Paving Small Matrices and The Kadison-Singer Extension Problem AIM Workshop Notes

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CHAPTER 1

Notation

$\mathbb{M}_n = n \times n$ complex matrices
$\mathbb{M}_n^0 = n \times n$ complex matrices with zero diagonal
$\mathbb{M}_{n,sa} = n \times n$ selfadjoint complex matrices
$\mathbb{M}_{n,sa}^0 = n \times n$ selfadjoint complex matrices with zero diagonal
$\mathbb{M}_{n,sym} = n \times n$ real symmetric matrices
$\mathbb{M}_{n,sym}^0 = n \times n$ real symmetric matrices with zero diagonal
$\mathbb{M}_{n,++} = n \times n$ non-negative matrices
$\mathbb{M}_{n,++}^0 = n \times n$ non-negative matrices with zero diagonal
$\mathbb{D}_n = n \times n$ diagonal matrices

If $A \in \mathbb{M}_n$, define

$$\alpha_k(A) = \underset{\text{diagonal projections } P_1+\cdots+P_k=I_n}{min} \quad \underset{1 \le j \le k}{max} \; ||P_j A P_j||$$

If $0 \neq A \in \mathbb{M}_n$, define

$$\tilde{\alpha}_k(A) = \frac{\alpha_k(A)}{\|A\|}.$$

If $\mathcal{S} \subset \mathbb{M}_n$, define

$$\tilde{\alpha}_k(\mathcal{S}) = \sup_{0 \neq A \in \mathcal{S}} \tilde{\alpha}_k(A).$$

CHAPTER 2

2-Pavings

THEOREM 2.1 (2-pavings).

n	$\tilde{\alpha}_2(\mathbb{M}_n^0)$	$\tilde{\alpha}_2(\mathbb{M}_{n,sa}^0)$	$\tilde{\alpha}_2(\mathbb{M}_{n,sym}^0)$
3	1	$\frac{1}{\sqrt{3}}$ 0.5773	$\frac{1}{2}$ 0.5000
4	″	″	$[?, \frac{1}{\sqrt{3}}]$ $[0.5493, 0.5773]$
5	″	$\frac{2}{\sqrt{5}}$ 0.8944	$\frac{2}{\sqrt{5}}$ 0.8944

1. Selfadjoint

PROPOSITION 2.2 (3×3 selfadjoint). $\tilde{\alpha}_2(\mathbb{M}_{3,sa}^0) = \frac{1}{\sqrt{3}} \approx 0.5773$.

PROOF. Suppose

$$A = \begin{bmatrix} 0 & a & b \\ \overline{a} & 0 & c \\ \overline{b} & \overline{c} & 0 \end{bmatrix} \in \mathbb{M}_{3,sa}^0 \text{ with } \alpha_2(A) = 1.$$

Then $|a|, |b|, |c| \geq 1$. By the Universal Selfadjoint 3-Identity (Lemma 7.1),

$$1 = \frac{|a|^2 + |b|^2 + |c|^2}{\|A\|^2} + \frac{2|\operatorname{Re}(a\overline{b}c)|}{\|A\|^3} \geq \frac{3}{\|A\|^2}.$$

Thus, $\|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{3}}$. This bound is attained by

$$A = \begin{bmatrix} 0 & 1 & i \\ 1 & 0 & 1 \\ -i & 1 & 0 \end{bmatrix}$$

because $\alpha_2(A) = 1$ and $\|A\| = \sqrt{3}$ by Corollary 7.2. □

PROPOSITION 2.3 (4×4 selfadjoint). $\tilde{\alpha}_2(\mathbb{M}_{4,sa}^0) = \frac{1}{\sqrt{3}}$.

PROOF. Suppose $A \in \mathbb{M}_{4,sa}^0$, with $\alpha_2(A) = 1$. Create a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if $|a_{ij}| < 1$. We have the following axioms:

(1) $G11$ is not a subgraph of G. Otherwise, A admits a 2-2 paving of norm < 1, violating the assumption $\alpha_2(A) = 1$.
(2) For all i, the degree of i is greater than 0. Otherwise, row i of A has three entries of absolute value $\geq 1 \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}}$.
(3) By removing a vertex from G, one cannot arrive at $G4$. Otherwise, A has a 3-compression of norm $\geq \sqrt{3} \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{3}}$.

This exhausts all possible 4-graphs and hence proves the inequality. □

PROPOSITION 2.4 (5×5 selfadjoint). *Let* $\tilde{\alpha}_2(\mathbb{M}^0_{5,sa}) = \frac{2}{\sqrt{5}} \approx 0.8944$.

PROOF. Suppose $A \in \mathbb{M}^0_{5,sa}$, with $\alpha_2(A) = 1$. Create a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4, 5\}$ and $(i, j) \in E$ if $|a_{ij}| < 1$. We may assume the following axiom:

(1) For all i, $deg(i) \geq 3$. Otherwise, row i of A has at least two entries of absolute value $\geq 1 \Rightarrow \|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{2}} \approx 0.7071$.

This leaves graphs $G50$, $G51$, and $G52$.

Case $G50$: Only two 2-compressions have norm ≥ 1, and they are disjoint. Without loss of generality, $\|A_{12}\|, \|A_{34}\| \geq 1$. We claim that every 3-compression has norm ≥ 1. Indeed, $\|A_{125}\| \geq \|A_{12}\| \geq 1$, $\|A_{345}\| \geq \|A_{34}\| \geq 1$, and the remaining 3-compressions have norm ≥ 1 because their complementary 2-compressions have norm < 1. It follows that $\|A\| \geq \frac{\sqrt{5}}{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{2}{\sqrt{5}}$.

Case $G51$: Only one 2-compression has norm ≥ 1. Without loss of generality, $\|A_{12}\| \geq 1$. It follows that

$$\begin{aligned}
\|A\|^2 &\geq \frac{1}{4}\|A\|^2_{HS} \\
&= \frac{1}{4}\left[\|A_{12}\|^2_{HS} + \frac{1}{2}\sum_{1\in B, 2\notin B}^{3} \|B\|^2_{HS} + \frac{1}{2}\sum_{2\in B, 1\notin B}^{3} \|B\|^2_{HS}\right] \\
&\geq \frac{1}{4}\left[2 + \frac{1}{2}\cdot 3 \cdot \frac{3}{2} + \frac{1}{2}\cdot 3\cdot\frac{3}{2}\right] = \frac{13}{8}.
\end{aligned}$$

Thus, $\|A\| \geq \sqrt{\frac{13}{8}} \Rightarrow \tilde{\alpha}_2(A) \leq \sqrt{\frac{8}{13}} \approx 0.7845$.

Case $G52$: Every 2-compression has norm $< 1 \Rightarrow$ every 3-compression has norm ≥ 1 $\Rightarrow \|A\| \geq \frac{\sqrt{5}}{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{2}{\sqrt{5}}$.

The matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 \end{bmatrix}$$

shows that the inequality is sharp. The unimodular circulant

$$B = \begin{bmatrix} 0 & e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} \\ e^{-2\pi i/5} & 0 & e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} \\ e^{\pi i/5} & e^{-2\pi i/5} & 0 & e^{2\pi i/5} & e^{-\pi i/5} \\ e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} & 0 & e^{2\pi i/5} \\ e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} & 0 \end{bmatrix}$$

also works. Note: A and B are unitarily equivalent. □

Alternate Proof. Suppose $A \in \mathbb{M}^0_{5,sa}$, with $\alpha_2(A) = 1$.

(1) Assume that all 3-compressions of A have norm ≥ 1. Then $\tilde{\alpha}_2(A) \leq \frac{2}{\sqrt{5}}$ (see the previous proof).

(2) Assume that exactly one 3-compression, say A_{345}, has norm < 1, then $\|A_{12}\| \geq 1 \Rightarrow \tilde{\alpha}_2(A) \leq \sqrt{\frac{8}{13}}$ (see the previous proof).

(3) Assume that exactly two 3-compressions have norm < 1. We may assume that the complementary 2-compressions are disjoint. Otherwise, $\|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{2}}$. Without loss of generality, $\|A_{12}\|, \|A_{34}\| \geq 1$ and $\|A_{345}\|, \|A_{125}\| < 1$. This is a contradiction.

(4) Assume that more than two 3-compressions have norm < 1. Then their complementary 2-compressions cannot be disjoint. Thus, $\|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{2}}$.

$\square$

2. Real Symmetric

Proposition 2.5 (3×3 real symmetric). $\tilde{\alpha}_2(\mathbb{M}^0_{3,sym}) = \frac{1}{2}$.

