Problems from the Workshops on
Low Eigenvalues of Laplace and Schrödinger Operators
at AIM 2006 (Palo Alto) and MFO 2009 (Oberwolfach)

Version 6, April 2009

Following are brief statements of some problems raised during the AIM Workshop on *Low Eigenvalues of Laplace and Schrödinger Operators*, May 22–26, 2006, and the MFO Oberwolfach, February 9–13, 2009. The name of the participant who mentioned the problem is stated in most cases, along with a brief reference to more information. This participant is not necessarily the original proposer of the problem in the literature, of course.

The problem statements given below include some editorial additions by the organizers, which may not reflect the views of the person who mentioned the problem.

Further open problems can be found in some of the Participant Contributions. [1]

1. **Pólya and Related Inequalities**

Consider eigenvalues of the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} -\Delta u_j = E_j u_j & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume $n \geq 2$.

(1) (Michael Loss, Timo Weidl) The Pólya Conjecture claims that the Weyl asymptotic formula provides a lower bound:

$$E_j \geq (2\pi)^2 (n/|S^{n-1}||\Omega|)^{2/n} j^{2/n}, \qquad j = 1, 2, 3, \ldots.$$

The conjecture remains open even for $j = 3$.

The best partial result known is with a factor of $n/(n+2)$ (which is less than 1) on the right hand side, as one deduces by estimating $E_j \leq E_J$ in the following inequality due to Li and Yau,

$$\sum_{j=1}^{J} E_j \geq \frac{n}{n+2} (2\pi)^2 (n/|S^{n-1}||\Omega|)^{2/n} J^{(n+2)/n}, \qquad J = 1, 2, 3, \ldots.$$

Berezin proved in 1972 that

$$\sum_j (E - E_j)_+^\sigma \leq \frac{|\Omega|}{(2\pi)^n} \int_{\mathbb{R}^n} (E - |p|^2)_+^\sigma \, dp, \qquad \sigma \geq 1, \quad E > 0.$$

The cases $0 \leq \sigma < 1$ remain open. The Pólya conjecture is exactly the case $\sigma = 0$. The inequality for $\sigma = 1$ implies the Li–Yau inequality via the Legendre transform.

[1] Corrections and updates will be gratefully received at `rlfrank@math.princeton.edu` (for Pólya and Lieb–Thirring inequalities) and `Laugesen@illinois.edu` (for Gap and Laplace eigenvalue inequalities).

(2) (Timo Weidl) Can one strengthen the Li–Yau result by including a correction term, perhaps involving the surface area of the boundary? This has been done for the discrete Laplacian on domains in a lattice; J. K. Freericks, E. H. Lieb, D. Ueltschi, *Phase separation due to quantum mechanical correlations*, Phys. Rev. Lett. 88, 106401 1-4 (2002).

There is a result by Melas of Li–Yau type with corrections involving *moments of inertia* rather than surface area, see A. Melas, *A lower bound for sums of eigenvalues of the Laplacian*, Proc. Amer. Math. Soc. 131 (2003), 631–636.

February 2009: The result of Melas has been strengthened by inclusion of a correction term involving the surface area of the boundary; see H. Kovarik, S. Vugalter, T. Weidl, *Two-dimensional Berezin–Li–Yau inequalities with a correction term*, Comm. Math. Phys. 287 (2009), 959–981. An earlier improvement for $\gamma \geq 3/2$, involving a notion of effective boundary, is due to T. Weidl, *Improved Berezin–Li–Yau inequalities with a remainder term*, in: Spectral theory of differential operators, T. Suslina and D. Yafaev (eds.), Amer. Math. Soc. Transl. Ser. 2, 225 (2008). For $\gamma < 3/2$ further improvements seem possible and desirable.

(3) (Timo Weidl) There are analogues of the Pólya and Li–Yau inequalities under Neumann boundary conditions, with the inequality signs reversed. The Pólya Conjecture remains open for Neumann boundary conditions for $j \geq 2$, except it was recently proved for $j = 2$ in two dimensions by A. Girouard *et al.*, J. Diff. Geometry, to appear. The analogue of Li–Yau was proved by Pawel Kröger; see P. Kröger, *Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space*, J. Funct. Anal. 106 (1992), no. 2, 353–357, and also A. Laptev *Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces*, J. Funct. Anal. 151 (1997), 531–545.

Can one strengthen the Kröger result by including a correction term?

(4) (Timo Weidl) The questions raised above are meaningful in the presence of a magnetic field. For more information and some progress see Item (10) below.

