# KOSTKA POLYNOMIALS AND ROOT PARTITION FUNCTIONS

This problem was contributed by John Stembridge (and transcribed by Nick Loehr, who takes full responsibility for any errors appearing herein).

## 1. Notation

We use the usual standard notation associated with root systems. Thus let $\Phi$ be a finite crystallographic root system with positive roots $\Phi^+$, dual root system $\Phi^\vee$, weight lattice $\Lambda$, dominant weights $\Lambda^+$, Weyl group $W$, group ring $\mathbb{Z}[\Lambda] = \mathbb{Z}\{e^\lambda : \lambda \in \Lambda\}$, antisymmetrization operators $J = \sum_{w \in W} \operatorname{sgn}(w)w$, and character functions $\chi(\lambda) = J(e^{\lambda+\rho})/J(e^\rho)$, where $\rho = \frac{1}{2}\sum_{\phi \in \Phi^+} \phi$. The Kostka-Foulkes polynomials associated to this root system will be denoted $K_{\lambda,\mu}(t)$. See Stembridge's lecture notes for more details.

## 2. Motivation

Define *Kostant's partition function* by

$$\prod_{\alpha>0} \frac{1}{1-te^\alpha} = \sum_\gamma \mathcal{P}(\gamma;t)e^\gamma,$$

where the summation runs over all $\gamma$ that can be expressed as a positive sum of simple roots. Note that $\mathcal{P}(\gamma;1)$ is the number of partitions of $\gamma$ into positive roots, while $\mathcal{P}(\gamma;t)|t^\ell$ is the number of partitions of $\gamma$ into exactly $\ell$ positive roots. The degree of $\mathcal{P}(\gamma;t)$ in $t$ is the height of $\gamma$.

*Kato's theorem* expresses the Kostka-Foulkes polynomials in terms of the polynomials $\mathcal{P}(\gamma;t)$. Kato's formula is

$$K_{\lambda,\mu}(t) = \sum_{w \in W} \operatorname{sgn}(w)\mathcal{P}(w(\lambda+\rho)-(\mu+\rho);t).$$

## 3. Problems

We can generalize Kostant's partition function as follows. Let $\Psi$ be *any* finite multiset of weights, and define elements $\mathcal{P}_\Psi(\gamma;t)$ by the formula

$$\prod_{\nu \in \Psi} \frac{1}{1-te^\nu} = \sum_\gamma \mathcal{P}_\Psi(\gamma;t)e^\gamma.$$

(Note that the $\mathcal{P}_\Psi$'s are not polynomials unless the cone generated by $\Psi$ is pointed.) Next, by analogy with Kato's formula, define *Kostka-Foulkes functions associated to* $\Psi$ by setting

$$K^\Psi_{\lambda,\mu}(t) = \sum_{w \in W} \operatorname{sgn}(w)\mathcal{P}_\Psi(w(\lambda+\rho)-(\mu+\rho);t).$$

We would like to have the *positivity result* $K^{\Psi}_{\lambda,\mu} \in \mathbb{N}[[t]]$ for all dominant weights $\lambda$ and $\mu$. Of course, this will not be true for arbitrary multisets $\Psi$.

**Problem 1:** Find a framework to explain which multisets $\Psi$ have positive Kostka-Foulkes functions $K^{\Psi}_{\lambda,\mu}(t)$. For example, for the root system $A_1$ corresponding to the Lie algebra $sl_2$, a *sufficient* condition for positivity is to take $\Psi$ to be the set of weights of a Borel submodule of an $sl_2$-module. However, this does not work for higher ranks. One conjecture is that the weights coming from Demazure modules may work, but some have expressed concern that these might not be robust enough to explain positivity in all cases of conjectural interest.

**Problem 2:** Understand the linear term (the coefficient of $t^1$) in the various Kostka-Foulkes functions. In particular, requiring the linear term to be nonnegative means that the multiplicities of weights in $\Psi$, say $M(\nu)$ for $\nu \in \Lambda$, must live in the cone defined by the constraints $M(\nu) \geq 0$ and

$$\sum_{w \in W} \operatorname{sgn}(w) M(w(\lambda+\rho) - (\mu+\rho)) \geq 0 \qquad (\lambda, \mu \in \Lambda^+).$$

Of course, $M(\cdot)$ must also have finite support. A description of the extreme rays or walls of this cone would provide *necessary* conditions on the multisets $\Psi$.