Proof. Suppose

$$A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \in \mathbb{M}^0_{3,sym} \text{ with } \alpha_2(A) = 1.$$

Then $|a|, |b|, |c| \geq 1$. By the Universal Selfadjoint 3-Identity (Lemma 7.1),

$$1 = \frac{a^2 + b^2 + c^2}{\|A\|^2} + \frac{2|abc|}{\|A\|^3} \geq \frac{3}{\|A\|^2} + \frac{2}{\|A\|^3}$$

which implies $\|A\| \geq 2$, hence $\tilde{\alpha}_2(A) \leq \frac{1}{2}$. This bound is attained by

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in \mathbb{M}^0_{3,sym}$$

since $\alpha_2(A) = 1$ and $\|A\| = 2$ by Corollary 7.2. □

Lemma 2.6. *Let*

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & d & e \\ 1 & d & 0 & f \\ 1 & e & f & 0 \end{bmatrix} \in \mathbb{M}^0_{4,sym}.$$

If

$$\left\| \begin{bmatrix} 0 & d & e \\ d & 0 & f \\ e & f & 0 \end{bmatrix} \right\| \geq 1,$$

then $\|A\| \geq (9.75)^{1/4} \approx 1.767$.

Proof. Let $x = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and

$$B = \begin{bmatrix} 0 & d & e \\ d & 0 & f \\ e & f & 0 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} 0 & x \\ x^* & B \end{bmatrix} \Rightarrow A^*A = \begin{bmatrix} xx^* & xB \\ B^*x^* & x^*x + B^*B \end{bmatrix}.$$

Thus

$$\begin{aligned} \|A\|^4 &= \|A^*A\|^2 \geq \left\| \begin{bmatrix} xx^* & xB \end{bmatrix} \right\|^2 \\ &= 9 + (d+e)^2 + (d+f)^2 + (e+f)^2. \end{aligned}$$

We claim that

$$(d+e)^2 + (d+f)^2 + (e+f)^2 \geq d^2 + e^2 + f^2.$$

Indeed, let $F(d,e,f) = (d+e)^2 + (d+f)^2 + (e+f)^2$ and $G(d,e,f) = d^2 + e^2 + f^2$. Using the Method of Lagrange Multipliers, we minimize $F(d,e,f)$ subject to the constraint $G(d,e,f) = r^2$:

$$
\begin{aligned}
2(d+e)+2(d+f) &= 2\lambda d \\
2(d+e)+2(e+f) &= 2\lambda e \\
2(d+f)+2(e+f) &= 2\lambda f \\
\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} &= \lambda \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\
\Rightarrow \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} x \\ x \\ x \end{bmatrix} \text{ or } \begin{bmatrix} d \\ e \\ f \end{bmatrix} &= \begin{bmatrix} x+y \\ x-y \\ -2x \end{bmatrix}.
\end{aligned}
$$

In the former case,

$$3x^2 = d^2+e^2+f^2 = r^2 \Rightarrow (d+e)^2+(d+f)^2+(e+f)^2 = 12x^2 = 4r^2.$$

In the later case,

$$
\begin{aligned}
(x+y)^2+(x-y)^2+(-2x)^2 = d^2+e^2+f^2 = r^2 \\
\Rightarrow (d+e)^2+(d+f)^2+(e+f)^2 = (2x)^2+(-x+y)^2+(-x-y)^2 = r^2.
\end{aligned}
$$

Thus, $r^2 \le (d+e)^2+(d+f)^2+(e+f)^2 \le 4r^2$, which proves the claim. Now

$$\|B\| \ge 1 \Rightarrow \|B\|_{HS}^2 \ge 1.5 \Rightarrow d^2+e^2+f^2 \ge 0.75.$$

Hence, $\|A\|^4 \ge 9.75$, which proves the lemma. □

PROPOSITION 2.7 (4×4 real symmetric). $\tilde{\alpha}_2(\mathbb{M}^0_{4,sym}) \in [0.5493, 0.5773]$.

PROOF. Suppose $A \in \mathbb{M}^0_{4,sym}$, with $\alpha_2(A) = 1$. Create a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if $|a_{ij}| < 1$. We have the following axioms:

(1) $G11$ is not a subgraph of G. Otherwise, A admits a 2-2 paving of norm < 1, violating the assumption $\alpha_2(A) = 1$.
(2) By removing a vertex from G, one cannot arrive at $G4$. Otherwise, A has a 3-compression of norm $\ge 2 \Rightarrow \|A\| \ge 2 \Rightarrow \tilde{\alpha}_2(A) \le \frac{1}{2}$.

This leaves only graph $G12$. Thus,

$$A = \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & d & 0 & f \\ c & e & f & 0 \end{bmatrix},$$

where $|a|, |b|, |c| \ge 1$, $|d|, |e|, |f| < 1$, and

$$\left\| \begin{bmatrix} 0 & d & e \\ d & 0 & f \\ e & f & 0 \end{bmatrix} \right\| \ge 1.$$

Lower bound:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -0.3946 & 0.6854 \\ 1 & -0.3946 & 0 & -0.3986 \\ 1 & 0.6854 & -0.3986 & 0 \end{bmatrix}.$$

□

CHAPTER 3

3-Pavings

In 1987 the 3-paving problem was posed: whether or not 3-pavings suffice for Anderson's Paving Conjecture and hence for Kadison-Singer. To date we have heard of no refutation to this. Recall also the $\frac{2}{3}$-challenge from then: whether or not $\tilde{\alpha}_3(\mathbb{M}_n^0) \leq \frac{2}{3}$, which the following table refutes.

THEOREM 3.1 (3-pavings).

n	$\tilde{\alpha}_3(\mathbb{M}_n^0)$	$\tilde{\alpha}_3(\mathbb{M}_{n,sa}^0)$	$\tilde{\alpha}_3(\mathbb{M}_{n,++}^0)$
4	$\boldsymbol{\frac{2}{1+\sqrt{5}}}$ 0.6180	$\frac{1}{\sqrt{3}}$ 0.5773	κ 0.5550
5	″	″	$[\kappa, \frac{2}{1+\sqrt{5}}]$ $[0.5550, 0.6180]$
6	$\boldsymbol{\frac{1}{\sqrt{2}}}$ **0.7071**	″	″
7	$[?, 1)$ $\mathbf{[0.8231, 1)}$	$\boldsymbol{[\frac{2}{3}, \frac{2}{\sqrt{7}})}$ $\mathbf{[0.6667, 0.7559)}$	$[\kappa, \frac{2}{3}]$ $[0.5550, 0.6667]$
8	$[?, 1]$ $[0.8231, 1]$	$\boldsymbol{[\frac{2}{3}, \frac{2}{\sqrt{5}}]}$ $\mathbf{[0.6667, 0.8944]}$	″
10	″	$\boldsymbol{[\frac{\sqrt{5}}{3}, 1]}$ $\mathbf{[0.7454, 1]}$	″

where

$$\kappa = \sqrt{\frac{3}{5 + 2\sqrt{7}\cos(\tan^{-1}(3\sqrt{3})/3)}},$$

boldface signifies what we feel are the most interesting facts, "?" signifies a lack of a closed form, and " signifies "ditto from above".

1. General

LEMMA 3.2. *Let*

$$A = \begin{bmatrix} r_1 e^{i\theta_1} & r_2 e^{i\theta_2} \\ 0 & r_3 e^{i\theta_3} \end{bmatrix} \in \mathbb{M}_2 .$$

Then there exist unitaries $U, V \in \mathbb{D}_2$ such that

$$UAV = \begin{bmatrix} r_1 & r_2 \\ 0 & r_3 \end{bmatrix}.$$

PROOF. Let

$$U = \begin{bmatrix} e^{-i\theta_2} & 0 \\ 0 & e^{-i\theta_3} \end{bmatrix}, V = \begin{bmatrix} e^{i(\theta_2-\theta_1)} & 0 \\ 0 & 1 \end{bmatrix}.$$

□

COROLLARY 3.3. *Let*

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathbb{M}_2 .$$

If $|a|, |b|, |c| \geq 1$, then $\|A\| \geq \frac{1+\sqrt{5}}{2}$.

PROOF. By the previous lemma,

$$\|A\| = \left\| \begin{bmatrix} |a| & |b| \\ 0 & |c| \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\| = \frac{1+\sqrt{5}}{2}.$$

□

PROPOSITION 3.4 (4×4 general). $\tilde{\alpha_3}(\mathbb{M}_4^0) = \frac{2}{1+\sqrt{5}} \approx 0.6180.$

PROOF. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & -\frac{2}{1+\sqrt{5}} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}_4^0 .$$

Then $\tilde{\alpha}_3(A) = \frac{2}{1+\sqrt{5}}$ ($\alpha_3(A) = 1$ and $\|A\| = \frac{1+\sqrt{5}}{2}$ by applying to the upper-right 3×3 corner either Parrott's Completion Lemma with Formula, or factoring the characteristic polynomial of the square of its absolute value, or Matlab).

