

L^2 HARMONIC FORMS IN GEOMETRY AND PHYSICS

The American Institute of Mathematics

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A.1 Ammann, Bernd

Overview over my research interests

I am interested in the spectrum of the Dirac operator on Riemannian manifolds. On compact manifolds the spectrum is discrete. In this case, one wants to find connections between geometric properties of Riemannian manifolds such as volume, systolic lengths, curvature bounds and quantities that measure the non-triviality of the parallel transport along closed loops and the eigenvalues of the Dirac operator. Among the growing literature on this subject, I want to give some examples for problems I worked at:

- * Construction of non-isometric families of Riemannian manifold with the same spectrum of the Dirac operator (collaboration with C. Baer)
- * Manifolds with sufficiently many small eigenvalues of the Dirac operator are diffeomorphic to nilmanifolds (collaboration with Chadwick Sprouse)

An on-going project of mine is the following: Assume that M^n is a fixed spin manifold. We want to study the eigenvalues of the Dirac operator as a function on the space of volume 1 Riemannian metrics on M . In particular, it might be interesting to find and study the stationary points of these functionals.

As a first step in this program, I studied the first positive eigenvalue as a function on a fixed conformal class on M . The problem has many analogies and relations to the Yamabe problem and to the positive mass theorem of general relativity.

I proved that the first positive eigenvalue attains its infimum λ_{\min}^M among all volume-1 metrics in the fixed conformal class if

$$\lambda_{\min}^M < \lambda_{\min}^{S^n}. \quad (+)$$

Here S^n denotes the sphere with the standard conformal structure. In collaboration with E. Humbert and B. Morel, Nancy I showed that (+) holds if $n \geq 7$, and if M is not conformally flat. A similar result holds if M is conformally flat, and if the spinorial mass endomorphism does not vanish.

If $n = \dim M = 2$, the Euler-Lagrange equation of the corresponding variational problem has applications to constant mean curvature surfaces: the spinorial version of the Weierstrass representation translates these Euler-Lagrange equations into the (local) equations for conformal immersions of M into \mathbb{R}^3 with constant mean curvature. One obtains the existence of new periodic constant mean curvature surfaces.

The study of the spinorial mass endomorphism in the conformally flat case leads naturally to questions on non-compact manifolds and to General relativity. In contrast to the case of compact manifolds, where the elliptic theory is already well studied, the analytical properties of non-compact, complete Riemannian manifolds is much more challenging. As it seems very hard to establish deep analytical theorems for arbitrary non-compact, complete manifolds, it makes sense to specialize to those manifolds that have a controlled behavior at infinity.

A very effective way to derive analytical properties of certain complete manifolds is provided by Melrose's b-calculus. This calculus is well suited to describe manifolds with asymptotically cylindrical ends. Similar analytical statements could be established for the 0-calculus, the edge-calculus, the double-edge calculus, the scattering calculus and many

others. An on-going project with Victor Nistor, Penn State and others, is to unify these different asymptotic geometries into a unified theory as far as possible. We studied the class of Riemannian manifolds with a Lie structure at infinity, which contains all previously listed geometries. Several important questions have been successfully solved. In particular, we defined a vector space of pseudodifferential operators which shares many of the properties of the algebra of pseudodifferential operators on a compact manifold. It is an involved theorem to show that this vector space is closed under composition of operators, or in other words, it is an “algebra of pseudodifferential operators”. Other questions remain open, for example the question “Are there manifolds with a Lie structure at infinity with $\inf_{p \in M} \text{injrad}_p = 0$?”. However, some strong sufficient conditions for $\inf_{p \in M} \text{injrad}_p > 0$ are established in a recent publication.

A.2 Bruening, Jochen

My interest in L^2 – *cohomology* stems from its role in spectral theory and geometric analysis associated with noncompact (singular) manifolds. Thus I am looking at the two main questions in the field, like many other researchers, namely

- a) to find conditions for the finiteness of L^2 – *cohomology*,
- b) to find suitable interpretations.

