PROBLEMS ON PAVING AND THE KADISON-SINGER PROBLEM

ABSTRACT. For a background and up to date information about paving see the posted note: *Paving and the Kadison-Singer Problem.* Please send problems which should be posted in this section to Pete Casazza at pete@math.missouri.edu.

1. NOTATION

Notation 1.1. *Given an orthonormal basis $\{e_i\}_{i\in I}$ for a Hilbert space $\mathbb{H}$, for any $A \subset I$ we denote by P_A the* **diagonal projection** *whose matrix has entries all zero except for the (i,i)-entries for $i \in A$ which are all one. For a matrix $A = (a_{ij})_{i,j=1}^n$ set $\delta_p = max\{|a_{ii}| : i = 1, 2, \cdots, n\}$. A* **diagonal symmetry** *is a diagonal matrix $(a_{ij}\}_{i,j=1}^n$ with $|a_{ii}| = 1$ for $1 \leq i \leq n$ and $a_{ij} = 0$ otherwise.*

Notation 1.2. *Given an orthonormal basis $\{e_i\}_{i\in I}$ for a Hilbert space $\mathbb{H}$, a* **k diagonal decomposition of the identity** *(k-d.d. for short) is a family of diagonal projections with disjoint ranges and*

$$\sum_{i=1}^{k} P_i = I.$$

Notation 1.3. *If A is an $n \times n$ matrix, define*

$$\alpha_k(A) = min\{max\ ||P_iAP_i|| : \{P_i\}_{i=1}^k \text{ is a k-d.d.}\}$$

and if A is an infinite matrix bounded in the operator norm, define

$$\alpha_k(A) = inf\{max\ ||P_iAP_i|| : \{P_i\}_{i=1}^k \text{ is a k-d.d.}\}.$$

If $A \neq 0$, in both cases define

$$\tilde{\alpha}_k(A) = \frac{\alpha_k(A)}{\|A\|}.$$

2. The Paving Conjecture

Anderson's Paving Conjecture (PC) [2]

For every $\epsilon > 0$, there is some "universal" positive integer $k = k(\epsilon)$ so that for every zero-diagonal finite matrix A, there exists a k-d.d. $\{P_i\}_{i=1}^k$ so that for all $i = 1, 2, \cdots, k$ we have

$$\|P_iAP_i\| \le \epsilon\|A\|. \tag{2.1}$$

When we have the inequality in Equation 2.1 for a matrix A, we say that A is (k, ϵ)**-pavable**.

Given a fixed matrix A, if for every $\epsilon < 1$ there is a $k \in \mathbb{N}$ so that A is (k, ϵ)-pavable then we say that A is **pavable**. If we have a class of matrices, we say this **class is pavable** (respectively, (k, ϵ)**-pavable**) if every member of the class is pavable (respectively, $(k\epsilon)$-pavable).

Paving for an arbitrary matrix A means paving for $A - E(A)$ where $E(A)$ denotes the diagonal of A with respect to a fixed basis. A simple iteration argument shows that PC is equivalent to the existence of a universal k working for just one fixed $\epsilon < 1$.

There are a number of restricted classes of matrices for which paveability with respect to any fixed basis is equivalent to PC:

1. Self-adjoint matrices [5, 6].
2. Unitary operators [5, 6].
3. Positive operators [5, 6].
4. Invertible operators (or invertible operators with zero diagonal) [5, 6]
5. Orthogonal projections [5, 6]
6. Gram matrices [5, 6].
7. Lower (respectively upper) triangular matrices [7].

3. Paving Conjectures Equivalent to PC

Conjecture 3.1. *For every zero-diagonal matrix A (finite or infinite) there is a k and $\epsilon < 1$ so that A is (k,ϵ)-pavable. I.e., For every zero-diagonal matrix A we have $\tilde{\alpha}_k(A) < 1$ for some k.*

The equivalence of Conjecture 3.1 and PC is a simple diagonal process. Note that Conjecture 3.1 does not require the universality of k and the bound below one depends on A.

Conjecture 3.2. *There is a universal k so that for all zero-diagonal matrices A (finite and infinite), there is a $\epsilon < 1$ so that A is (k,ϵ)-pavable. (I.e., For every zero-diagonal matrix A we have $\tilde{\alpha}_k(A) < 1$.*

Note: The rest of the conjectures in this section require universal constants k and $\epsilon < 1$ which are independent of $n \in \mathbb{N}$ and the matrix.

Conjecture 3.3. [4] *There exists an $\epsilon < 1$ and a natural number k so that all orthogonal projections A on ℓ_2^{2n} with $1/2's$ on the diagonal are (k,ϵ)-pavable.*

Note that Conjecture 3.3 does not require zero-diagonal. In [4] it is shown that Conjecture 3.3 fails for $k = 2$. I.e. There is no $\epsilon < 1$ so that this class of projections is $(2,\epsilon)$-pavable.

Conjecture 3.4. [4] *There is a universal k and an $\epsilon < 1$ so that every norm one self-adjoint operator U with zero diagonal and satisfying $U^2 = I$ is (k,ϵ)-pavable.*

Conjecture 3.5. [9] *There exist universal constants $0 < \delta, \epsilon < 1$ and $k \in \mathbb{N}$ so that all orthogonal projections P on ℓ_2^n with $\delta(P) \le \delta$ are (k,ϵ)-pavable.*

Weaver also posed an equal norm version of this conjecture.

Conjecture 3.6. [9] *There exist universal constants $0 < \delta, \sqrt{\delta} \le \epsilon < 1$ and $k \in \mathbb{N}$ so that all orthogonal projections P on ℓ_2^n with $\delta(P) \le \delta$ and $\|Pe_i\| = \|Pe_j\|$ for all $i, j = 1, 2, \cdots, n$ are (k,ϵ)-pavable.*

4. Related Problems

Problem 4.1. *If PC is true, then are the upper triangular invertibles on $\ell_2(\mathbb{N})$ path connected? (Larson, Paulsen, Orr, Weiss, Zhang).*

The problem of the connectedness of the upper triangular invertible matrices is well-known in the non-self-adjoint operator community.

Problem 4.2. *Are there analogs for Problem 4.1 in other classes of operators such as Laurent, Toeplitz, analytic toeplitz, etc.? (Weiss).*

Problem 4.3. *Are the Laurent operators paveable? (Bourgain-Tzafriri* [3], *Halpern, Kaftal and Weiss* [8]*).*

Problem 4.4. *Are all Laurent operators with H^∞-symbol paveable? (Paulsen, Weiss).*

Problem 4.5. *Is PC equivalent to PC for some k-d.d. $\{P_i\}_{i=1}^k$ where*

$$|rank\ P_i - rank\ P_j| \leq 1, \quad for\ all\ 1 \leq i, j \leq k?$$

(Casazza, Edidin, Weiss).

5. The Akemann-Anderson Conjecture

Akemann and Anderson [1] posed a conjecture which would imply a positive solution to KS.

Conjecture B ([1], 7.1.3) There exists $\gamma, \epsilon > 0$ (and independent of n) such that for any projection P with $\delta_p < \gamma$ there is a diagonal symmetry S such that $\|PSP\| < 1 - \epsilon$.

Conjecture B is still open despite considerable effort having been expended on it. Weaver [10] states that a counterexample to Conjecture B would probably lead to a negative solution to the Paving Conjecture.

References

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