Now suppose $A \in \mathbb{M}_4^0$, with $\alpha_3(A) = 1$. Create a digraph $D = (V, E)$ as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if $|a_{ij}| \geq 1$. We may assume the following axioms:

(1) For all $i \neq j$, either $(i, j) \in E$ or $(j, i) \in E$. Otherwise A admits a 1-1-2 paving of norm < 1, violating the assumption $\alpha_3(A) = 1$.
(2) For all i, the in-degree of i and the out-degree of i are less than 3. Otherwise, $\|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \approx 0.5774$.

This leaves only digraphs $D149$, $D185$, $D186$, and $D218$ as labeled in [**1**]. Now each of these digraphs has $D12$ as a subgraph [ibid.]. Thus, $\|A\| \geq \frac{1+\sqrt{5}}{2}$ (Corollary 3.3) $\Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1+\sqrt{5}}$. □

Proposition 3.5 (5×5 general). $\tilde{\alpha}_3(\mathbb{M}_5^0) = \frac{2}{1+\sqrt{5}} \approx 0.6180$.

Proof. Clearly,

$$\tilde{\alpha}_3(\mathbb{M}_5^0) \geq \tilde{\alpha}_3(\mathbb{M}_4^0) = \frac{2}{1+\sqrt{5}}.$$

Now let $A \in \mathbb{M}_5^0$, with $\alpha_3(A) = 1$. Construct a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4, 5\}$ and $(i, j) \in E$ if $|a_{ij}|, |a_{ji}| < 1$. We may assume the following axioms:

(1) $G11$ is not a subgraph of G. Otherwise, G has a 1-2-2 paving of norm < 1, violating the fact that $\alpha_3(A) = 1$.
(2) By removing a vertex from G one cannot arrive at $G8$. Otherwise, there exists a 4-compression B of A such that $\alpha_3(B) \geq 1$. Since $\tilde{\alpha}_3(\mathbb{M}_4^0) = \frac{2}{1+\sqrt{5}}$, this would imply $\|B\| \geq \frac{1+\sqrt{5}}{2} \Rightarrow \|A\| \geq \frac{1+\sqrt{5}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1+\sqrt{5}}$.

This leaves $G23$. After permuting indices, we may assume that

$$A = \begin{bmatrix} 0 & s_{12} & s_{13} & b_{14} & b_{15} \\ s_{21} & 0 & s_{23} & b_{24} & b_{25} \\ s_{31} & s_{32} & 0 & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & 0 & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & 0 \end{bmatrix},$$

where $|s_{ij}| < 1$ and $\max\{|b_{ij}|, |b_{ji}|\} \geq 1\}$ for all $i \neq j$. Permuting the indices 4 and 5, if necessary, we may assume $|b_{45}| \geq 1$. If b_{51}, b_{52}, and b_{53} all have magnitude ≥ 1, then $\|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} < \frac{2}{1+\sqrt{5}}$. Thus, we may assume that one of them has magnitude $< 1 \Rightarrow$ either b_{15}, b_{25}, or b_{35} has magnitude ≥ 1. Permuting the indices 1, 2, and 3, if necessary, we may assume $|b_{35}| \geq 1$. If $|b_{34}| \geq 1$, then

$$\|A\| \geq \left\| \begin{bmatrix} b_{34} & b_{35} \\ 0 & b_{45} \end{bmatrix} \right\| \geq \frac{1+\sqrt{5}}{2}.$$

Likewise, if $|b_{43}| \geq 1$, then

$$\|A\| \geq \left\| \begin{bmatrix} 0 & b_{35} \\ b_{43} & b_{45} \end{bmatrix} \right\| \geq \frac{1+\sqrt{5}}{2}.$$

It follows that $\tilde{\alpha}_3(A) \leq \frac{2}{1+\sqrt{5}}$. □

Proposition 3.6 (6×6 general). $\tilde{\alpha}_3(\mathbb{M}_6^0) = \frac{1}{\sqrt{2}} \approx 0.7071$.

Proof. Construct a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4, 5, 6\}$ and $(i, j) \in E$ if $|a_{ij}|, |a_{ji}| < 1$. We may assume the following axioms:

(1) $G61$ is not a subgraph of G. Otherwise A would have a 2-2-2 paving of norm < 1, violating the fact that $\alpha_3(A) = 1$.
(2) By removing vertices from G, one cannot arrive at $G8$. Otherwise A would have a 4-compression B such that $\alpha_3(B) \geq 1$. Since $\tilde{\alpha}_3(\mathbb{M}_4^0) = \frac{2}{1+\sqrt{5}}$, this would imply $\|B\| \geq \frac{1+\sqrt{5}}{2} \Rightarrow \|A\| \geq \frac{1+\sqrt{5}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1+\sqrt{5}} < \frac{1}{\sqrt{2}}$.
(3) For all vertices i, $deg(i) \geq 3$. Otherwise, if $deg(i) \leq 2$, then either row i or column i of A would have at least two entries of magnitude $\geq 1 \Rightarrow \|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{2}}$.

This eliminates all graphs. Now let

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -\frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \in \mathbb{M}_6^0.$$

Then $\alpha_3(A) = 1$ and $A^*A = 2I$. □

Proposition 3.7 (7×7 general). $\tilde{\alpha}_3(\mathbb{M}_7^0) \in [0.8231, 1)$.

Proof. The following matrix was discovered by searching among 7×7 unitary circulants for bad pavers. The starting point for the search was a 7×7 unitary circulant with the eigenvalue distribution $(1, e^{\pi i/3}, e^{-\pi i/3}, i, -i, -1, -1)$.

$$A = \begin{bmatrix} 0 & a & b & c & d & e & f \\ f & 0 & a & b & c & d & e \\ e & f & 0 & a & b & c & d \\ d & e & f & 0 & a & b & c \\ c & d & e & f & 0 & a & b \\ b & c & d & e & f & 0 & a \\ a & b & c & d & e & f & 0 \end{bmatrix},$$

where

$$\begin{aligned} a &= -0.19104830537481 - 0.18571483276728i \\ b &= 0.03404378754044 + 0.00110165928527i \\ c &= -0.13926357252448 + 0.42165365488402i \\ d &= 0.21474405201775 - 0.42217403069332i \\ e &= -0.28337369310887 - 0.48101315713848i \\ f &= 0.29151538363540 - 0.33115367910212i. \end{aligned}$$

Then $\alpha_3(A) = 0.82305627367962$ and $A^*A = I$, i.e. $\tilde{\alpha}_3(A) = 0.82305627367962$.

It remains to show that $\tilde{\alpha}_3(\mathbb{M}_7^0) \neq 1$. To that end, let $A \in \mathbb{M}_7^0$, with $\alpha_3(A) = 1$. If every 3-compression of A has norm ≥ 1, then $\|A\| > 1$ (Corollary 7.10). If, on the other hand, some 3-compression of A has norm < 1, then the complementary 4-compression B satisfies $\alpha_2(B) \geq 1$. In particular, every 2-2 paving of B has norm ≥ 1. By Lemma 7.11, we may assume that

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & * & * \\ * & 0 & 0 & 0 & * & 0 & * \\ * & 0 & 0 & 0 & * & * & 0 \end{bmatrix},$$

where $|a| = |b| = |c| = 1$ and $\|A_{567}\| < 1$. Since $\|A_{12}\| = \|A_{35}\| = 0$, $\|A_{467}\| = 1 \Rightarrow \|A_{67}\| = 1 \Rightarrow \|A_{567}\| = 1$, a contradiction. □

2. Selfadjoint

PROPOSITION 3.8 (4×4 selfadjoint). $\tilde{\alpha}_3(\mathbb{M}^0_{4,sa}) = \frac{1}{\sqrt{3}} \approx 0.5773$.

PROOF. Suppose $A \in \mathbb{M}^0_{4,sa}$, with $\alpha_3(A) = 1$. Then $|a_{ij}| \geq 1$ for all $i \neq j$. Thus, $\|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}}$. Now let

$$A = \begin{bmatrix} 0 & i & 1 & 1 \\ -i & 0 & 1 & -1 \\ 1 & 1 & 0 & i \\ 1 & -1 & -i & 0 \end{bmatrix} \in \mathbb{M}^0_{4,sa}\,.$$

Then $\tilde{\alpha}_3(A) = \frac{1}{\sqrt{3}}$ ($\alpha_3(A) = 1$ and $A^*A = 3I$). □

PROPOSITION 3.9 (5×5 selfadjoint). $\tilde{\alpha}_3(\mathbb{M}^0_{5,sa}) = \frac{1}{\sqrt{3}}$.

