PAVING AND THE KADISON-SINGER PROBLEM

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Abstract.
This is an introduction to problems surrounding the Paving Conjecture.

1. Paving Parameters and Notation

Notation 1.1. *Given an orthonormal basis* $\{e_i\}_{i\in I}$ *for a Hilbert space* $\mathbb{H}$*, for any* $A \subset I$ *we denote by* P_A *the* **diagonal projection** *whose matrix has entries all zero except for the* (i,i)*-entries for* $i \in A$ *which are all one.*

Notation 1.2. *Given an orthonormal basis* $\{e_i\}_{i\in I}$ *for a Hilbert space* $\mathbb{H}$*, a* **k diagonal decomposition of the identity** *(k-d.d. for short) is a family of diagonal projections with disjoint ranges and*

$$\sum_{i=1}^{k} P_i = I.$$

Notation 1.3. *If* A *is an* $n \times n$ *matrix, define*

$$\alpha_k(A) = min\{max\ ||P_i A P_i|| : \{P_i\}_{i=1}^k \text{ is a } k\text{-d.d.}\}$$

and if A *is an infinite matrix bounded in the operator norm, define*

$$\alpha_k(A) = inf\{max\ ||P_i A P_i|| : \{P_i\}_{i=1}^k \text{ is a } k\text{-d.d.}\}.$$

If $A \neq 0$*, in both cases define*

$$\tilde{\alpha}_k(A) = \frac{\alpha_k(A)}{\|A\|}.$$

Define

$$\alpha_k(\mathcal{S}) = \sup_{0\neq A\in\mathcal{S}} \tilde{\alpha}_k(A),$$

and if $\mathcal{S} \subset \mathbb{M}_n$*, this supremum is attained by extremal bad pavers* A_n *making*

$$\alpha_k(\mathcal{S}) = \max_{0\neq A\in\mathcal{S}} \tilde{\alpha}_k(A) = \tilde{\alpha}_k(A_n).$$

As n increases, for a fixed category of matrices $S \subset \mathbb{M}_n$ such as selfadjoint or unitary operators (so the matrix sizes change but not the class), one has $0 \leq \alpha_k(\mathcal{S}) \leq 1$ and is increasing in both k and n, although as the table in Section 7 indicates, not necessary strictly in n (but strict monotonicity yet unknown for k).

2. **Anderson's Paving Conjecture**

A remarkable connection between the Kadison-Singer Extension Problem (KS) and finite matrices was given by J. Anderson [A].

Anderson's Paving Conjecture (PC) [A]

For every $\epsilon > 0$, there is some "universal" positive integer $k = k(\epsilon)$ so that for every zero-diagonal finite matrix A, there exists a k-d.d. $\{P_i\}_{i=1}^k$ so that for all $i = 1, 2, \ldots, k$ we have

$$\|P_i A P_i\| \leq \epsilon \|A\|. \tag{1}$$

Note that this is equivalent to the existence of a universal such k for some fixed $\epsilon < 1$. When we have the inequality in Equation 1 for a matrix A, we say that A is (k, ϵ)**-pavable**.

Given a fixed matrix A, if for every $\epsilon < 1$ there is a $k \in \mathbb{N}$ so that A is (k, ϵ)-pavable then we say that A **pavable**. If we have a class of matrices, we say this **class is pavable** (respectively, (k, ϵ)-pavable) if every member of the class is pavable (respectively, $(k\epsilon)$-pavable).

It is known [BHKW] that the class of operators satisfying PC (the **pavable operators**) is a closed subspace of $B(\ell_2)$. The only large classes of operators which have been shown to be pavable are "diagonally dominant" matrices [BCHL, BCHL2, BHKW, G], matrices with all entries real and positive [HKW] and Laurant operators over Riemann integrable functions (See also [HKW2]). Also, in [BHKW2] there is an analysis of the paving problem for certain Schatten C_p-norms.

Akemann and Anderson [AA] posed several conjectures which would imply a positive solution to PC.

Conjecture A ([AA], 7.1.1) For any projection P, there exists a diagonal symmetry S so that $\|PSP\| \leq 2\delta_p$.