Question a) I have attacked in several papers via calculation of the so-called L^2 – *index* which involves subtle (and generally non computable) quantities in the non-Fredholm case, a well known fact since the seminal paper of Atiyah, Patodi, and Singer in the cylindrical case. Recently, in joint work with Ballmann and Carron, we have tried to develop a simple model of an operator valued ode with little regularity, called “Dirac system”, which ensures finiteness of the L^2 – *cohomology* for basic functional analytic reasons. This model covers a good part of the known cases, notably manifolds of pinched negative curvature with finite volume, but it is limited by the fact that the manifolds which can be covered by it have to be “almost one dimensional” near infinity. It remains to be seen whether the ideas involved admit suitable extensions.

The workshop promises to provide new insight into the subject, based on rather exciting ideas from physics and bringing together a substantial group of experts, and I am looking very much forward to learning about new developments.

A.3 Carron, Gilles

I’ve two questions about the topological interpretation of the space of harmonic L^2 forms.

The first one is not mine it is a conjecture of E. Bueler [Trans. Am. Math. Soc. 351, No.2, 683-713 (1999)]. When (M, g) is a complete Riemannian manifold and $\omega : M \rightarrow \mathbb{R}$ is a positive function on M , we can consider the weighted L^2_ω cohomology and the associated Hodge-DeRham operator Δ_ω . E. Bueler asked the following question about complete Riemannian manifold with Ricci curvature bounded from below : if one take the weight given by the heat kernel associated to the Laplace-Beltrami operator, is it true that 0 is isolated in the spectrum of Δ_ω and that the associated weighted L^2 cohomology spaces are isomorphic to the usual De Rham cohomology groups of the manifold. A related question is to find for a large class of complete Riemannian manifolds (for instance manifold with bounded geometry), a good geometrical weight insuring that the associated weighted L^2 cohomology spaces are isomorphic to the usual De Rham cohomology groups.

The second question is about complete Riemannian manifold with non negative curvature. The geometry of this class of manifolds has been described by Cheeger and Gromoll. And it is natural to ask if on such a Riemannian manifold, the spaces of L^2 harmonic forms are finite dimensional, and if there is some topological interpretation for these spaces. According to the Bochner formula, the answer is known for 1 forms, and there is a work of Escobar and Freire [Duke Math. J. 69, No.1, 1-41 (1993)] which gives conditions (on the curvature decay) insuring that there is no L^2 harmonic form in certain degree. We can also asked these questions for manifold with a controlled on the decay of curvature similar to the one introduced by U. Abresch [Ann. Sci. Ec. Norm. Supr., IV. Sr. 18, 651-670 (1985)].

A.4 Cherkis, Sergey

There are numerous geometric implications of String theory dualities. These are particularly strong when the corresponding physical theory preserves eight supersymmetry charges. In mathematical terms it translates into hyperkählerity.

When the real dimension of a complete hyperkähler manifold is equal to four, it is referred to as a Self-dual Gravitational Instanton. I am particularly interested in Gravitational instantons, as well as the Yang-Mills instantons on them.

These three mathematical problems that I would like to mention in particular:

1. Classification of Gravitational Instantons.

a) What are the possible ends, i.e. asymptotic metrics of noncompact gravitational instantons?

b) For a given asymptotic, what are the possible topologies of hyperkähler manifolds with this asymptotic metric?

c) What are the explicit hyperkähler metrics on these manifolds.

2. Construction of Yang-Mills instantons on Gravitational instantons.

a) Kronheimer and Nakajima provided the ADHM construction for instantons on ALE spaces. An ALE space is a gravitational instanton with the asymptotic metric given by the flat metric on \mathbb{R}^4/Γ with Γ being a finite subgroup of $SU(2)$. These data are algebraic. Another example of a gravitational instanton is $\mathbb{R}^3 \times S^1$. Yang-Mills instantons on this manifold are also called calorons. ADHM data for these is given in terms of solutions to Nahm equations, which form a system of nonlinear first order differential equations.

$$\begin{cases} \partial_t T_1 + i[T_0, T_1] & = -i[T_2, T_3] \\ \partial_t T_2 + i[T_0, T_2] & = -i[T_3, T_1] \\ \partial_t T_3 + i[T_0, T_3] & = -i[T_1, T_2] \end{cases}$$

A natural question to pose is how to describe instantons on a Taub-NUT space.

b) A generalization of the above question is how to describe instantons on all Asymptotically Locally Flat spaces.