PROOF. Clearly,

$$\tilde{\alpha}_3(\mathbb{M}^0_{5,sa}) \geq \tilde{\alpha}_3(\mathbb{M}^0_{4,sa}) = \frac{1}{\sqrt{3}}.$$

Now let $A \in \mathbb{M}^0_{5,sa}$, with $\alpha_3(A) = 1$. Construct a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4, 5\}$ and $(i, j) \in E$ if $|a_{ij}| < 1$ ($\Rightarrow |a_{ji}| < 1$). We may assume the following axioms:

(1) $G11$ is not a subgraph of G. Otherwise, A would have a 1-2-2 paving of norm < 1, violating the assumption $\alpha_3(A) = 1$.
(2) By removing a vertex from G, one cannot arrive at $G8$. Otherwise, A would have a 4-compression B such that $\alpha_3(B) \geq 1$. Since $\tilde{\alpha}_3(\mathbb{M}^0_{4,sa}) = \frac{1}{\sqrt{3}}$, this would imply $\|B\| \geq \sqrt{3} \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}}$.
(3) For every vertex i, $deg(i) \geq 2$. Otherwise, if $deg(i) \leq 1$, then row i of A has at least three entries of magnitude $\geq 1 \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}}$.

This eliminates all graphs. □

PROPOSITION 3.10 (6×6 selfadjoint). $\tilde{\alpha}_3(\mathbb{M}^0_{6,sa}) = \frac{1}{\sqrt{3}}$.

PROOF. Clearly,

$$\tilde{\alpha}_3(\mathbb{M}^0_{6,sa}) \geq \tilde{\alpha}_3(\mathbb{M}^0_{5,sa}) = \frac{1}{\sqrt{3}}.$$

Now let $A \in \mathbb{M}^0_{6,sa}$, with $\alpha_3(A) = 1$. Construct a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4, 5, 6\}$ and $(i, j) \in E$ if $|a_{ij}| < 1$ ($\Rightarrow |a_{ji}| < 1$). We may assume the following axioms:

(1) $G61$ is not a subgraph of G. Otherwise, A would have a 2-2-2 paving of norm < 1, violating the assumption $\alpha_3(A) = 1$.
(2) By removing a vertices from G, one cannot arrive at $G8$. Otherwise, A would have a 4-compression B such that $\alpha_3(B) \geq 1$. Since $\tilde{\alpha}_3(\mathbb{M}^0_{4,sa}) = \frac{1}{\sqrt{3}}$, this would imply $\|B\| \geq \sqrt{3} \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}}$.
(3) For every vertex i, $deg(i) \geq 3$. Otherwise, if $deg(i) \leq 2$, then row i of A has at least three entries of magnitude $\geq 1 \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}}$.

This eliminates all graphs. □

Preliminaries for 7 × 7 Selfadjoints

Notation: $F = \left[1 - \delta_{ij}\right] \in \mathbb{M}_{n,sa}^0$ (the "fat" operator)

LEMMA 3.11. *Let $0 \neq A \in \mathbb{M}_{n,sa}^0$. Then the following are equivalent:*

i. $\|A\|^2 = \frac{n-1}{n}\|A\|_{HS}^2$.

ii. *There exists a nonzero $\alpha \in \mathbb{R}$ such that*

$$\sigma(\alpha^{-1}A) = \left(1, \overbrace{-\frac{1}{n-1}, -\frac{1}{n-1}, ..., -\frac{1}{n-1}}^{n-1}\right).$$

iii. *There exists a diagonal unitary $U \in \mathbb{D}_n$ and a nonzero $\beta \in \mathbb{R}$ such that*

$$U^*AU = \beta F.$$

PROOF. (i $\Leftrightarrow$ ii): We have seen that $\|A\|^2 = \frac{n-1}{n}\|A\|_{HS}^2$ if and only if

$$\sigma(A) = \pm\|A\|\left(1, -\frac{1}{n-1}, -\frac{1}{n-1}, ..., -\frac{1}{n-1}\right).$$

(ii $\Leftrightarrow$ iii): Set $\tilde{A} = \alpha^{-1}A$. If $\sigma(\tilde{A}) = \left(1, -\frac{1}{n-1}, -\frac{1}{n-1}, ..., -\frac{1}{n-1}\right)$, then there exists a unitary $U \in \mathbb{M}_n$ such that

$$\tilde{A} = V \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & -\frac{1}{n-1} & 0 & ... & 0 \\ 0 & 0 & -\frac{1}{n-1} & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & -\frac{1}{n-1} \end{bmatrix} V^*.$$

Letting v stand for the first column of V, we have that

$$A = \frac{n}{n-1}vv^* - \frac{1}{n-1}I = \left[\tfrac{n}{n-1}v_i\overline{v_j} - \tfrac{1}{n-1}\delta_{ij}\right].$$

Since $\tilde{A} \in \mathbb{M}_{n,sa}^0$,

$$\frac{n}{n-1}|v_i|^2 - \frac{1}{n-1} = 0 \Rightarrow v_i = \frac{1}{\sqrt{n}}e^{i\theta_i}$$

for some $\theta_i \in \mathbb{R}$. It follows that

$$\tilde{A} = \frac{1}{n-1}\left[e^{i(\theta_i - \theta_j)} - \delta_{ij}\right] = \frac{1}{n-1}UFU^*,$$

where

$$U = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n}) \in \mathbb{D}_n.$$

Thus, $U^*AU = \beta F$, where $\beta = \frac{\alpha}{n-1}$. (iii $\Rightarrow$ ii): Clearly

$$F = nE - I,$$

where all the off-diagonal entries of $E \in \mathbb{M}_n$ equal $\frac{1}{n}$. Since E is a rank-one projection,

$$\sigma(F) = (n-1, -1, -1, ..., -1).$$

The result follows. $\square$

LEMMA 3.12. *Let $0 \neq A \in \mathbb{M}^0_{n,sa}$. Fix $k \geq 3$ and assume $\|B\|^2 = \frac{k-1}{k}\|B\|^2_{HS}$ for all k-compressions B of A. Then there exists a diagonal unitary $U \in \mathbb{D}_n$ and an $\alpha > 0$ such that*

$$U^*AU = \alpha S,$$

where all the off-diagonal entries of $S \in \mathbb{M}^0_{n,sa}$ equal ± 1.

PROOF. Let B be a k-compression of A. By Lemma 3.11, all the off-diagonal entries of B have the same modulus. It follows that all the off-diagonal entries of A have the same modulus, say α (here we use $k \geq 3$). Set $C = \alpha^{-1}A$. Then all the off-diagonal entries of C have modulus 1, and $\|B\|^2 = \frac{k-1}{k}\|B\|^2_{HS}$ for all k-compressions B of C. We claim that $c_{rs}c_{st} = \pm c_{rt}$ for all $r < s < t$. Indeed, this follows from Lemma 3.11 applied to any k-compression B of C containing r, s, and t (again we use $k \geq 3$). Now let $\phi_1, \phi_2, ..., \phi_{n-1} \in \mathbb{R}$ be such that $c_{i,i+1} = e^{i\phi_i}$, $i = 1, 2, ..., n-1$. For $j = 1, 2, ..., n$, define $\theta_j = -\sum_{i=1}^{j-1}\phi_i$. We claim that

$$c_{rs} = \pm e^{i(\theta_r - \theta_s)}, \ r < s.$$

Indeed,

$$\begin{aligned} c_{rs} &= \pm c_{r,r+1}c_{r+1,r+2} \cdot s c_{s-1,s} = \pm e^{i\phi_r}e^{i\phi_{r+1}} \cdots e^{i\phi_{s-1}} \\ &= \pm e^{i\sum_{i=r}^{s-1}\phi_i} = \pm e^{i\left(\sum_{i=1}^{s-1}\phi_i - \sum_{i=1}^{r-1}\phi_i\right)} = \pm e^{i(\theta_r - \theta_s)}. \end{aligned}$$

Setting

$$U = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n}) \in \mathbb{D}_n,$$

we have that $U^*CU = S \in \mathbb{M}^0_{n,sa}$, where all the off-diagonal entries of S are ± 1. $\square$

PROPOSITION 3.13 (**7×7 selfadjoint**). $\tilde{\alpha}_3(\mathbb{M}^0_{7,sa}) \in \left[\frac{2}{3}, \frac{2}{\sqrt{7}}\right) \approx [0.6667, 0.7559)$.

PROOF. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 0 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 0 \end{bmatrix} \in \mathbb{M}^0_{7,sa}.$$

Then $\tilde{\alpha}_3(A) = \frac{2}{3}$ ($\alpha_3(A) = 2$ and $\|A\| = 3$). Thus, $\tilde{\alpha}_3(\mathbb{M}_{7,sa}) \geq \frac{2}{3}$. Now let $A \in \mathbb{M}^0_{7,sa}$, with $\alpha_3(A) = 1$.