Weaver [W2] gives a counterexample to Conjecture A. Recently, Casazza, Edidin, Kalra and Paulsen [CEKP] showed that for every equal-norm Parseval frame $\{f_i\}_{i=1}^n$ for $\mathbb{R}^k$ with $n > 4k$, its correlation matrix $P = (\langle f_i, f_j \rangle)$ is a counterexample to Conjecture A.

Conjecture B ([AA], 7.1.3) There exists $\gamma, \epsilon > 0$ (and independent of n) such that for any projection P with $\delta_p < \gamma$ there is a diagonal symmetry S such that $\|PSP\| < 1 - \epsilon$.

Conjecture B is still open despite considerable effort having been expended on it. Weaver [W2] states that a counterexample to Conjecture B would probably lead to a negative solution to the Paving Conjecture.

Current research of Weiss and Zarikian [WZ] presented in Sections 3-5 quantifies this fact in terms of "paving parameters" and computing sharp bounds attained by optimal matrices. These paving parameters and the Schatten-norm analogs were introduced and investigated in [HKW]-[HKW2] and later in [BHKW]-[BHKW2].

3. Some Paving Results

An obvious question concerns the connection between the k and ϵ in (k,ϵ)-pavability. In [CEKP] it is shown:

Theorem 3.1. *Assume that every zero-diagonal, real unitary matrix is (k,ϵ)-pavable. Then $1 \le k\epsilon^2$.*

The most powerful result on paving is due to Bourgain-Tzafriri [BT]

Theorem 3.2 (Bourgain-Tzafriri)**.** *There is a constant $C < \infty$ and, for each $0 < \delta < 1$ and $a > 0$, an integer $n(\delta, a)$ such that ,whenever $n \ge n(\delta, a)$ and $A = (a_{i,j})_{i,j=1}^n$ is a norm one $n \times n$ matrix whose entries satisfy the condition*

$$|a_{i,j}| \le \frac{1}{(log\ n)^(1+a)},$$

for all $1 \le i, j \le n$, then A is (k,ϵ)-pavable where $k = [\delta^{-1}]$ and $\epsilon = C\delta^{a/C}$.

Recall that an $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ is called a **conference matrix** if $A = A^*$, $A^2 = (n-1)I$, $a_{ii} = 0$ and for $1 \le i \ne j \le n$ we have $a_{ij} = \pm 1$. It was recently shown, using Theorem 3.2, that the conference matrices are pavable. [CEKP].

Theorem 3.3. *There is an $\epsilon < 1$ so that all conference matrices for all $n \in \mathbb{N}$ are $(2,\epsilon)$-pavable. Hence, the class of conference matrices is pavable.*

4. Paving Conjectures Equivalent to PC

Conjecture 4.1. *For every zero-diagonal matrix A (finite or infinite) there is a k and $\epsilon < 1$ so that A is (k,ϵ)-pavable. I.e., For every zero-diagonal matrix A we have $\tilde{\alpha}_k(A) < 1$ for some k.*

The equivalence of Conjecture 4.1 and PC is a simple diagonal process. Note that in Conjecture 4.1 does not require the universality of k and the bound below one depends on A.

Conjecture 4.2. *There is a universal k so that for all zero-diagonal matrices A (finite and infinite), there is a $\epsilon < 1$ so that A is (k,ϵ)-pavable. (I.e., For every zero-diagonal matrix A we have $\tilde{\alpha}_k(A) < 1$.*

Note: The rest of the conjectures in this section require universal constants k and $\epsilon < 1$ which are independent of $n \in \mathbb{N}$ and the matrix.

Conjecture 4.3. [CEKP] *There exists an $\epsilon < 1$ and a natural number k so that all orthogonal projections A on ℓ_2^{2n} with $1/2'$s on the diagonal are (k,ϵ)-pavable.*

Note that Conjecture 4.3 does not require zero-diagonal. In [**?**] it is shown that Conjecture 4.3 fails for $k = 2$. I.e. There is no $\epsilon < 1$ so that this class of projections is $(2,\epsilon)$-pavable.