3. There is a simple relation between Bogomolny Equations, Hitchin System, and Nahm Equations. Namely, they are reductions of Self-Duality equations of Yang-Mills theory to respectively three, two, and one dimension. Moreover, the Nahm transform establishes a one-to-one map between monopoles, i.e. gauge equivalence classes of solutions of the Bogomolny equation, and classes of gauge equivalent solutions of Nahm equations. Via Hitchin's scattering problem in a monopole background, each monopole configuration corresponds to a spectral curve S in minitwistor space $T\mathbb{P}^1$. This is the same curve as the spectral curve of

the corresponding solution of Nahm equations:

$$S : \det(T_2 - iT_3 + 2\zeta T_1 - \zeta^2(T_2 + iT_3)\eta) = 0.$$

Each solution of Hitchin equations on some Riemann surface Σ consists of a connection $A(s)$ and a Higgs field $\Phi(s)$, with $s \in \Sigma$. It also leads to a holomorphic curve C in the cotangent space of Σ . If Σ is an elliptic curve, there is a natural limit in which Hitchin system degenerates into the system of Nahm equations. Thus there is a relation between the spectral curves C and S .

This relation has a deep physical meaning. The curve C in $T^*\Sigma$ can be interpreted as a physical position of an M theory five-brane. The curve S on the other hand contains information about nonperturbative contributions to the low energy dynamics of this five-brane.

A.5 Dai, Xianzhe

I am interested in the asymptotic geometries of manifolds with special holonomy and the L^2 cohomology on these spaces. I am also interested in learning about the physics related to it. Years ago, Cheeger and I have looked at L^2 -signature for nonisolated conical singularity.

A.6 Degeratu, Anda

I am interested in studying index theory/ L^2 -cohomology of non-compact manifolds which arrive as crepant resolutions of Calabi-Yau orbifolds. The kind of geometries involved are the so called Quasi Asymptotically Locally Euclidean (QALE) geometries introduced by Dominic Joyce.

A.7 Etesi, Gabor

My main interest in the theory of L^2 harmonic forms comes from Yang-Mills theory. As it is well-known, an L^2 harmonic 2-form can be regarded as the curvature of a reducible, self-dual $SU(2)$ connection of finite action. I can formulate two questions on the subject.

1. The class of geometries over which we study the above problem is as follows. Let (M, g) be a “gravitational instanton”. In a narrow sense by a gravitational instanton we mean a connected, non-compact, four-dimensional, complete hyper-Kähler manifold with an end of prescribed asymptotical behaviour. These asymptotical conditions can be either asymptotically locally Euclidean (ALE) or asymptotically locally flat (ALF) or ALG (this last abbreviation is by induction). Such hyper-Kähler geometries naturally appear in recent investigations in string theory as it was pointed out by Cherkis and Kapustin. Understanding the L^2 cohomology over them is especially interesting in connection with Sen’s conjecture, also motivated by string theory.

A recent theorem of Hausel, Hunsicker and Mazzeo shows that the full L^2 cohomology over these spaces comes from degree two; therefore by the first observation above, the problem of constructiong L^2 harmonic forms can be naturally incorporated into finding (reducible) self-dual $SU(2)$ connections over these spaces and the whole problem becomes conformally invariant. In a joint paper with Tamas Hausel, we constructed explicitly the full L^2 cohomology of the Gibbons–Hawking spaces: gravitational instantons of type A_k ALE and ALF. Our technique is based on constructing self-dual $SU(2)$ connections via conformal rescalings of the metric with harmonic functions and then identifying the reducible ones.

