FUSION PRODUCT PROBLEMS

This document is essentially a transcript of remarks by Rinat Kedem at the third open problem session of the Kostka workshop, which was moderated by Bogdan Ion. The transcriber (Nick Loehr) takes full responsibility for any errors and garbles appearing below.

Consider two fixed irreducible modules V_λ and V_μ for $\mathcal{A} = gl_n$, with highest weight vectors v_λ and v_μ. We have the fusion product

$$V_\lambda * V_\mu = gr(\mathcal{U}(\mathcal{A}^-[t])(v_\lambda \otimes v_\mu)),$$

whose m'th graded piece is given by

$$\mathcal{U}^m(\mathcal{A}^-[t]) \cdot (v_\lambda \otimes v_\mu)/\mathcal{U}^{<m}(\mathcal{A}^-[t]) \cdot (v_\lambda \otimes v_\mu).$$

Recall that the action here is $(x \otimes t) \cdot (v_\lambda \otimes v_\mu) = z_1(x \cdot v_\lambda) \otimes v_\mu + z_2 v_\lambda \otimes (x v_\mu)$. See Kedem's lecture notes for more details. We define elements $c^\nu_{\lambda,\mu}(q)$ by decomposing $V_\lambda * V_\mu$ into a direct sum of irreducible representations:

$$V_\lambda * V_\mu = \bigoplus_\nu c^\nu_{\lambda,\mu}(q) V_\nu.$$

When $q = 1$, we recover the ordinary Littlewood-Richardson coefficients $c^\nu_{\lambda,\mu}$.

Problem 1. Find a nice combinatorial description of $c^\nu_{\lambda,\mu}(q)$. One approach is to look for a suitable statistic on the set of Littlewood-Richardson tableaux associated to λ, μ, ν.

Problem 2. Describe explicit bases for the graded pieces $(V_\lambda * V_\mu)_m$. Procesi et al. solved this problem in the case where $\lambda = a\omega_1$ and $\mu = b\omega_2$ (where the ω_i are fundamental weights). In this case, we get the Schur coefficients in the Frobenius character of the ring $R_{(a,b)}$ (see Haiman's lecture notes).

Problem 3. When λ is a single row ($\lambda = \omega_i$), it is known that $c^\nu_{\lambda,\mu}(q) = q^M$ for some power M. What is this power in terms of μ and ν?

Problem 4. Answer similar questions for k-level-restricted fusion products (which are related to Gromov-Witten invariants). These are obtained by forming the quotient with respect to a suitable ideal I_k of "integrability relations". One such relation is $(f_i \otimes 1)^{k+1} = 0$; thus, $(f_i \otimes 1)^{k+1} \in I_k$. We have

$$V_\lambda *_k V_\mu = gr\left(\frac{\mathcal{U}(\mathcal{A}^-[t])}{I_k} \cdot (v_\lambda \otimes v_\mu)\right).$$

Decomposing into irreducible pieces, we get

$$V_\lambda *_k V_\mu = \bigoplus_\nu R^{\nu,k}_{\lambda,\mu}(q) V_\nu,$$

where $R^{\nu,k}_{\lambda,\mu}(q)$ are Gromov-Witten invariants. We would like to have a basis for each graded piece of $V_\lambda *_k V_\mu$, combinatorial interpretations for each $R^{\nu,k}_{\lambda,\mu}(q)$, etc.

Problem 5. Answer the analogues of the questions above in the two-variable case, where we are studying

$$gr\left(\mathcal{U}(\mathcal{A}^-[t_1,t_2])(v_\lambda \otimes v_\mu)\right).$$

There should be a connection between this construction and the doubly-graded Garsia-Haiman modules R_μ (see Haiman's lecture notes). Note that the fusion product of n fundamental weights, namely $(V_{\omega_1})^{*n}$, gives the ring R_n of diagonal coinvariants. This is obvious after three pages of calculations.

Reference: B. Feigin and S. Loktev, "On generalized Kostka polynomials and the quantum Verlinde rule," *Differential topology, infinite-dimensional Lie algebras, and applications*, Amer. Math. Soc. Transl. Ser. 2 **194**, pp. 61—79.