If every 3-compression B of selfadjoint A has norm ≥ 1, then $\|B\|_2^2 \geq \frac{3}{2}\|B\|^2$ by selfadjointness using Proposition 7.5 ($p = 2$, $n = 3$).
General identity: $\sum_B \|B\|^2_{HS} = 5\|A\|^2_{HS}$ by a counting argument.
From general selfadjoint trace zero inequality for odd rank: $\|A\|^2_{HS} \leq 6\|A\|^2$ by Corollary 7.4 ($n = 7$). Thus

$$35 \leq \sum_B \|B\|^2 \leq \frac{2}{3}\sum_B \|B\|^2_{HS} = \frac{10}{3}\|A\|^2_{HS} \leq 20\|A\|^2$$

and hence $\|A\| \geq \frac{\sqrt{7}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{\sqrt{7}}$.

That $\|A\| \geq \frac{\sqrt{7}}{2}$ is a special case of Corollary 7.6 ($n = 7,\ k = 3$), so the above internal proof of this can alternatively be referenced.

If, on the other hand, some 3-compression of A has norm < 1, then the complementary 4-compression B satisfies $\alpha_2(B) \geq 1$. Since $\tilde{\alpha}_2(\mathbb{M}^0_{4,sa}) = \frac{1}{\sqrt{3}}$, $\|B\| \geq \sqrt{3}$ $\Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} < \frac{2}{\sqrt{7}}$.

Now assume $\alpha_3(A) = 1$ and $\|A\| = \frac{\sqrt{7}}{2}$. By the previous discussion, every 3-compression B of A has norm ≥ 1. Thus

$$35 \leq \sum_B \|B\|^2 \leq \frac{2}{3} \sum_B \|B\|^2_{HS} = \frac{10}{3}\|A\|^2_{HS} \leq 20\|A\|^2 = 35.$$

It follows that $\|B\|^2 = \frac{2}{3}\|B\|^2_{HS}$ for all 3-compressions B of A. By Lemma **??**, there exists a diagonal unitary $U \in \mathbb{D}_n$ and an $\alpha > 0$ such that $U^*AU = \alpha S$, where all the off-diagonal entries of $S \in \mathbb{M}^0_{n,sa}$ are ± 1. Searching exhaustively among all such S, we see that $\tilde{\alpha}_3(A) \leq \frac{2}{3} < \frac{2}{\sqrt{7}}$, a contradiction. □

Proposition 3.14 (8 × 8 selfadjoint). $\tilde{\alpha}_3(\mathbb{M}^0_{8,sa}) \in \left[\frac{2}{3}, \frac{2}{\sqrt{5}}\right] \approx [0.6667, 0.8944]$.

Proof. Clearly,

$$\tilde{\alpha}_3(\mathbb{M}^0_{8,sa}) \geq \tilde{\alpha}_3(\mathbb{M}^0_{7,sa}) \geq \frac{2}{3}.$$

Now let $A \in \mathbb{M}^0_{8,sa}$, with $\alpha_3(A) = 1$. If every 3-compression of A has norm ≥ 1, then $\|A\| \geq \frac{\sqrt{7}}{2}$ (by proof of 3.13 every 7-compression has norm $\geq \frac{\sqrt{7}}{2}$) $\Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{\sqrt{7}} < \frac{2}{\sqrt{5}}$. If, on the other hand, some 3-compression of A has norm < 1, then the complementary 5-compression B satisfies $\alpha_2(B) \geq 1$. Since $\tilde{\alpha}_2(\mathbb{M}^0_{5,sa}) = \frac{2}{\sqrt{5}}$, $\|B\| \geq \frac{\sqrt{5}}{2} \Rightarrow \|A\| \geq \frac{\sqrt{5}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{\sqrt{5}}$. □

Proposition 3.15 (10 × 10 selfadjoint). $\tilde{\alpha}_3(\mathbb{M}^0_{10,sa}) \in \left[\frac{\sqrt{5}}{3}, 1\right] \approx [0.7454, 1]$.

Proof. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 0 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 0 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & 1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 0 \end{bmatrix} \in \mathbb{M}^0_{10,sa}.$$

Then $\tilde{\alpha}_3(A) = \frac{\sqrt{5}}{3}$ ($\alpha_3(A) = \sqrt{5}$ and $A^*A = 9I$). □

Remark: A is a *conference matrix*.

3. Nonnegative

LEMMA 3.16. *Let $A \in \mathbb{M}^0_{4,++}$. If $\alpha_3(A) = 1$ and a row or column of A has three entries ≥ 1, then $\|A\| \geq 2$. This inequality is sharp.*

PROOF. We may assume the first row of A has three entries ≥ 1. Then

$$\|A\| \geq \left\| \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & b_{23} & b_{24} \\ 0 & b_{32} & 0 & b_{34} \\ 0 & b_{42} & b_{43} & 0 \end{bmatrix} \right\|,$$

where $\max\{b_{ij}, b_{ji}\} \geq 1$ for all $i \neq j$. Since

$$\min \left\{ \left\| \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & \delta_{23} & \delta_{24} \\ 0 & 1-\delta_{23} & 0 & \delta_{34} \\ 0 & 1-\delta_{24} & 1-\delta_{34} & 0 \end{bmatrix} \right\| : \delta_{23}, \delta_{24}, \delta_{34} \in \{0,1\} \right\} = 2,$$

we have that $\|A\| \geq 2$. A sharp example is furnished by the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

□

PROPOSITION 3.17 (4 × 4 nonnegative). $\tilde{\alpha}_3(\mathbb{M}^0_{4,++}) = \kappa \approx 0.5550$.

PROOF. Suppose $A \in \mathbb{M}^0_{4,++}$, with $\alpha_3(A) = 1$. Create a digraph $D = (V, E)$ as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if $a_{ij} \geq 1$. We may assume the following axioms:

(1) For all $i \neq j$, either $(i, j) \in E$ or $(j, i) \in E$. Otherwise, A admits a 1-1-2 paving of norm < 1, violating the assumption $\alpha_3(A) = 1$.
(2) For all vertices i, the in-degree of i and the out-degree of i are less than 3. Otherwise, row i or column i of A has three entries $\geq 1 \Rightarrow \|A\| \geq 2$ (Lemma 3.16) $\Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{2} < \kappa$.

This leaves digraphs $D149$, $D185$, $D186$, and $D218$, which all have $D149$ as a subgraph. Thus,

$$\|A\| \geq \left\| \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\| = \frac{1}{\kappa} \Rightarrow \tilde{\alpha}_3(A) \leq \kappa.$$

Now let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\tilde{\alpha}_3(A) = \kappa \Rightarrow \tilde{\alpha}_3(\mathbb{M}^0_{4,++}) \geq \kappa$. □

PROPOSITION 3.18 (6×6 nonnegative). $\tilde{\alpha}_3(\mathbb{M}^0_{6,++}) \in \left[\kappa, \frac{2}{1+\sqrt{5}}\right] \approx [0.5550, 0.6180]$.

PROOF. Suppose $A \in \mathbb{M}^0_{6,++}$, with $\alpha_3(A) = 1$. Create a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4, 5, 6\}$ and $(i, j) \in E$ if $a_{ij}, a_{ji} < 1$. We may assume the following axioms:

(1) $G61$ is not a subgraph of G. Otherwise, A has a 2-2-2 paving of norm < 1, violating the assumption $\alpha_3(A) = 1$.
(2) By removing vertices, one cannot arrive at $G8$. Otherwise, A has a 4-compression B with $\alpha_3(B) \geq 1 \Rightarrow \|B\| \geq \frac{1}{\kappa} \Rightarrow \|A\| \geq \frac{1}{\kappa} \Rightarrow \tilde{\alpha}_3(A) \leq \kappa$.
(3) G has no isolated vertices. Otherwise, if vertex i is isolated, then either row i or column i of A has at least three entries $\geq 1 \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}}$.
(4) There does not exist a partition $V = \{i, j, k\} \bigsqcup \{i', j', k'\}$ such that $(r, s') \notin E$, $r, s \in \{i, j, k\}$. Otherwise, some 3×3 submatrix of A has at least five entries $\geq 1 \Rightarrow \|A\| \geq \frac{1}{\kappa} \Rightarrow \tilde{\alpha}_3(A) \leq \kappa$ (by exhaustive search of 0-1 3×3 matrices with five 1's).