Our aim is to extend the method to the most recent gravitational instantons of D_k ALF type constructed by Cherkis and Hitchin quite explicitly. The theorem of Hausel–Hunsicker–Mazzeo is based on a compactification X of the original manifold M . In case of the ALF condition X is a closed, smooth four-manifold. We conjecture that the conformal structure of the original space (M, g) extends to X (up to some mild singularity) and the whole Yang–Mills theory, hence L^2 cohomology comes from that of X . This conjecture is supported by the experience with the A_k ALF Gibbons–Hawking spaces, i.e. multi-Taub-NUT geometries: self-dual $SU(2)$ connections we found over these spaces have integer action (not fractional). Since hyper-Kähler spaces are self-dual, twistor theory naturally appears; one may try to construct the twistor space Z of X . Via Penrose transform, harmonic functions, hence self-dual connections over X (i.e., M) can be derived from $H^1(Z, O(-2))$. Finally, the reducible self-dual connections can be identified via topological methods: there is a one-to-one correspondence between the generators of the second cohomology of X and reducible self-dual connections through topological splittings of the Yang–Mills bundle into Whitney sums of line bundles.

2. Our second question is more general. Assume \overline{M} is a smooth, connected, compact four-manifold with a connected boundary. Let (M, g) be a complete Riemannian four-manifold with $M := \overline{M} \setminus \partial M$. Take an asymptotic boundary condition for the metric, for definiteness we can think one of the ALE, ALF, or ALG condition (or more generally, one of the boundary conditions considered in the Hausel–Hunsicker–Mazzeo paper). Take a smooth, self-dual, irreducible $SU(2)$ connection over (M, g) of finite action (or energy, L^2 norm). Is it true that its action is congruent mod \mathbb{Z} to one of the Chern–Simons invariants of the boundary ∂M ? Notice that the conjecture is true for the explicitly found self-dual connections over the Gibbons–Hawking spaces, moreover the Euclidean Schwarzschild-manifold (an ALF space with non-self-dual curvature). Also notice that all conditions (smoothness, irreducibility) is essential.

A.8 Grieser, Daniel

My current main interest lies in understanding the analysis of geometric differential operators on Riemannian manifolds that arise as smooth parts of (compact, singular, real or complex) algebraic varieties, or more generally semi-algebraic sets. Here, it is assumed that the singular space is embedded in a smooth space, and that the Riemannian metric arises from restriction of a Riemannian metric on the ambient space.

A central goal here would be to calculate the Hodge theory of, say, L^2 -harmonic forms in terms of singularity data. An outstanding problem in this direction is the Cheeger-Goresky-McPherson conjecture, which states that, for complex projective algebraic varieties, the L^2 Hodge theory of the smooth part of the variety should be described by the intersection cohomology of the variety itself.

Closely related problems are the asymptotic structure of solutions to geometric elliptic equations on these spaces, and consequences for spectral theory. For example, one might hope that harmonic functions (or forms) should be conormal on a suitable resolution of the space in question. A related problem would be to establish an asymptotic expansion of the heat trace in this context.

A.9 Gualtieri, Marco

a) I would like to have an overview of the current state of knowledge concerning the kind of information contained in the L^2 cohomology of Riemannian manifolds, and how this information changes in the presence of special geometry.

b) I am interested in the effect of T-duality on the L^2 cohomology.

c) I am interested in the L^2 cohomology of generalized Kahler, Calabi-Yau, and G2xG2 metrics.

A.10 Hausel, Tamas

My main research interest is to understand the cohomological properties of complete hyperkähler manifolds of non-compact type and find applications of this in other parts of mathematics and physics. The spaces I am interested in include moduli spaces of instantons on ALE spaces, more generally Nakajima's quiver varieties, toric hyperkähler varieties, moduli spaces of magnetic monopoles on \mathbb{R}^3 and moduli spaces of Higgs bundles on a Riemann surface.

The cohomological properties I am interested are partly to find what analogues of the Hodge-theoretical Kähler package survive for these hyperkähler manifolds. In the (compact) Kähler case this package contains Hodge theory on L^2 harmonic forms and its consequences like the strong Hard Lefschetz theorem. Thus I am naturally led (e.g. in our joint paper with Hunsicker and Mazzeo) to questions about Hodge theory on hyperkähler manifolds of this non-compact type.