Conjecture 4.4. [CEKP] *There is a universal k and an $\epsilon < 1$ so that every norm one self-adjoint operator U with zero diagonal and satisfying $U^2 = I$ is (k,ϵ)-pavable.*

Conjecture 4.5. [W] *There exist universal constants $0 < \delta, \epsilon < 1$ and $k \in \mathbb{N}$ so that all orthogonal projections P on ℓ_2^n with $\delta(P) \le \delta$ are (k,ϵ)-pavable.*

Weaver also posed an equal norm version of this conjecture.

Conjecture 4.6. [W] *There exist universal constants* $0 < \delta, \sqrt{\delta} \leq \epsilon < 1$ *and* $k \in \mathbb{N}$ *so that all orthogonal projections* P *on* ℓ_2^n *with* $\delta(P) \leq \delta$ *and* $\|Pe_i\| = \|Pe_j\|$ *for all* $i, j = 1, 2, \ldots, n$ *are* (k, ϵ)*-pavable.*

([CFTW], Theorem 2.4) provides a number of restricted classes for which paveability with respect to any fixed basis (of their matrices with diagonal subtracted) is equivalent to KS: selfadjoint operators, unitary operators, positive operators, invertible operators (or invertible operators with zero diagonal), orthogonal projections, and Gram matrices. Also, [CT] shows that that PC is equivalent to paving triangular matrices. The theorems above, as they indicate in the papers, apply to classes of either finite or infinite matrices. But it is important to keep in mind that paving finite matrices in any of the restricted classes requires paving norms bounded away from 1 for KS-equivalence, while for infinite matrices only strictly less than 1 suffices.

5. **Problems**

Problem 5.1. *If KS is true, then are the upper triangular invertibles on* $\ell^2(\mathbb{N})$ *path connected? (Larson, Paulsen, Orr, Weiss, Zhang)*

The problem of the connectedness of the upper triangular invertible matrices is well-known in the non-self-adjoint operator algebra community.

Are there analogs for this question in other classes of operators such as Laurent, Toeplitz, analytic Toeplitz, etc.? (Weiss)

Problem 5.2. *Are all Laurent operators paveable?*
These are multiplication operators on $L^2(\mathbb{T})$ *with* L^∞ *symbol. ([BT2] (1991), [HKW] (1986))*

Problem 5.3. *Are all Laurent operators with* H^∞ *symbol paveable? (Paulsen, Weiss)*
Are all bounded upper triangular matrices paveable? (Probably weaker than KS.) (Weiss)

Problem 5.4. *Limited experimental evidence suggests that extremal bad pavers' or near extremals' best (minimal) pavings occur at least for equally sized projections, e.g., for paving extremal* 7×7 *matrices the optimal paving projections that we found have rank* $2, 2, 3$. *Is this often or always true? Can bad pavers have best pavings that are both equally sized and not equally sized. Even for infinitely many* n *without consideration for* k. *(*[WZ]*)*

Problem 5.5 (Paving parameter behavior-see Section 4)**.** *(*[WZ]*)*
(a) Beat the current 3-paving parameter challenges (see Section 7) or prove they are sharp: $.8231$ *for general finite matrices or* $\frac{\sqrt{5}}{3} \approx .7454$ *for selfadjoint finite matrices. Tighten up, or better, find the precise bound for* $\alpha_3(\mathbb{M}^0_{7,sa}) \in [\frac{2}{3}, \frac{2}{\sqrt{7}})$ *from the table in Section 7.*
(b) Is $\underset{n}{Lim}\ \alpha_k(\mathcal{S}) < 1$ *or* $\underset{n}{Lim}\ \alpha_k(\mathcal{S}) = 1$ *for any* $k > 2$*? In particular, for* $k = 3$*?*
Even more basic, is there some zero-diagonal matrix for which $\tilde{\alpha}_3(A) = 1$*?*
What about other $k > 2$ *(for* $k = 2$ *the* 3×3 *cyclic shift achieves this)?*

The rest of this note is a survey of the research of Weiss and Zarikian on paving small matrices as of January 1, 2007. Enhancements and modifications are in progress.