(5) (Evans Harrell, Joachim Stubbe) For $\sigma \geq 2$ the mapping

$$r_\sigma : E \mapsto E^{-\sigma-d/2}\left(\frac{|\Omega|}{(2\pi)^n}\int_{\mathbb{R}^n}(E-|p|^2)^\sigma dp - \sum_j (E-E_j)_+^\sigma\right)$$

is non-increasing. This was proved in E. M. Harrell and L. Hermi, *Differential inequalities for Riesz means and Weyl-type bounds for eigenvalues*, J. Funct. Analysis **254** (2008), 3173–3191, using the trace identities of E. M. Harrell and J. Stubbe, *On trace identities and universal eigenvalue estimates for some partial differential operators*, Trans. Amer. Math. Soc. **349** (1997), 1797–1809, and *Universal bounds and semiclassical estimates for eigenvalues of abstract Schrödinger operators*, preprint 2008, available as arXiv:0808.1133.

According to Weyl's asymptotic formula $r_\sigma(E)$ tends to zero as E tends to infinity and therefore $r_\sigma(E) \geq 0$, which is the Berezin–Li–Yau-inequality. Can one strengthen this bound in the trace identity of Harrell–Stubbe to obtain correction terms involving the surface area of the boundary? For the Laplacian with periodic boundary conditions a similar monotonicity property holds (for details see E. M. Harrell and J. Stubbe, *Trace identities for commutators, with applications to the distribution of eigenvalues*, preprint 2009, available as arXiv:0903.0563). In this case the search for the correction term is related to the famous Gauss circle problem (or lattice point problem).

(6) (Evans Harrell, Joachim Stubbe) Prove monotonicity results like in Item (5) for higher order operators (e.g. clamped plate problem) and fractional powers of Laplacians (for some results on $\sqrt{-\Delta}$ see E. M. Harrell and S. Yıldırım Yolcu, *Eigenvalue inequalities for Klein-Gordon Operators*, accepted for publication in J. Funct. Analysis) leading to Berezin-Li-Yau inequalities for these operators.

(7) (Evans Harrell, Joachim Stubbe) For $p > 0$ let

$$M_p(J) := \left(\frac{n+2p}{n} \frac{1}{J} \sum_{j=1}^{J} E_j^p \right)^{\frac{1}{p}} \tag{1}$$

and for $p = 0$ define

$$M_0(J) := e^{\frac{2}{n}} \left(\prod_{j=1}^{J} E_j \right)^{\frac{1}{J}}. \tag{2}$$

According the the Weyl asymptotic formula, for all $p \geq 0$,

$$M_p(J) \sim (2\pi)^2 (n/|S^{n-1}||\Omega|)^{2/n} J^{2/n}$$

as $J \to \infty$. In E. M. Harrell and J. Stubbe, *On trace identities and universal eigenvalue estimates for some partial differential operators*, Trans. Amer. Math. Soc. **349** (1997), 1797–1809, it has been shown that

$$M_1^2(J) - M_2(J) \geq \frac{1}{4}(E_{J+1} - E_J)^2 (\geq 0)$$

and

$$M_1(J) - \sqrt{M_1^2(J) - M_2(J)} \leq E_J \leq E_{J+1} \leq M_1(J) + \sqrt{M_1^2(J) - M_2(J)}.$$

Both inequalities are sharp in the Weyl limit. For extensions to other $M_p(J)$ see E. M. Harrell and J. Stubbe, *Universal bounds and semiclassical estimates for eigenvalues of abstract Schrödinger operators*, preprint 2008, available as arXiv:0808.1133. For $p > 0$ find an upper bound of the form

$$M_p^{2p}(J) - M_{2p}^p(J) \leq C(p, \Omega) E_1^{2p} J^{2p\kappa}$$

with $\kappa < 2/n$.

(8) (Evans Harrell, Joachim Stubbe) With the above notations does

$$E_J \le M_p(J)$$

hold for all J and all $p \ge 0$? Can one find Ω and J such that the inequality

$$M_1^2(J) - M_2(J) \ge \frac{1}{4}(E_{J+1} - E_J)^2$$

is saturated ?

2. **Lieb–Thirring Inequalities**

Write $E_1 < E_2 \le E_3 \le \cdots \le 0$ for the eigenvalues of $-\Delta - V$ on $L^2(\mathbb{R}^n)$, meaning

$$(-\Delta - V)u_j = E_j u_j.$$

The eigenfunctions u_j represent bound states with energies E_j. For simplicity we assume $\boxed{V \ge 0}$. Assume $n \ge 1$.