This leaves $G114$ and $G133$, both of which have a 5-compression of the form

$$\begin{bmatrix} 0 & * & * & * & * \\ * & 0 & * & * & * \\ * & * & 0 & \cdot & \cdot \\ * & * & \cdot & 0 & \cdot \\ * & * & \cdot & \cdot & 0 \end{bmatrix},$$

where a "$*$" in the (i, j) position indicates that $a_{ij} \geq 1$ or $a_{ji} \geq 1$, and a "$\cdot$" in the (i, j) position indicates that $a_{ij} < 1$. Searching exhaustively over all 0-1 5×5 matrices satisfying this pattern yields $\|A\| \geq \frac{1+\sqrt{5}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1+\sqrt{5}}$. □

CHAPTER 4

2,3-Pavings Summary Table

n	$\alpha_2(\mathbb{M}_n^0)$	$\alpha_2(\mathbb{M}_{n,sa}^0)$	$\alpha_2(\mathbb{M}_{n,sym}^0)$	$\alpha_3(\mathbb{M}_n^0)$	$\alpha_3(\mathbb{M}_{n,sa}^0)$	$\alpha_3(\mathbb{M}_{n,++}^0)$
3	1	$\frac{1}{\sqrt{3}}$ .5773	$\frac{1}{2}$ .5000	0	0	0
4	″	″	$[?, \frac{1}{\sqrt{3}}]$ $[.5493, .5773]$	$\mathbf{\frac{2}{1+\sqrt{5}}}$ **.6180**	$\frac{1}{\sqrt{3}}$ .5773	κ .5550
5	″	$\frac{2}{\sqrt{5}}$ .8944	$\frac{2}{\sqrt{5}}$ .8944	″	″	$[\kappa, \frac{2}{1+\sqrt{5}}]$ $[.5550, .6180]$
6	″	$[\frac{2}{\sqrt{5}}, 1]$	$[\frac{2}{\sqrt{5}}, 1]$	$\mathbf{\frac{1}{\sqrt{2}}}$ **.7071**	″	″
7	″	″	″	$[?, 1)$ $\mathbf{[.8231, 1)}$	$\mathbf{[\frac{2}{3}, \frac{2}{\sqrt{7}})}$ $\mathbf{[.6667, .7559)}$	$[\kappa, \frac{2}{3}]$ $[.5550, .6667]$
8	″	″	″	$[?, 1]$ $[.8231, 1]$	$\mathbf{[\frac{2}{3}, \frac{2}{\sqrt{5}}]}$ $\mathbf{[.6667, .8944]}$	″
10	″	″	″	″	$[\frac{\sqrt{5}}{3}, 1]$ $[.7454, 1]$	″

Part 2

Supplementary Material and Tools

CHAPTER 5

Supplementary Material: 2-Pavings

CHAPTER 6

Supplementary Material: 3-Pavings

1. 4×4 General

LEMMA 6.1. *Let $A \in \mathbb{M}_4^0$. If $\alpha_3(A) = 1$ and $\|A\| < \sqrt{3}$, then there exists a permutation matrix $U \in \mathbb{M}_4$ such that*

$$U^*AU = \begin{bmatrix} 0 & \hat{a} & \hat{b} & \tilde{c} \\ \tilde{a} & 0 & \hat{d} & \hat{e} \\ \tilde{b} & \tilde{d} & 0 & \hat{f} \\ \hat{c} & \tilde{e} & \tilde{f} & 0 \end{bmatrix},$$

where $|\tilde{x}| \le |\hat{x}|$ for all $x \in \{a, b, c, d, e, f\}$. The result remains true if $A \gg 0$ and $\|A\| < 2$.

PROOF. Let

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$

The condition $\alpha_3(A) = 1$ implies that $\max\{|a_{ij}|, |a_{ji}|\} \ge 1$ for all $i < j$. The condition $\|A\| < \sqrt{3}$ (resp. $A \gg 0$ and $\|A\| < 2$) ensures that each row and each column has at most two entries of magnitude greater than or equal to 1 (see Lemma 6.1). Conjugating by $U_{(12)}$, if necessary, we may assume that $|a_{12}| \ge |a_{21}|$, which we indicate as follows:

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & a_{13} & a_{14} \\ \tilde{a}_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$

Case 1: Suppose $|a_{13}| \ge |a_{31}|$. Then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \tilde{a}_{14} \\ \tilde{a}_{21} & 0 & a_{23} & a_{24} \\ \tilde{a}_{31} & a_{32} & 0 & a_{34} \\ \hat{a}_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$

Conjugating by $U_{(23)}$, if necessary, we may assume that $|a_{23}| \ge |a_{32}|$. Then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \tilde{a}_{14} \\ \tilde{a}_{21} & 0 & \hat{a}_{23} & a_{24} \\ \tilde{a}_{31} & \tilde{a}_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & a_{42} & \tilde{a}_{43} & 0 \end{bmatrix}.$$

If $|a_{24}| \geq |a_{42}|$, then we are done. Thus, we may assume the opposite. That is,

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \tilde{a}_{14} \\ \tilde{a}_{21} & 0 & \hat{a}_{23} & \tilde{a}_{24} \\ \tilde{a}_{31} & \tilde{a}_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & \hat{a}_{42} & \tilde{a}_{43} & 0 \end{bmatrix}.$$

Conjugating by $U = U_{(1432)}$ yields

$$U^*AU = \begin{bmatrix} 0 & \hat{a}_{41} & \hat{a}_{42} & \tilde{a}_{43} \\ \tilde{a}_{14} & 0 & \hat{a}_{12} & \hat{a}_{13} \\ \tilde{a}_{24} & \tilde{a}_{21} & 0 & \hat{a}_{23} \\ \hat{a}_{34} & \tilde{a}_{31} & \tilde{a}_{32} & 0 \end{bmatrix}.$$

Case 2: Suppose $|a_{13}| < |a_{31}|$. Then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \tilde{a}_{13} & a_{14} \\ \tilde{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$

Case 2.1: If $|a_{14}| \geq |a_{41}|$, then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \tilde{a}_{13} & \hat{a}_{14} \\ \tilde{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & a_{34} \\ \tilde{a}_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$

Conjugating by $U_{(34)}$ yields

$$U_{(34)}^*AU_{(34)} = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{14} & \tilde{a}_{13} \\ \tilde{a}_{21} & 0 & a_{24} & a_{23} \\ \tilde{a}_{41} & a_{42} & 0 & a_{43} \\ \hat{a}_{31} & a_{32} & a_{34} & 0 \end{bmatrix},$$

and we may proceed as in Case 1.

Case 2.2: If $|a_{14}| < |a_{41}|$, then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \tilde{a}_{13} & \tilde{a}_{14} \\ \tilde{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & a_{34} \\ \hat{a}_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$

Conjugating by $U_{(34)}$ if necessary, we may assume that $|a_{34}| \geq |a_{43}|$. Then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \tilde{a}_{13} & \tilde{a}_{14} \\ \tilde{a}_{21} & 0 & \hat{a}_{23} & a_{24} \\ \hat{a}_{31} & \tilde{a}_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & a_{42} & \tilde{a}_{43} & 0 \end{bmatrix}.$$

Case 2.2.1: If $|a_{24}| \geq |a_{42}|$, then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \tilde{a}_{13} & \tilde{a}_{14} \\ \tilde{a}_{21} & 0 & \hat{a}_{23} & \hat{a}_{24} \\ \hat{a}_{31} & \tilde{a}_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & \tilde{a}_{42} & \tilde{a}_{43} & 0 \end{bmatrix}.$$

Conjugating by $U = U_{(1234)}$ yields

$$U^*AU = \begin{bmatrix} 0 & \hat{a}_{23} & \hat{a}_{24} & \tilde{a}_{21} \\ \tilde{a}_{32} & 0 & \hat{a}_{34} & \hat{a}_{31} \\ \tilde{a}_{42} & \tilde{a}_{43} & 0 & \hat{a}_{41} \\ \hat{a}_{12} & \tilde{a}_{13} & \tilde{a}_{14} & 0 \end{bmatrix}.$$

Case 2.2.2: If $|a_{24}| < |a_{42}|$, then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \tilde{a}_{13} & \tilde{a}_{14} \\ \tilde{a}_{21} & 0 & \hat{a}_{23} & \tilde{a}_{24} \\ \hat{a}_{31} & \tilde{a}_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & \hat{a}_{42} & \tilde{a}_{43} & 0 \end{bmatrix}.$$

Conjugating by $U = U_{(13)(24)}$ yields

$$U^*AU = \begin{bmatrix} 0 & \hat{a}_{34} & \hat{a}_{31} & \tilde{a}_{32} \\ \tilde{a}_{43} & 0 & \hat{a}_{41} & \hat{a}_{42} \\ \tilde{a}_{13} & \tilde{a}_{14} & 0 & \hat{a}_{12} \\ \hat{a}_{23} & \tilde{a}_{24} & \tilde{a}_{21} & 0 \end{bmatrix}.$$