These questions were originally motivated by physics (e.g. in my thesis), in the form of Sen's conjectures and its relatives, but recently some new interesting applications of these problems arose in the theory of Yang-Mills instantons on various gravitational instantons (e.g. in joint papers with Etesi), string theoretical mirror symmetry (e.g. in joint papers with Thaddeus, and in recent joint work with Rodriguez-Villegas), combinatorics of matroids (in joint papers with Sturmfels and Swartz), the arithmetic of some of these hyperkähler spaces and in turn representation theory of finite groups of Lie type (both in recent joint work with Rodriguez-Villegas), and recently I am working on applications to knot theory. In all these applications Hodge theory on the above hyperkähler manifolds plays the central role.

I would finally highlight three problems which penetrated my research from many directions:

1. This is the quest to find the "right" compactification of my favourite spaces, e.g. whose intersection cohomology would calculate the Hodge cohomology of the original space.

2. Is there a Hodge theoretical L^2 -index interpretation of a certain signature, which can be defined on circle compact manifolds using the integration technique in our recent paper with Proudfoot? I conjecture that for a hyper-compact hyperkähler manifold this signature is always non-positive if the quaternionic dimension is odd and is non-negative if the quaternionic dimension is even; a result which is known for the L^2 -signature by a result of Hitchin. My conjecture is known to be true for toric hyperkähler manifolds from a result in matroid theory.

3. Understand if a strong version of the Hard Lefschetz theorem could be proven in a large class of these non-compact hyperkähler manifolds; a version of which is recently conjectured for the moduli space of Higgs bundles, as an outcome of our work with Rodriguez-Villegas on the number theory of these varieties. A weak version of the Hard Lefschetz

theorem has been proven in our paper with Sturmfels and in my paper on the quaternionic geometry of matroids.

A.11 Hunsicker, Eugenie

My primary research area has been in L^2 Hodge and signature theorems for noncompact manifolds. As one of the organizers, my goals for the workshop are essentially those outlined in the workshop description. I am interested in discussing with other mathematicians working in the field how our recent work relates. There have been many advances recently, and I feel it would be useful to discuss if there are ways in which different techniques that have been used could be combined to develop a more general theory than currently exists. For instance, is there a way to relate the recent work of Carron on ALE spaces and the recent work of my collaborators and myself on fibred boundary metrics with Saper's recent work on L-modules? One difficulty I have seen in developing further Hodge results of possible interest to the physics community is that it is not obvious how to generalize the topological side of the existing theorems to manifolds with more complicated ALE type metrics. Saper's recent work develops a wide new set of cohomology theories which seem like good candidates, and I believe it would be useful to see how they may relate to the existing Hodge theorems for ALE and fibred boundary spaces. If we can do this, I believe it could give us a better sense of how a more general theory might look.

On the physics side, I am interested first in understanding the origin of the S-duality type conjectures. I believe that understanding these well will help me direct my research in ways of interest to physicists. I am also curious if there might be a direct mathematical way to understand these conjectures. In particular, could the physical intuition behind the conjectures inspire new mathematical approaches to the questions of L^2 Hodge, signature, and more general index theory on noncompact manifolds?

A.12 Jardim, Marcos

I hope to discuss the following to conjectures:

Conjecture 1. Let M and N be two connected 4-dimensional hyperkähler manifolds. If N is a component of the moduli space of instantons on M and M is a component of the moduli space of instantons on N , then:

- M is diffeomorphic to N ;
- the Nahm transform taking instantons on M into instantons on N is invertible.

This is known to be true for $M = T^4, K3$, while the first part is true for M being an ALE space, with strong evidence supporting the second part. What about, say, ALF spaces?

(For details about the Nahm transform for instantons over hyperkähler manifolds, see my paper with Claudio Bartocci “Hyperkähler Nahm transform” math.DG/0312045)

Conjecture 2. Let M a connected 4-dimensional hyperkähler manifold, $M \neq \mathbb{R}^4$. Then there is a 4-dimensional component of the moduli space of instantons on M which is diffeomorphic to M .