The Lieb–Thirring inequality can be written as

$$\sum_j |E_j|^\gamma \le L_{n,\gamma} \int_{\mathbb{R}^n} V^{\gamma+n/2}\, dx,$$

This inequality holds (with a constant $L_{n,\gamma}$ independent of V) iff the parameter γ satisfies $\gamma \ge 1/2$ if $n = 1$, $\gamma > 0$ if $n = 2$ and $\gamma \ge 0$ if $n \ge 3$. The case $\gamma = 0$ (counting eigenvalues) is the Cwikel–Lieb–Rozenblum Inequality (CLR).

In other words

$$\mathrm{Tr}\,(-\Delta - V)_-^\gamma \le \frac{C_{n,\gamma}}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (|p|^2 - V(x))_-^\gamma \, dp dx,$$

where

$$C_{n,\gamma} = \frac{L_{n,\gamma}}{L_{n,\gamma}^{\mathrm{cl}}} \qquad \text{and} \quad L_{n,\gamma}^{\mathrm{cl}} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (|p|^2 - 1)_-^\gamma \, dp.$$

The constant $L_{n,\gamma}^{\mathrm{cl}}$ is called the semiclassical Lieb–Thirring constant.

Note that $C_{n,\gamma} \ge 1$ always, by the Weyl asymptotics, and that $C_{n,\gamma}$ is decreasing in γ for each fixed n, by the Aizenman–Lieb monotonicity result.

To start with, let us summarize some known results on the constants $C_{n,\gamma}$, along with conjectures about best (smallest) values of $C_{n,\gamma}$.

n	γ	Best known $C_{n,\gamma}$	Best constant?	status	last updated
1	$\frac{1}{2}$	2	2	known	
	$(\frac{1}{2},\frac{3}{2})$	2*	$2\left(\frac{\gamma-1/2}{\gamma+1/2}\right)^{\gamma-1/2}$	conjectured	
	$[\frac{3}{2},\infty)$	1	1	known	
2	$(0,\frac{1}{2})$	?			
	$[\frac{1}{2},1)$	3.64			Feb. 2009
	$[1,\frac{3}{2})$	1.82			Feb. 2009
	$[\frac{3}{2},\infty)$	1	1	known	
$=3$	$[0,\frac{1}{2})$	6.87	$8/\sqrt{3}\simeq 4.62$	conjectured	
	$[\frac{1}{2},1)$	3.64			Feb. 2009
	$[1,\frac{3}{2})$	1.82	1	conjectured	Feb. 2009
	$[\frac{3}{2},\infty)$	1	1	known	
≥ 4	$[0,\frac{1}{2})$	10.34			Feb. 2009
	$[\frac{1}{2},1)$	3.64			Feb. 2009
	$[1,\frac{3}{2})$	1.82	1	conjectured	Feb. 2009
	$[\frac{3}{2},\infty)$	1	1	known	

*better is known for $\gamma\in[1,\frac{3}{2})$, *e.g.* $C_{1,1}\leq\frac{\pi}{\sqrt{3}}\simeq 1.82$ via work of Eden–Foias.

Remark. References to the results in the table and to many of the questions below can be found in the lecture notes by Michael Loss and Timo Weidl, and in the survey paper by Dirk Hundertmark (which further states some better estimates on $C_{n,\gamma}$ for special values of n and γ).

For February 2009 updates see J. Dolbeault, A. Laptev, M. Loss, *Lieb–Thirring inequalities with improved constants*, J. Eur. Math. Soc. 10 (2008), and R. L. Frank, E. H. Lieb, R. Seiringer, *Number of bound states of Schrödinger operators with matrix-valued potentials*, Lett. Math. Phys. 82, 107 (2007).

Now we state open problems on Lieb–Thirring inequalities.

(1) (Richard Laugesen) Must an optimal potential V exist, for those Lieb–Thirring inequalities in which the best constant is not known? In particular this question is open for $n=1$ and $\frac{1}{2}<\gamma<\frac{3}{2}$.

A restricted version of the problem asks: within the class of potentials having m bound states (where $m\geq 1$ is given), does an optimal potential exist?

(2) (Richard Laugesen) If an optimal potential exists, then does it have just a single bound state? (In other words, does $-\Delta - V$ have just a single eigenvalue?) When $n=1$ and $\frac{1}{2}<\gamma<\frac{3}{2}$, the natural conjecture is that the optimal potential is the one found by J. B. Keller when he determined the best constant in $|E_1|^\gamma \leq C\int_{\mathbb{R}} V^{\gamma+1/2}\,dx$ (see J. Mathematical Phys. 2:262–266, 1961).