6. **Paving Classes and Parameters**

Important paving classes with simpler structure for investigating $\alpha_k(\mathcal{S})$:

$\mathbb{M}_n^0 = n \times n$ zero-diagonal complex matrices
$\mathbb{M}_{n,sa}^0 = n \times n$ zero-diagonal selfadjoint complex matrices
$\mathbb{M}_{n,sym}^0 = n \times n$ zero-diagonal real symmetric matrices
$\mathbb{M}_{n,++}^0 = n \times n$ zero-diagonal non-negative entried matrices
$\mathbb{M}_{n,\Delta}^0 = n \times n$ zero-diagonal upper (or lower) triangular matrices

Paving Parameter KS-equivalents:

$\underset{n}{Lim}\ \alpha_k(\mathcal{S}) < 1$ for some k and any of the classes (with diagonals subtracted if necessary): $\mathbb{M}_n^0, \mathbb{M}_{n,sa}^0, , \mathbb{M}_{n,\Delta}^0$, finite unitaries (or finite unitaries with zero-diagonal), finite positive matrices, finite orthogonal projections, finite orthogonal projections with constant diagonal $\frac{1}{2}$ (or those with constant diagonal or near constant diagonal), and finite triangular matrices. Also important are finite versions of analytic Toeplitz or analytic Laurent operators (finite blocks of upper triangular Toeplitz or Laurent operators). By [BT] only Laurents with non-Besov symbol remain unpaved.

Main focus of [WZ]:
To begin a bottom-up approach towards determining whether or not

$$\boldsymbol{\alpha_3(\mathbb{M}_n^0) \quad \text{or} \quad \alpha_3(\mathbb{M}_{n,sa}^0) \uparrow 1.}$$

7. **Paving small matrices**

[WZ] is devoted to 3-paving small matrices by developing methods to find optimal "bad pavers" and the consequent paving parameters. For starters, the 1987 $\frac{2}{3}$ challenge in [HKW3] is beaten in [WZ] by proving that $\alpha_3(\mathbb{M}_6^0) = \frac{\sqrt{2}}{2} > \frac{2}{3}$, in part, via constructing an obtimal bad paver A_6 given below. 2-pavings fail to settle KS because, as observed in [Ibid.], $\alpha_2(\mathbb{M}_3^0) = 1$ so 2-pavings fail. Indeed, clearly $\alpha_2(\mathbb{M}_3^0) = \frac{\|\sum P_i U_3 P_i\|}{\|U_3\|} = 1$ for every 3-d.d. for the simple unitary shift $e_1 \to e_2 \to e_3 \to e_1$ with matrix $U_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Notwithstanding $\alpha_2(\mathbb{M}_n^0) = 1\ \forall\ n \geq 3$, α_2 constrained to selfadjoint and other classes of matrices is studied by others (e.g., Casazza and Paulsen) and are [WZ]-bootstrapping tools for determining $\alpha_3(\cdot)$.

2,3-Small Matrix Pavings Summary Table

n	$\alpha_2(\mathbb{M}_n^0)$	$\alpha_2(\mathbb{M}_{n,sa}^0)$	$\alpha_2(\mathbb{M}_{n,sym}^0)$	$\alpha_3(\mathbb{M}_n^0)$	$\alpha_3(\mathbb{M}_{n,sa}^0)$	$\alpha_3(\mathbb{M}_{n,++}^0)$	$\alpha_3(\mathbb{M}_{n,\Delta}^0)$
3	1	$\frac{1}{\sqrt{3}}$ .5773	$\frac{1}{2}$ .5000	0	0	0	0
4	″	″	$[?, \frac{1}{\sqrt{3}}]$ [.5493, .5773]	$\mathbf{\frac{2}{1+\sqrt{5}}}$ **.6180**	$\frac{1}{\sqrt{3}}$ .5773	κ .5550	
5	″	$\frac{2}{\sqrt{5}}$ .8944	$\frac{2}{\sqrt{5}}$ .8944	″	″	$[\kappa, \frac{2}{1+\sqrt{5}}]$ [.5550, .6180]	
6	″	$[\frac{2}{\sqrt{5}}, 1]$	$[\frac{2}{\sqrt{5}}, 1]$	$\mathbf{\frac{1}{\sqrt{2}}}$ **.7071**	″	″	
7	″	″	″	$[?, 1)$ **[.8231, 1)**	$\mathbf{[\frac{2}{3}, \frac{2}{\sqrt{7}})}$ [.6667, .7559)	$[\kappa, \frac{2}{3}]$ [.5550, .6667]	
8	″	″	″	$[?, 1]$ [.8231, 1]	$\mathbf{[\frac{2}{3}, \frac{2}{\sqrt{5}}]}$ **[.6667, .8944]**	″	
10	″	″	″	″	$\mathbf{[\frac{\sqrt{5}}{3}, 1]}$ **[.7454, 1]**	″	

The main focus is on the paving parameters $\alpha_3(\mathbb{M}_n^0)$ and $\alpha_3(\mathbb{M}_{n,sa}^0)$. Their classes possess the most structure and our methods are best suited for them.