This is known to be true for M being $T^4, T^2 \times \mathbb{R}^2, K3$ and an ALE space. There is evidence that it is also true for $T^3 \times \mathbb{R}$, though it is false for $S^1 \times \mathbb{R}^3$. What about ALF spaces?

Conjecture 3. Let Γ be a finite subgroup of $SO(3-d)$, $d = 0, \dots, 3$. Assume that Γ acts on $T^d \times \mathbb{R}^{4-d}$ in such a way that all singularities are simple. The minimal resolution of $T^d \times \mathbb{R}^{4-d}/\Gamma$ admits a hyperkähler metric, and compute its asymptotics.

This is true in the case $d = 0$ via Kronheimer’s work; do all non-compact 4-dimensional hyperkähler manifolds whose metrics have the given asymptotic behaviour arise in this way?

Finally, I would also like to discuss the construction of the moduli space of “spatially periodic instantons” i.e. instantons on $T^3 \times \mathbb{R}$, as well as existence and non-existence results for different asymptotic behaviour (similar to those obtained for doubly-periodic instantons in *math.DG/0108004*). It would also be nice to discuss the Nahm transform and spectral curves.

A.13 Lee, Kimyeong

Title: Normalizable Middle Forms on Instanton and Monopole Moduli Space

I am interested in the L^2 -harmonic forms in the moduli space of BPS solitons. The issues I would like to discuss is whether the following three conjectures are true or not. They are motivated by physics.

I) On the nonsingular relative moduli space of self-dual k instantons of the $U(n)$ gauge theory on noncommutative R^4 space, there exists unique normalizable harmonic middle form.

II) On the relative moduli space of some set of BPS magnetic monopoles in the $N = 4$ supersymmetric Yang-Mills theory where the gauge group is not maximally broken to abelian subgroup, there exists a unique normalizable harmonic form if the total magnetic charge belongs to a singlet representation of the unbroken magnetic dual group.

III) On the relative moduli space of self-dual k calorons of the $U(n)$ gauge theory on noncommutative $R^3 \times S^1$ space, where the Wilson loop symmetry breaking leads to unbroken $U(1)^r \times K$ with semi-simple Lie group K , the number of normalizable harmonic middle forms is equal to r .

The first conjecture is motivated by considering a circle compactification of the tensionless $N = 2$ supersymmetric string theories on six dimension. The instantons on five dimensional Yang-Mills theories are the momentum modes along the compact circle of this string theory and so we expect the unique state for each momentum. The conjecture has shown to be true when $k = 2, n = 1$ for which the relative moduli space is the Eguchi-Hanson space and $k = 1, n$ case where the moduli space is Calabi space. The normalizable middle forms are found explicitly in these cases.

The second conjecture is motivated by the S-duality in the $N = 4$ supersymmetric Yang-Mills theories in four dimensional space time. The global color problem arises when the gauge group, which was maximally broken to abelian subgroups, has unbroken nonabelian subgroup and so some of magnetic monopoles become massless. The massless magnetic monopoles cannot exist by their own, but can be present when there are massive monopoles.

The third conjecture is motivated in part by the S -duality in the maximally supersymmetric Yang-Mills theories in $R^{1+3} \times S^1$, and by the T -dual picture. This conjecture is less well-developed compared with the first two. The existence of the normalizable middle form in the Calabi space also supports this conjecture as it can be considered as the moduli space of a caloron with a trivial Wilson loop. At least for a single caloron with a maximal symmetry breaking to $U(1)^n$, one can argue that there are n normalizable bound states.

To test the above conjectures, I am working on some specific examples which may be doable as they involve a few number of solitons.

There are side issues not directly related to the L^2 harmonic form but still puzzling aspects of the moduli space of instantons and monopoles.

A.14 Martens, Johan

Recently, following work by Nekrasov, Hausel-Proudfoot and Nakajima-Yoshioka have used formal integrals of equivariant forms on (non-compact) spaces, defined as sums of contributions of fixed-point components for a group action. By the localization theorem in equivariant cohomology, these correspond to the expected integrals for compact spaces. Can a similar justification be made in the case of L^2 forms (where one uses an appropriately modified Cartan or Weil model for the equivariant cohomology)?