This "single bound state" conjecture is due to Lieb and Thirring, 1976. In dimension $n = 1$, the conjecture is known to be true in the endpoint cases $\gamma = 1/2$ (in which case V is a delta function) and $\gamma = 3/2$ (in which case V is a transparent or reflectionless potential).

(3) (Eric Carlen) Does there exist a bound of the form $\sum_j |E_j|^\gamma \leq C|E_1|^\gamma$? Here the factor C could depend on n, γ, and on the integrability of a power of V sufficient to guarantee that the lefthand side is finite.

(4) (Rafael Benguria) The use of Korteweg–de Vries (KdV) integrable system methods when $n = 1, \gamma = 3/2$, suggests that one might similarly study Lieb–Thirring inequalities for the linear equation associated with the Benjamin–Ono equation (another integrable system). Tomas Ekholm, Rupert Frank and Dirk Hundertmark made progress during the Workshop already, by obtaining the analog of the Aizenman–Lieb "monotonicity toward best constants" result. The Lax pair for the Benjamin–Ono equation can be found for example in R.L. Anderson and E. Tafflin, *The Benjamin-Ono equation -Recursivity of linearization maps- Lax pairs*, Letters in Mathematical Physics, 9 (1985), 299–311. See also, D.J. Kaup and Y. Matsuno, *The inverse scattering transform for the Benjamin–Ono equation*, Studies in applied mathematics 101 (1998), 73–98.

(5) (Rupert Frank) The best constant when $n = 1, \gamma = 1$, is due to Eden–Foias (see A. Eden and C. Foias, *A simple proof of the generalized Lieb–Thirring inequalities of one–space dimension*, Journal of mathematical analysis and applications, 162 (1991), 250–254.) More precisely, they proved a Sobolev inequality, which then gives a Lieb–Thirring inequality via the Legendre transform. So a question is: can one find a more direct proof of this Lieb–Thirring inequality?

Also, can one sharpen the Eden–Foias bound by including correction terms in their argument?

February 2009: An operator-valued version of the Eden–Foias bound has been proved by J. Dolbeault, A. Laptev, M. Loss, *Lieb-Thirring inequalities with improved constants*, J. Eur. Math. Soc. 10 (2008). By the 'lifting of dimension'-argument this result leads to the best known values for the constants in the Lieb–Thirring inequalities for $\gamma \geq 1$ if $n = 1$ and for $\gamma \geq 1/2$ if $n \geq 2$.

(6) (Timo Weidl) Can one find a way to directly estimate the sum of the eigenvalues, without going through the Birman–Schwinger transformation (which *counts* the eigenvalues rather than summing them)?

(7) (Almut Burchard) *The Ovals Problem.* Consider a smooth closed curve γ of length 2π in $\mathbb{R}^3$, and let $\kappa(s)$ be its curvature as a function of arclength. The curve determines the one-dimensional Schrödinger operator $H_C = -d^2/ds^2 + \kappa^2$ acting on 2π-periodic functions.

This operator appears in the equation for the tension of a smooth, elastic, inextensible loop [5], and in connection with a Lieb–Thirring inequality in one dimension [4]; similar Schrödinger operators with quadratic curvature potentials have been studied in connection with quantum mechanics on narrow channels [2], Dirac operators on the sphere [3], and curvature-driven flows describing the motion of interfaces in reaction-diffusion equations [1].

A natural conjecture is that the principal eigenvalue $e(\gamma)$ is minimal when γ is a circle, where it takes the value 1. This question is open even for planar loops that enclose convex sets (*'ovals'*). It is known that the value $e(\gamma) = 1$ is attained for an entire family of planar curves whose curvature is given by $\kappa(s) = \left(\alpha^2 \cos^2 s + \alpha^{-2} \sin^2 s\right)^{-1}$. When $\alpha \to 0$, these curves collapse onto two straight line segments of length π joined at the ends. The inequality $e(\gamma) \geq 1$ has recently been shown for curves in some neighborhood of the family [5], and for curves satisfying additional geometric constraints [6]. The best universal lower bound on $e(\gamma)$ that is currently known is $.6085$ [6].