Single entries in the table represent precise values of the α paving parameter. That is, the given upper bound is proven and concrete extremal matrices are found for which these bounds are attained (for $\alpha_3(\mathbb{M}_4^0), \alpha_3(\mathbb{M}_6^0)$, A_4, A_6 are presented below). The interval entries such as $[.8231, 1)$, $[\frac{2}{3}, .7559)$, and $[.7454, 1]$ are ranges for $\alpha_3(\cdot)$ for which we verify the upper bound and construct concrete examples for which the lower bound is attained. In the table, " ″ " denotes "ditto to above," "?" denotes the lack of a closed form for the decimal bound, blanks indicate work in progress based on the new triangular KS-equivalence, and the boldface numbers represent the most significant numerical and mathematical breakthroughs in this study. The constant $\kappa := \sqrt{\frac{3}{5+2\sqrt{7}\cos(\tan^{-1}(3\sqrt{3})/3)}}$. The interval entries are important because, as preliminary bounds, they are essential bootstrapping ingredients in the proofs for sharp bounds. Strengthening our methods to sharpen the boldface bounds for $n = 7$ in the general and selfadjoint cases is the next phase of this part of the project. The selfadjoints appear to show the most promise having more rigid structure and supporting more powerful methods developed in [WZ].

Best concrete bounds known by us: a 7×7 circulant matrix A_7 and a 10×10 selfadjoint circulant matrix S_{10} for which

$$\tilde{\alpha}_3(A_7) = .8231 \quad \text{and} \quad \tilde{\alpha}_3(S_{10}) = \frac{\sqrt{5}}{3} \approx .7574.$$

Extremality for these sizes (equivalently, bad paver status of these matrices) are not known.

Extremal bad pavers A_4, A_6 for $\alpha_3(\mathbb{M}_4^0)$, $\alpha_3(\mathbb{M}_6^0)$ and the not necessarily bad pavers A_7 (noncirculant) and S_{10}(conference). (Many other examples at [KZ], e.g., a circulant A_7.)

$$A_4 = \begin{pmatrix} 0 & 1 & 1 & -\frac{2}{1+\sqrt{5}} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad A_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -\frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 0 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 0 \end{pmatrix} \in \mathbb{M}_{7,sa}^0$$

$$S_{10} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 0 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 0 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & 1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 0 \end{pmatrix} \in \mathbb{M}_{10,sa}^0 \,.$$

Uniqueness of bad pavers is discussed at length in [WZ]. Uniqueness never occurs as the bad paver class is closed under scalar multiplication and basis permutations. However, as [WZ] finds in several instances, one has uniqueness in a graph theory sense. Additionally in [WZ] the eigenvalue distributions of bad pavers and near bad pavers are studied with variational methods developed for this.

8. **Universal identities for small matrices**

The strategy in [WZ] for its results summarized above in Section 7 is to normalize, not the norm of a target zero-diagonal matrix A, but its optimal paving norm. That is, multiply A by a suitable scalar (thereby not changing its $\tilde{\alpha}_3(A)$) so to assume without loss of generality that $\alpha_3(A) = 1$. To this end we found it useful to develop general formulas for $2, 3, 4$ size matrices, especially selfadjoint ones, to determine from their entries or various standard measurements including: determinants, traces and the various Schatten norms, whether or not they have norm 1 and more generally, whether they have norm less or greater than 1 (i.e., whether or not they are contractions and strict contractions). Explicit formulas for their norms involve solutions to polynomial equations, but information can be gleened more easily from certain universal identities.

