i = e^{jt_i}, \quad t_i = \frac{2\pi i}{n+1}, \quad i = 0, 1, \ldots, n,$$

where

$$f(z) = \frac{p(z)}{q(z)} = \sum_{t=0}^{\infty} f_t z^{-t} \quad (|z| \geq 1).$$

Theorem 2 can be interpreted in the following way: while cutting unit sample response of a proper m-th order transfer function $f(z) = p(z)/q(z)$ by removing the terms $f_t z^{-t}$ with $t > n$ yields, in general, a rational function $f^{[n]}$ of order n, the samples $f^{[n]}(z_i)$ of this function at all $(n+1)$-th order roots of 1 are matched exactly by the samples of another m-th order transfer function $\hat{f}^{[n]}(z) = (Lp)(z)/q(z)$.

Proof. We have

$$\frac{p(z)}{q(z)} = D + C(zI - A)^{-1}B,$$

where the m-by-m Schur matrix A and an m-vector B are defined by q, and D, C depend linearly on p. Then, using the identity $z_i^n = 1/z_i$, we have

$$\begin{aligned} f^{[n]}(z_i) &= D + \frac{1}{z_i} C \frac{I - (A/z_i)^n}{I - A/z_i} B \\ &= D + C \frac{I - z_i A^n}{z_i I - A} B = D - CA^n B \\ &= D - CA^n B + C(I - A^{n+1})(z_i I - A)^{-1} B. \end{aligned}$$

Hence $\hat{f}^{[n]}$ can be defined by

$$\hat{f}^{[n]}(z) = D - CA^n B + C(I - A^{n+1})(zI - A)^{-1} B,$$

which proves the theorem. ∎

The rest of the proof is based on a classical spectral factorization theorem. Let $\mathcal{X}_a^m$ be the set of all positive even trigonometric polynomials a of degree m such that $a(0) = 1$. Let $\mathcal{X}_q^m$ be the set of all real Schur polynomials $q \in \mathbf{R}[z]$ of degree m satisfying $q(1) = 1$.

Theorem 3 *There exists a one-to-one correspondence $\theta_m : \mathcal{X}_a^m \mapsto \mathcal{X}_q^m$ such that*

$$a(t) = |q(\exp(jt))|^2 \quad \text{whenever} \quad q = \theta_m(a). \tag{9}$$

In other words, positive even trigonometric polynomials of degree m are squares of absolute values of Schur polynomials on the unit circle.

Proof. For a given Schur polynomial q, (9) clearly defines a positive trigonometric polynomial a of same degree. Conversely, if

$$a(t) = \sum_{k=0}^{m} a_k \cos(mt)$$

is a positive trigonometric polynomial of degree m then

$$\phi(z) = \frac{1}{2} z^m \sum_{k=0}^{m} a_k (z^m + z^{-m})$$

is a polynomial of degree $2m$ with no roots on the unit circle. Since

$$z^{2m}\phi(1/z) = \phi(z),$$

the roots of ϕ are "symmetric" with respect to the unit circle (i.e. if s is a root then $1/s$ is a root as well). Hence $\phi(z) = q(z)q(1/z)$ for some Schur polynomial of degree m. Since $q(1)^2 = \phi(1) = a(0) = 1$, one can set $q(1) = 1$. ■

To complete the proof of Theorem 1, note that for $z = e^{jt}$, $t \in \mathbf{R}$, we have

$$\begin{aligned}
\frac{p(z)}{q(z)} + \frac{r(z)}{z^m q(1/z)} &= \frac{p(z)}{q(z)} + \frac{r(\bar{z})}{q(\bar{z})} \\
&= \frac{(p+r)\bar{q} + (\bar{p}+\bar{r})q}{2q\bar{q}} + \frac{(p-r)\bar{q} - (\bar{p}-\bar{r})q}{2q\bar{q}} \\
&= \frac{b(z) + jc(z)}{a(z)},
\end{aligned}$$

where

$$\begin{aligned}
a(t) &= q(z)q(\bar{z}), \\
b(t) &= \mathrm{Re}(p(z) + r(z))\bar{q}(z), \\
c(t) &= \mathrm{Im}(p(z) - r(z))\bar{q}(z).
\end{aligned}$$

This yields a mapping of (p, q, r) into (a, b, c). To show that the mapping is invertible, note that, for a given q, $p_1 = p + r$ is uniquely defined by the identity

$$2b(t) = p_1(z)q(1/z) + p_1(1/z)q(z),$$

and, for given q and $p + r$, $p_2 = p - r = p + r - 2r$ is uniquely defined by the identity

$$2c(t) = p_2(z)q(1/z) - p_2(1/z)q(z).$$