Several participants at the Workshop had worked on this problem previously (including Benguria, Loss, Burchard, Thomas, and Linde). All agreed that classical Calculus of Variations techniques may be exhausted at this point, and that rearrangement techniques seem to fail. Linde and Burchard claimed that minimizers can be shown to exist, and should be convex, but could conceivably contain one corner, or two corners joined by a straight line segment. Benguria pointed to the family of putative minimizers (which look like ellipses in polar coordinates) as evidence that the problem may have a hidden affine symmetry. Carlen, Mazzeo, and Benguria proposed to search for geometric flows that drive $e(\gamma)$ towards its minimum. The affine curvature flow [7] was mentioned as a promising candidate. Rapti and Lee proposed to analyze the Euler–Lagrange equation using ODE methods. Laugesen suggested applying the Birman–Schwinger transformation, after which the conjecture becomes that the largest eigenvalue of the operator $T = \kappa(d^2/ds^2 + \gamma)^{-1}\kappa$ is larger than 1, for each constant $0 < \gamma < 1$. Equivalently, take $\gamma = 1$ and try to show the largest eigenvalue of T is larger than 1, when T acts on functions ψ with $\kappa\psi$ orthogonal to $\sin s$ and $\cos s$. The hope is that a good choice of trial function (in the variational principle for the largest eigenvalue) might suffice to prove this conjecture.

References for the ovals problem

[1] E. M. Harrell and M. Loss. *On the Laplace operator penalized by mean curvature.* Commun. Math. Phys. 195:643-650 (1998).

[2] P. Exner, E. M. Harrell and M. Loss. *Optimal eigenvalues for some Laplacians and Schrödinger operators depending on curvature.* Oper. Theory Adv. Appl. 108:47-58 (1999).

[3] T. Friedrich. *A geometric estimate for a periodic Schrödinger operator.* Colloq. Math. 83:209-216 (2000).

[4] R. D. Benguria and M. Loss. *Connection between the Lieb–Thirring conjecture for Schrödinger operators and an isoperimetric problem for ovals on the plane.* Contemporary Math. 362:53-61 (2004).

[5] A. Burchard and L. E. Thomas. *On an isoperimetric inequality for a Schrödinger operator depending on the curvature of a loop.* J. Geometric Analysis 15:543-563 (2005).

[6] H. Linde. *A lower bound for the ground state energy of a Schrödinger operator on a loop.*, Proc. Amer. Math. Soc. 134 (2006), 3629–3635.

[7] B. Andrews. *The affine curve-lengthening flow.* Crelle J. Reine Angew. Math. 506:43-83 (1999).

(8) (Timo Weidl) For $n = 2, \gamma = 0$, can one prove a Cwikel–Lieb–Rozenblum Inequality that involves a logarithmic correction factor? Without some such correction factor, the inequality fails, since any nontrivial attractive potential has at least one bound state.

February 2009: This problem has been solved in H. Kovarik, S. Vugalter, T. Weidl, *Spectral estimates for two-dimensional Schrödinger operators with application to quantum layers*. Comm. Math. Phys. 275 (2007), no. 3, 827–838.

(9) (Timo Weidl) Can one obtain improved Lieb–Thirring constants when working on a domain Ω rather than on all of $\mathbb{R}^n$? For example, can one obtain a boundary correction term?

(10) (Timo Weidl) *Magnetic Schrödinger operators on a domain.* Consider the Dirichlet Laplacian in a domain in $\mathbb{R}^n$. The technique of iteration-in-dimension gives sharp Lieb–Thirring constants for arbitrary magnetic fields for $\gamma \geq 3/2$ and any $n \geq 2$. (See the final part of A. Laptev and T. Weidl, *Sharp Lieb–Thirring inequalities in high dimensions*, Acta Mathematica 184 (2000), 87-111.) For $1/2 \leq \gamma < 3/2$ one also gets estimates uniform in the magnetic field, but the constant is (probably) not sharp. With the same approach, the results of D. Hundertmark, A. Laptev and T. Weidl (*New bounds on the Lieb–Thirring constants*, Inventiones Math. 140 (2000), 693-704) carry over to magnetic operators; see the remark at the end of that paper.

The sharp Li–Yau bound (corresponding to $\gamma = 1$) has been proved by L. Erdös, M. Loss and V. Vougalter (*Diamagnetic behavior of sums of Dirichlet eigenvalues*, Ann. Inst. Fourier (Grenoble) 50 (2000), 891–907) for constant magnetic fields. Does this bound hold true for *arbitrary* magnetic fields for $1 \leq \gamma < 3/2$?

For $\gamma = 0$, does the Pólya conjecture hold true for tiling domains in the presence of magnetic fields?

February 2009: The answer to the latter question is negative for constant magnetic fields. Indeed, the sharp constant in the corresponding lower bound for $0 \leq \gamma < 1$ was found in R. L. Frank, M. Loss, T. Weidl, *Pólya's conjecture in the presence of a constant magnetic field*. J. Eur. Math. Soc., to appear.