□

D149: breadth-first labeling 2134

$$\begin{bmatrix} 0 & * & * & \cdot \\ \cdot & 0 & * & * \\ \cdot & \cdot & 0 & * \\ * & \cdot & \cdot & 0 \end{bmatrix}$$

$$\inf \left\{ \left\| \begin{bmatrix} 0 & \boxed{1} & \boxed{1} & \cdot \\ \cdot & \boxed{0} & \boxed{1} & 1 \\ \cdot & \cdot & 0 & 1 \\ 1 & \cdot & \cdot & 0 \end{bmatrix} \right\| \right\} = \left\| \begin{bmatrix} 0 & 1 & 1 & -\frac{2}{1+\sqrt{5}} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\| = \frac{1+\sqrt{5}}{2} \approx 1.6180$$

D185: breadth-first labeling 2341

$$\begin{bmatrix} 0 & * & * & \cdot \\ \cdot & 0 & * & * \\ \cdot & * & 0 & * \\ * & \cdot & \cdot & 0 \end{bmatrix}$$

$$\inf \left\{ \left\| \begin{bmatrix} 0 & \boxed{1} & \boxed{1} & \cdot \\ \cdot & \boxed{0} & \boxed{1} & 1 \\ \cdot & \boxed{1} & \boxed{0} & 1 \\ 1 & \cdot & \cdot & 0 \end{bmatrix} \right\| \right\} = \left\| \begin{bmatrix} 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\| = \sqrt{3} \approx 1.7321$$

REMARK 6.2. *Although this example doesn't satisfy the hypotheses of Lemma 6.1, it satisfies the conclusion. Also, the extreme example doesn't satisfy the graph theory, since* $|\cdot| < 1$.

D186: breadth-first labeling 3124

$$\begin{bmatrix} 0 & * & * & \cdot \\ \cdot & 0 & * & * \\ * & \cdot & 0 & * \\ * & \cdot & \cdot & 0 \end{bmatrix}$$

$$\inf\left\{\left\|\begin{bmatrix}0 & 1 & 1 & \cdot\\ \cdot & 0 & 1 & 1\\ 1 & \cdot & 0 & 1\\ 1 & \cdot & \cdot & 0\end{bmatrix}\right\|\right\} = \left\|\begin{bmatrix}0 & 1 & 1 & -1/2\\ -1/2 & 0 & 1 & 1\\ 1 & 0 & 0 & 1\\ 1 & 1/2 & 0 & 0\end{bmatrix}\right\| = \frac{\sqrt{11}}{2} \approx 1.6583$$

D218: breadth-first labeling 3124

$$\begin{bmatrix}0 & * & * & \cdot\\ \cdot & 0 & * & *\\ * & \cdot & 0 & *\\ * & * & \cdot & 0\end{bmatrix}$$

$$\inf\left\{\left\|\begin{bmatrix}0 & 1 & 1 & \cdot\\ \cdot & 0 & 1 & 1\\ 1 & \cdot & 0 & 1\\ 1 & 1 & \cdot & 0\end{bmatrix}\right\|\right\} = \left\|\begin{bmatrix}0 & 1 & 1 & -1/3\\ -1/3 & 0 & 1 & 1\\ 1 & -1/3 & 0 & 1\\ 1 & 1 & -1/3 & 0\end{bmatrix}\right\| = \frac{5}{3} \approx 1.6667$$

REMARK 6.3. *Notice that this is a circulant. Best among circulants?*

CHAPTER 7

Tools

1. Universal Selfadjoint 3-Identity and consequences

LEMMA 7.1 (Universal Selfadjoint 3-Identity). *Arbitrary* 3×3 *selfadjoint trace zero matrices* S *satisfy:*

$$\frac{||S||_2^2}{2||S||^2} + \frac{|Det\, S|}{||S||^3} = 1$$

PROOF. Since all trace zero finite (or trace class) matrices have a basis in which their representation has zero diagonal, without loss of generality we can assume S has the form:

$$S = \begin{pmatrix} 0 & a & b \\ \overline{a} & 0 & c \\ \overline{b} & \overline{c} & 0 \end{pmatrix}$$

and by computation, the characteristic polynomial:

$$\begin{aligned} c_\lambda(S) = \det(\lambda - S) &= \lambda^3 - 2\,Re\,\overline{a}b\overline{c} - \lambda(|a|^2 + |b|^2 + |c|^2) \\ &= \lambda^3 - (|a|^2 + |b|^2 + |c|^2)\lambda - 2\,Re\,a\overline{b}c \\ &= \lambda^3 - \frac{||S||_2^2}{2}\lambda - Det\,S. \end{aligned}$$

An alternative way to see this is that the characteristic polynomial has the form $\lambda^3 + p\lambda^2 + q\lambda + r$, with $p = 0$ because the sum of the roots is the trace of S, the latter also implying

$$q = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{1}{2}\left((\lambda_1 + \lambda_2 + \lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\right) = \frac{-||S||_2^2}{2}$$

where λ_j, $j = 1, 2, 3$ denotes its roots, and $r = -\lambda_1\lambda_2\lambda_3 = -\det S$.

Since S is selfadjoint, $\lambda = \pm||S||$ is an eigenvalue of S. Also, because this is the largest eigenvalue in modulus and S has trace zero, the other two real eigenvalues are opposite this in sign making their product, $Det\,S$, the same sign as λ. Hence $(\pm||S||)^3 = \frac{||S||_2^2}{2}(\pm||S||) + (\pm|Det\,S|)$, whence the Universal Selfadjoint 3-Identity in either case. □

COROLLARY 7.2 (Universal Selfadjoint 3-Identity consequences). *For arbitrary* 3×3 *selfadjoint trace zero matrices* S,

$$||S|| = 1 \Leftrightarrow \frac{||S||_2^2}{2} + |Det\,S| = 1.$$

For greater or less than 1, *the respective conditions are equivalent. A necessary condition for equality is* $3/2 \leq ||S||_2^2 \leq 2$.

PROOF. The Universal Selfadjoint 3-Identity, $\frac{||S||_2^2}{2||S||^2} + \frac{|Det\, S|}{||S||^3} = 1$, implies that if $||S|| > 1$ then $\frac{||S||_2^2}{2} + |Det\, S| > 1$, and likewise, if $||S|| < 1$ then $\frac{||S||_2^2}{2} + |Det\, S| < 1$. Therefore $||S|| = 1$ if and only if $\frac{||S||_2^2}{2} + |Det\, S| = 1$.

Moreover, if $\frac{||S||_2^2}{2} + |Det\, S| = 1$, then $||S||_2^2 \leq 2$. Also in this case when $||S|| = 1$, $||S||_2^2 \geq \frac{3}{2}||S||^2 = \frac{3}{2}$ is the $n = 3,\ p = 2$ case of Proposition 7.5. □

2. Universal Selfadjoint 4-Identity and consequences

Universal Selfadjoint 4-Identity (for 4×4 selfadjoint zero-trace):

$$\frac{||S||_2^2}{2||S||^2} + \frac{|Tr\, S^3|}{3||S||^3} - \frac{Det\, S}{||S||^4} = 1$$

Unpolished and unverified work (for proofs see file UniversalIdentities.Tex):

Consequence: Since $\frac{|Det\, S|}{||S||^4} \leq 1$

$$\frac{||S||_2^2}{2||S||^2} + \frac{|Tr\, S^3|}{3||S||^3} \leq 2$$

Separate Fact ($||S||_2^2 \geq \frac{n}{n-1}||S||^2$): $||S||_2^2 \geq \frac{4}{3}||S||^2$ so $\frac{||S||_2^2}{2||S||^2} \geq \frac{2}{3}$

Implying: $\frac{|Tr\, S^3|}{3||S||^3} \leq \frac{4}{3}$

(Trivially also follows generally from Hölder: $|Tr\, S^3|^{1/3} \leq ||S||_3 \leq 4^{1/3}||S||$)

Development of Universal Selfadjoint 4-Identity:

Let S denote a 4×4 selfadjoint zero-trace matrix with eigenvalues

$$1 = \lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq |\lambda_4|.$$

$$\begin{aligned} c_\lambda(S) &= (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) \\ &= \lambda^4 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\lambda^3 + (\sum_{i<j} \lambda_i\lambda_j)\lambda^2 - (\sum_{i<j<k} \lambda_i\lambda_j\lambda_j)\lambda + \lambda_1\lambda_2\lambda_3\lambda_4 \\ &= \lambda^4 + p\lambda^2 - q\lambda + r \end{aligned}$$

SUMMARY: NASC for $||S|| = 1$ (unverified)

1. $p \geq \frac{2}{3}$ 2. $p + |q| + r = 1$
3. $0 \leq p + |q| \leq 2$ (equivalent to |product of roots| ≤ 1)
4. When $p < 1$, $\frac{20}{27} - \frac{2}{3}p - \frac{2}{27}(3p-2)^{3/2}) \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p-2)^{3/2}$.
5. When $p \geq 1$, $0 \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p-2)^{3/2}$.