The universal general 2-identity is the well-known formula for the norm of an arbitrary rank 2 operator in terms of its Hilbert-Schmidt norm and C_4-norm: $\|S\| = \frac{\|S\|_2^2}{2} + \sqrt{\frac{\|S\|_4^4}{2} - \frac{\|S\|_2^4}{4}}$. Higher order universal identities express the norm only implicitly in terms of such other quantities (see [WZ] for details).

Universal Selfadjoint 3-Identity for arbitrary rank 3 self-adjoint trace zero operators S:

$$\frac{||S||_2^2}{2||S||^2} + \frac{|Det\, S|}{||S||^3} = 1$$

Norm 1 criteria: for arbitrary rank 3 selfadjoint trace zero operators S,

$$||S|| = 1 \Leftrightarrow \frac{||S||_2^2}{2} + |Det\, S| = 1.$$

For greater or less than 1, the respective conditions are equivalent.
A necessary condition for equality is $3/2 \leq ||S||_2^2 \leq 2$.

Universal Selfadjoint 4-Identity for rank 4 self-adjoint zero-trace operators:

$$\frac{||S||_2^2}{2||S||^2} + \frac{|Tr\, S^3|}{3||S||^3} - \frac{Det\, S}{||S||^4} = 1$$

Consequences: $\frac{|Det\, S|}{||S||^4} \leq 1$, $\frac{|Tr\, S^3|}{3||S||^3} \leq \frac{4}{3}$ and $\frac{||S||_2^2}{2||S||^2} + \frac{|Tr\, S^3|}{3||S||^3} \leq 2$

Development of Universal Selfadjoint 4-Identity: Let S denote a rank 4 selfadjoint zero-trace operator with eigenvalues: $1 = \lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq |\lambda_4|$.

$$\begin{aligned} c_\lambda(S) &= \lambda^4 + p\lambda^2 - q\lambda + r = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) \\ &= \lambda^4 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\lambda^3 + (\sum_{i<j} \lambda_i\lambda_j)\lambda^2 - (\sum_{i<j<k} \lambda_i\lambda_j\lambda_j)\lambda + \lambda_1\lambda_2\lambda_3\lambda_4 \end{aligned}$$

SUMMARY: NASC for $||S|| = 1$ (unvetted)

1. $p \geq \frac{2}{3}$ 2. $p + |q| + r = 1$
3. $0 \leq p + |q| \leq 2$ (equivalent to |product of roots| ≤ 1)
4. When $p < 1$, $\frac{20}{27} - \frac{2}{3}p - \frac{2}{27}(3p-2)^{3/2}) \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p-2)^{3/2}$.
5. When $p \geq 1$, $0 \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p-2)^{3/2}$.

(4-5: *max* $(0, \frac{20}{27} - \frac{2}{3}p - \frac{2}{27}(3p-2)^{3/2}) \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p-2)^{3/2})$

9. Operator Norm/p-Norm Comparisons

Because the p-norm plays such a central role in this work, finding relations to the operator norm proved useful. These relations depend on the eigenvalue distribution.

Proposition 9.1 (Operator Norm/p-Norm)**.** *If A is a finite rank selfadjoint trace 0 matrix and*

$$k = |\#\ \textit{strictly positive eigenvalues} - \#\ \textit{strictly negative eigenvalues}|,$$

then for $p \geq 1$,

$$||A||_p \leq (rank\ A - k)^{1/p}\ ||A||$$

(Sharp example: diag $(-1, 1)$)
(Sharp asymptotically: diag $(\pm 1, \ldots, \pm 1(\frac{rank\ A-k-2}{2}$ pairs of them$), 1, -\frac{k}{k+1}, -\frac{1}{k(k+1)}, \ldots, -\frac{1}{k(k+1)})$; note: rank $A - k$ must be even)

Corollary 9.2. *If A is an $n \times n$ selfadjoint trace 0 matrix with n odd, then*

$$||A||_2 \leq \sqrt{n-1}\, ||A||.$$

Proposition 9.3. *If A is an $n \times n$ selfadjoint trace 0 matrix and $p \geq 1$ (or more generally rank $A = n$), then*

$$||A||_p \geq [1 + \frac{1}{(n-1)^{p-1}}]^{1/p}\ ||A||$$

with equality iff $A = c\, diag(-1, \frac{1}{n-1}, \ldots, \frac{1}{n-1})$.

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