(11) (Timo Weidl) *Magnetic Schrödinger operators on* $\mathbb{R}^n$. Consider Lieb–Thirring bounds for magnetic Schrödinger operators on all of $\mathbb{R}^n$. In all cases where the sharp constant is known, either the magnetic field is not relevant (dimension $n = 1$) or the value of the constant is independent of the magnetic field ($\gamma \geq 3/2$ and $n \geq 2$ as above, where the sharp constant equals the classical constant).

Can the magnetic field change the optimal value of the Lieb–Thirring constant in the remaining cases? (February 2009: The magnetic field can change the optimal value at most by an explicit factor depending only on γ and d; see R. L. Frank *A simple proof of Hardy-Lieb-Thirring inequalities*. Comm. Math. Phys., to appear.)

This question is rather speculative, because we do not know the sharp constants even in the non-magnetic case. But let us put forward the following more specific version:

Can one construct a counterexample to the Lieb–Thirring conjecture that the optimal constant is the classical one for $n = 3, \gamma = 1$, by using a suitable magnetic field?

(12) (Eric Carlen) *Generalization to manifolds.* Do there exist Lieb–Thirring inequalities on manifolds? As a basic first question, do the critical exponents ($\gamma = \frac{1}{2}$ when $n = 1$, and $\gamma = 0$ when $n = 2$) depend on the geometry?

Some references to get started here are A. A. Ilyin, *Lieb–Thirring inequalities on the N-sphere and in the plane, and some applications*, Proc. London Math. Soc. (3) 67 (1993), 159–182; and *Lieb–Thirring integral inequalities and their applications to attractors of Navier-Stokes equations*, Sb. Math. 196 (2005), 29–61. A classic reference for applications to turbulence is E. Lieb, *On characteristic exponents in turbulence*, Comm. Math. Phys. 92 (1984), 473–480.

February 2009: Intuition from recent results on continuous trees suggest that the critical exponents depend on both the local and global dimension of the manifold (see T. Ekholm, R. L. Frank, H. Kovarik, *Eigenvalue estimates for Schrödinger operators on metric trees*, `arXiv:0710.5500v1`.)

Analogues of Lieb-Thirring inequalities on tori and spheres have been proved in E. Harrell and J. Stubbe, *Trace identities for commutators, with applications to the distribution of eigenvalues*, `arXiv:0903.0563v1`.

(13) (Mark Ashbaugh) *Reverse Lieb–Thirring Inequality.* For dimension $n = 1$, Damanik and Remling have proved a Reverse Lieb–Thirring Inequality in the subcritical range $0 < \gamma \leq \frac{1}{2}$ (*Schrödinger operators with many bound states*, Duke Math. J. 136 (2007), 51–80) Sharp constants seem not to be known. A Reverse Cwikel–Lieb–Rozenblum Inequality for the eigenvalue counting function for dimension $n = 2$ in the critical case $\gamma = 0$ has been proved by A. Grigor'yan, Yu. Netrusov, S.-T. Yau, *Eigenvalues of elliptic operators and geometric applications*, Surveys in Differential Geometry IX (2004), 147-218.

(14) (Rupert Frank) *Powers of the Laplacian.* Can one prove a critical Lieb–Thirring inequality for arbitrary powers of the Laplacian? That is, one wants

$$\mathrm{tr}\ ((-\Delta)^s - V)_-^\gamma \leq L_{\gamma,n} \int_{\mathbb{R}^n} V_+^{\gamma+n/2s}\, dx$$

for $\gamma = 1 - n/2s > 0$. Such an inequality is known for s a positive integer by work of Netrusov–Weidl.

Timo Weidl remarked that regardless of whether these operators have physical significance, the higher order situation can help shed light on what makes the second-order case work.

(15) (Rupert Frank) *Hardy–Lieb–Thirring Inequality.* Can one prove a Lieb–Thirring bound with a Hardy weight, on the half-line? That is, one wants

$$\mathrm{tr}\left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} - V\right)_-^{\theta/2} \leq C_\theta \int_0^\infty V(r) r^{1-\theta}\, dr$$

for $0 < \theta \leq 1$. The inequality is known for $\theta = 1$ (Lieb–Thirring). For $\theta = 0$ it fails (although note that if it were true, it would resemble Bargmann's inequality).

February 2009: The inequality for all $0 < \theta \leq 1$ has been proved in T. Ekholm, R. L. Frank, *Lieb-Thirring inequalities on the half-line with critical exponent*. J. Eur. Math. Soc. 10 (2008), no. 3, 739 - 755. The sharp constant C_θ is not known, and there is not even a conjecture for it.