A.15 Saper, Leslie

I am interested in the relation between analytic and topological invariants of singular spaces whose smooth locus has a Riemannian metric. One example of such spaces are projective algebraic varieties where one either constructs a complete Kähler metric on the smooth locus or considers the incomplete metric induced from the ambient projective space. Another example are the various compactifications (reductive Borel-Serre, Satake, toroidal) of locally symmetric spaces with the natural metric. A particularly nice example, common to the two preceding classes, is the Baily-Borel-Satake compactification of a Hermitian locally symmetric space; this is known to be a projective algebraic variety. The primary invariants I consider are (on the analytic side) the L^2 -cohomology and the space of L^2 -harmonic forms, and (on the topological side) the intersection cohomology for various perversities as well as the ordinary cohomology of the interior. My work is primarily motivated by number theory however the methods should have applicability elsewhere.

There is a generalization of Hermitian locally symmetric spaces, natural from the point of view of representation theory, the equal-rank locally symmetric spaces. There is even a generalization of the Baily-Borel-Satake compactification in this context, the real equal-rank Satake compactifications. For these spaces, a number of conjectures have been resolved which identify some of these invariants: Zucker's conjecture and Borel's conjecture (the L^2 -cohomology and the L^2 -harmonic forms agree with the middle-perversity intersection cohomology of a real equal-rank Satake compactification) and the Goresky-MacPherson-Rapoport conjecture (the middle-perversity intersection cohomology of the reductive Borel-Serre compactification agrees with the middle-perversity intersection cohomology of a real equal-rank Satake compactification). These global isomorphisms are proved by verifying a (very non-trivial!) local isomorphism near singular points of the Satake compactification.

Some of my recent work has focused on the reductive Borel-Serre compactification of non-equal-rank locally symmetric spaces. Again global isomorphisms of invariants may be proved in some cases. However, in contrast with the previous equal-rank theorems, these new isomorphisms do not hold locally. It is an interesting open question whether one can find smaller compactifications (perhaps Satake compactifications) where the isomorphisms become local.

A.16 Singer, Michael

I am interested in the metric structure of the moduli spaces of monopoles for arbitrary gauge group and symmetry breaking. Many of the unresolved issues are described in my recent note with Michael Murray. Following the work of Jarvis, one knows that these moduli spaces (suitably framed) can be identified with spaces of holomorphic maps of fixed degree from the Riemann sphere into a generalized flag manifold. The picture is that such a moduli space contains various strata, which are fibred by spaces of dimension a multiple of 4. The first conjecture to be proved is that the L^2 metric on these fibres really is well defined and gives a complete hyperkaehler metric.

Then there are a number of natural questions, such as the dependence of the metric on the natural parameters of the moduli space (mass m and charge k). Of particular interest, perhaps, is to allow ‘degenerations’ of m , which is an element of the lie algebra of the gauge group, by taking it to a wall of a Weyl chamber. Presumably such questions are related to understanding the way the strata fit together in the full moduli space.

Many of these questions have no real analogue for the case of $SU(2)$ -monopoles, which is accordingly much better understood, thanks to the work over many years of (among others) Atiyah, Donaldson, Hitchin, Hurtubise, Manton, Murray, Taubes,.... It seems to me that some genuinely new work needs to be done in order to answer them.

A.17 Yeganefar, Nader

I’m interested in Hodge theory on complete manifolds. For a number of interesting geometric situations, Hodge type theorems already exist, but for some others, not much is known.

- A. On nonnegatively curved manifolds, there are only some partial vanishing results in special cases. Is it true that on a nonnegatively curved manifold, the spaces of L^2 harmonic forms are all finite dimensional? It is probably easier to deal first with manifolds with a pole, with curvature going to zero at infinity. A powerful tool for proving such finiteness theorems seems to be G. Carron’s notion of non parabolicity at infinity.
- B. Mazzeo and Phillips computed the Hodge cohomology of geometrically finite hyperbolic manifolds. Is it possible to find a simpler proof of their result? What about the variable curvature case or the complex hyperbolic case?