(16) (Carlo Morpurgo) *Cwikel–Lieb–Rozenblum bounds and heat kernel inequalities.*

Let Y be the Yamabe operator, or conformal Laplacian, on the euclidean "round" sphere (S^n, g). That is, $Y = \Delta_{S^n} + \frac{n}{2}\left(\frac{n}{2} - 1\right)$, where Δ_{S^n} denotes the Laplace–Beltrami operator on S^n.

Consider a positive smooth function W on S^n, normalized so that $\int_{S^n} W^{n/2}$ =volume of the round sphere. Define $Y_W = W^{-1/2} Y W^{-1/2}$, acting on $L^2(S^n, g)$.

CONJECTURE 1. For $n \geq 3$,

$$\max_{t>0} \left\{ t^{n/2} \mathrm{Tr}[e^{-tY_W}] \right\} \leq \max_{t>0} \left\{ t^{n/2} \mathrm{Tr}[e^{-tY}] \right\}. \tag{3}$$

(Note that the eigenvalues of Y_W are the same as the eigenvalues of $W^{-(n+2)/4} Y W^{(n-2)/4}$ acting on $L^2(S^n, Wg)$, which is the natural Yamabe operator in the metric Wg.)

In other words we are looking for the best constant $C(W)$ in the inequality

$$\mathrm{Tr}[e^{-tY_W}] \leq \frac{C(W)}{t^{n/2}}, \qquad t > 0, \tag{4}$$

and the conjecture states that this constant is attained precisely by the right side of (3), which is the best constant in (4) for $W \equiv 1$.

If Conjecture 1 is true then we can considerably improve the known CLR bounds, at least in low dimensions, noting that for a given positive potential V, the eigenvalues of the Birman–Schwinger operator $V^{-1/2}\Delta V^{-1/2}$ are the same as those of Y_W, with $W = (V \circ \pi)|J_\pi|^{2/n}$, π being the stereographic projection and J_π its Jacobian.

CONJECTURE 2. If $n \geq 4$ then the function $f_W(t) = t^{n/2}\mathrm{Tr}[e^{-tY_W}]$ is decreasing in t.

An asymptotic expansion $f_W(t) \sim a_0(W) + ta_1(W) + \ldots$ holds as $t \to 0$, with $a_0(W) = (4\pi)^{-n/2}\int_{S^n} W^{n/2}$ and with $a_1(W)$ written explicitly in terms of the total curvature. Hence Conjecture 2 would imply (equality in) Conjecture 1 for $n \geq 4$, because Conjecture 1 normalizes the constant term $a_0(W)$ in the expansion.

It is known that $a_1(W)$ is negative for $n \geq 5$, zero for $n = 4$, and positive for $n = 3$, so that Conjecture 2 fails for small t when $n = 3$.

On the other hand, Conjecture 2 holds for large t and any $n \geq 3$, since the known sharp lower bound $\lambda_0(W) \geq \lambda_0(1) = \frac{n}{2}\left(\frac{n}{2} - 1\right)$ for the lowest eigenvalue of Y_W implies that $f_W(t)$ is decreasing when $t > \left(\frac{n}{2} - 1\right)^{-1}$.

Conjecture 2 is true if $W \equiv 1, n \geq 4$.

3. **Gap Inequalities**

Consider eigenvalues of the Dirichlet Laplacian on a bounded convex domain $\Omega \subset \mathbb{R}^n$ with *convex* potential V:

$$\begin{cases} (-\Delta + V)u_j = \lambda_j u_j & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume $n \geq 1$. Notice the operator is written with $+V$, not $-V$ like in the previous section.

Van den Berg's Gap Conjecture is that

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{d^2}, \qquad d = \mathrm{diam}(\Omega),$$

with equality holding when $n = 1, V \equiv 0$. (In dimensions $n \geq 2$, the inequality should be strict, with equality holding only in the limit as the domain degenerates to an interval.) For an extended treatment of the problem and many references, see Mark Ashbaugh's introduction *The Fundamental Gap* on the AIM website. Also see the overhead transparencies of Rodrigo Bañuelos's talk.

In dimension $n = 1$ the conjecture has been completely proved by Richard Lavine (1994).

In dimensions $n \geq 2$, the best partial result says that $\lambda_2 - \lambda_1 \geq \pi^2/d^2$, which is missing the desired factor of 3 on the righthand side. The first proof of this result used P-function techniques based on the maximum principle. The second proof adapted the methods of Weinberger, who resolved the analogous Neumann gap problem long ago.

Now we state open problems, beginning with one dimension and then considering higher dimensional problems.

(1) (Richard Lavine) Can one expand the class of potentials for which the gap inequality holds, in one dimension? It is known for convex potentials, but also for single well potentials with a centered transition point. See the write-up by Mark Ashbaugh.

(2) (Richard Lavine) Normalize the eigenfunctions u_j in L^2 and define $\langle V \rangle_j = \int_\Omega V u_j^2 \, dx$. Are these means $\langle V \rangle_j$ an increasing sequence as j increases? The question is already interesting in one dimension.

(3) (Richard Lavine) Can one strengthen the gap inequality by adding to its righthand side a term that involves V? The question is already interesting in one dimension.

(4) (Rodrigo Bañuelos) Can Lavine's approach be extended to higher dimensions?

(5) (Mark Ashbaugh) In dimensions $n \geq 2$, one should try to understand whether genuine barriers exist to pushing the P-function techniques beyond the known π^2/d^2 bound. One seems to need to improve the log-concavity bound on the groundstate u_1 (due to Brascamp–Lieb). That is, instead of just discarding the Hessian of $\log u_1$ when it arises, on the grounds that it is ≤ 0, one seems to want to bound the Hessian strictly away from 0. Can this be achieved by the methods of Brascamp–Lieb, or of Korevaar?

(6) (Antoine Henrot) The Gap Conjecture is already very interesting in the case of vanishing potential $V \equiv 0$. A possible approach is as follows.
   (a) Prove the gap infimum $\inf_{\Omega \in \mathcal{O}} (\lambda_2 - \lambda_1)$ is not attained, when $\mathcal{O}$ is the class of convex domains with diameter 1.
   (b) Prove that minimizing sequences shrink to a segment of length 1.
   (c) Prove that the gap for a sequence of shrinking domains behaves like the gap of a one-dimensional Schrödinger operator with convex potential (semiclassical limit arguments).
   (d) Complete the proof using the results in the one dimensional case (Lavine's Theorem).

   It seems that points (b), (c) and (d) are OK. It remains to prove point (a)!

(7) (Helmut Linde) *Operator-valued potentials.* In order to prove the gap conjecture one could consider the Laplacian on a two-dimensional domain as being a one-dimensional operator with a matrix-valued potential. This makes it possible to approach the problem via a sequence of simplified "toy models". For example, one can try to prove the gap conjecture first for very special classes of matrix-valued potentials, like potentials that have constant eigenvectors and whose eigenvalues are convex functions. Then one could gradually generalize this theorem to approach the "real" gap conjecture.

(8) (Timo Weidl and Richard Laugesen) *Magnetic Schrödinger operators.* For magnetic Schrödinger operators, the Gap Conjecture cannot hold as stated because the eigenvalue gap can be reduced to zero by the introduction of a magnetic field.

   Can one still obtain a valid gap inequality by subtracting from the righthand side a term depending on the magnetic potential A?

(9) (Rodrigo Bañuelos) *Powers of the Laplacian.* Is the groundstate of $\sqrt{-\Delta}$ log-concave? See also the comments above on log-concavity of the groundstate of $-\Delta$.

(10) (Rodrigo Bañuelos) *Properties of the eigenfunction ratio.* The *Hot Spots* conjecture of Bernhard Kawohl says that the first nontrivial eigenfunction of the Neumann Laplacian attains its maximum and mimimum values on the boundary of the convex domain Ω. This has been proved only for some special classes of domains. The analogous conjecture for the Dirichlet Laplacian would be that the ratio u_2/u_1 attains its maximum and mimimum values on the boundary of Ω. Note u_2/u_1 satisfies Neumann boundary conditions (by explicit calculation, assuming the boundary is smooth) and satisfies a certain elliptic equation.

(11) (Robert Smits) *Robin boundary conditions.* Turn now from the Dirichlet boundary condition to the Robin condition $\partial u/\partial\nu = -\alpha u$ (for some given constant $\alpha > 0$, with ν denoting the outward normal). Is the gap $\lambda_2 - \lambda_1$ minimal when $V = 0$ and Ω degenerates to a segment having the same diameter as Ω?

In one dimension, is the gap minimal when $V = 0$ and Ω is a segment? Can Lavine's methods be adapted to Robin boundary conditions, in one dimension?

If one could prove the groundstate u_1 is log-concave, then existing methods could be adapted to imply $\lambda_2 - \lambda_1 \geq \pi^2/d^2$, like is already known for the Neumann and Dirichlet situations. Incidentally, the Rayleigh quotient for the gap can be shown (like in the Dirichlet case) to equal

$$\lambda_2 - \lambda_1 = \min_{\int_\Omega f u_1^2\,dx = 0} \frac{\int_\Omega |\nabla f|^2 u_1^2\,dx}{\int_\Omega f^2 u_1^2\,dx},$$

with the potential entering implicitly through the dependence of u_1 on V.