(4-5: $max\ (0, \frac{20}{27} - \frac{2}{3}p - \frac{2}{27}(3p-2)^{3/2}) \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p-2)^{3/2})$

3. Operator Norm/p-Norm Comparisons

Proposition 7.3 (Operator Norm/p-Norm). *If A is a finite rank selfadjoint trace 0 matrix and*

$$k = |\#\ \textit{strictly positive eigenvalues} - \#\ \textit{strictly negative eigenvalues}|,$$

then for $p \geq 1$,

$$||A||_p \leq (rank\ A - k)^{1/p}\ ||A||$$

(Sharp example: $diag\,(-1,1)$)
(Sharp asymptotically: $diag\,(\pm 1, \ldots, \pm 1(\frac{rank\ A-k-2}{2}\ pairs\,of\ them), 1, -\frac{k}{k+1}, -\frac{1}{k(k+1)}, \ldots, -\frac{1}{k(k+1)})$; note: rank $A - k$ must be even)

Proof. Easy proof for $p \geq 2$ based on the $p = 2$ case:
If $< \lambda_j >$ are the (real) eigenvalues of A, then

$$\sum_1^n |\lambda_j|^p = \sum_1^n |\lambda_j|^{p-2}|\lambda_j|^2 \leq |\lambda_1|^{p-2}\sum_1^n |\lambda_j|^2 \leq |\lambda_1|^{p-2}(n-k)|\lambda_1|^2 = (n-k)|\lambda_1|^p.$$

For all $p \geq 1$, we describe informally the following variational approach:

Maximize $\sum |\lambda_j|^p$ subject to $\lambda_1 + \cdots + \lambda_n = 0$.

Without loss of generality, $A \neq 0$, $||A|| \leq 1$ and $tr\,A \neq 0$ implies that for some $n > m \geq 1$ the eigenvalues of A have the $[-1,1]$ distribution:

$$-1 \leq \lambda_n \leq \cdots \leq \lambda_{m+1} < 0 < \lambda_m \leq \cdots \leq \lambda_1 \leq 1,$$

We induct on $n - k$. Since $A \neq 0$, $n - k > 0$ and is even and so $n - k \geq 2$.

Increase λ_1 and decrease λ_n equally so to preserve the trace, until one of them reaches 1 or -1, respectively. (Increasing both moduli increases the sum $\sum |\lambda_j|^p$ and so permits reduction of the proof to this case.) If they both reach 1 or -1, then dropping them leaves k invariant and reduces to the $n - k - 2$ case.

If now $\lambda_1 = 1$ and $\lambda_n > -1$ (handle the reverse case the same), decrease λ_n and increase λ_{n-1} equally to preserve their sum. Elementary calculus shows that this will increase $|\lambda_n|^p + |\lambda_{n-1}|^p$. Continue this until either λ_n reaches -1 or λ_{n-1} reaches λ_{n-2}. If the former, then drop λ_n and λ_1, and again apply the induction hypothesis. If the latter, then decrease both until λ_n reaches -1 or both λ_{n-1} and λ_{n-2} reaches λ_{n-3}, and so on. This process will increase $\sum |\lambda_j|^p$ and unless $m = 1$, one has $m > 1$ or equivalently, $\lambda_n + \cdots + \lambda_{m+1} < -1$ implying that eventually in this process λ_n will reach -1 so we can apply again the induction hypothesis while preserving k. If $m = 1$, then this process ends in one pair of ± 1 with sum 2 so $\sum_1^n |\lambda_j|^p \leq 2 \leq n - k$. □

Corollary 7.4. *If A is an $n \times n$ selfadjoint trace 0 matrix with n odd, then $||A||_2 \leq \sqrt{n-1}\,||A||$.*

PROPOSITION 7.5. *If A is an $n \times n$ selfadjoint trace 0 matrix and $p \geq 1$ (or more generally rank $A = n$), then*

$$||A||_p \geq [1 + \frac{1}{(n-1)^{p-1}}]^{1/p} \; ||A||$$

with equality iff $A = c\, diag(-1, \frac{1}{n-1}, \ldots, \frac{1}{n-1})$.

PROOF. Suffices to show the sequence analog for $\lambda_1 + \cdots + \lambda_n = 0$, all λ_j real. Since the inequality is obvious for $p = 1$, needing selfadjoint with trace 0 to see it, we can assume without loss of generality that $p > 1$. Then

$$|\lambda_1| = | - \sum_2^n \lambda_j| \leq ||\mathbf{1}||_{p'} ||\lambda||_p$$

where $\lambda := < \lambda_j >_{2 \leq j \leq n}$, $\mathbf{1} := < 1 >_{2 \leq j \leq n}$, and $\frac{1}{p} + \frac{1}{p'} = 1$, i.e., $\frac{p}{p'} = p - 1$. Equality holds if and only if λ is a constant multiple of $\mathbf{1}$. (*This is the p-case for Cauchy-Schwartz equality which I presume holds true for $p \neq 2$ like it does for $p = 2$-except I don't know a reference.*) So

$$|\lambda_1|^p \leq (n-1)^{p/p'} \sum_2^n |\lambda_j|^p = (n-1)^{p-1} \sum_2^n |\lambda_j|^p.$$

Adding $(n-1)^{p-1}|\lambda_1|^p$ to both sides yields: $[1+(n-1)^{p-1}]||A||^p \leq (n-1)^{p-1}||A||_p^p$, from which (iii) follows. The case for equality also follows from the previous comment about equality. □

COROLLARY 7.6. *If every k-compression of $A \in \mathbb{M}^0_{n,sa}$ has norm ≥ 1, then*

$$\|A\| \geq \begin{cases} \frac{\sqrt{n-1}}{k-1} & n \text{ even} \\ \frac{\sqrt{n}}{k-1} & n \text{ odd} \end{cases}.$$

PROOF. Denote by Π_k the set of all k-compressions of A.
Then $\|B\|^2 \leq \frac{k-1}{k}\|B\|_2^2$ for all $B \in \Pi_k$ by Proposition 7.5 ($p = 2$ & take n to be k).
Then

$$\binom{n}{k} \leq \sum_{B\in\Pi_k} \|B\|^2 \leq \frac{k-1}{k} \sum_{B\in\Pi_k} \|B\|_{HS}^2 = \frac{k-1}{k}\binom{n-2}{k-2}\|A\|_{HS}^2 \leq (n \text{ or } n{-}1)\frac{k-1}{k}\binom{n-2}{k-2}\|A\|^2.$$

Thus,

$$\|A\|^2 \geq \frac{\binom{n}{k}}{(n \text{ or } n-1)\frac{k-1}{k}\binom{n-2}{k-2}} = \frac{\sqrt{n-1}}{k-1} \text{ or } \frac{\sqrt{n}}{k-1}.$$

□

COROLLARY 7.7. *If $\widetilde{\alpha}_2(\mathbb{M}^0_{n-k,sa}) < \widetilde{\alpha}_3(\mathbb{M}^0_{n,sa})$ and*

$$\widetilde{\alpha}_3(\mathbb{M}^0_{n,sa} \cap \{\textit{all zero-diagonals with } \pm 1 \textit{ off diagonal entries}\}) < \begin{cases} \frac{k-1}{\sqrt{n-1}} & n \text{ even} \\ \frac{k-1}{\sqrt{n}} & n \text{ odd} \end{cases},$$

then

$$\widetilde{\alpha}_3(\mathbb{M}^0_{n,sa}) < \begin{cases} \frac{k-1}{\sqrt{n-1}} & n \text{ even} \\ \frac{k-1}{\sqrt{n}} & n \text{ odd} \end{cases}.$$

PROOF. Fix an extremal $A = A_n$, that is, $\widetilde{\alpha}_3(\mathbb{M}^0_{n,sa}) = \frac{\alpha_3(A)}{\|A\|}$ and without loss of generality assume $\alpha_3(A) = 1$ and $\|A\| = \frac{1}{\widetilde{\alpha}_3(\mathbb{M}^0_{n,sa})}$.

Either $\|B\| < 1$ for some k-compression or every k-compression B of A has norm ≥ 1.

Assume first $\|B\| < 1$ for some k-compression $B = PAP$. Because $\alpha_3(A) = 1$, every 3-paving has norm ≥ 1 and by definition, $\widetilde{\alpha}_2(\mathbb{M}^0_{n-k,sa}) \geq \frac{\alpha_2((I-P)A(I-P))}{\|(I-P)A(I-P)\|}$ so $\|(I-P)A(I-P)\| \geq \frac{\alpha_2((I-P)A(I-P))}{\widetilde{\alpha}_2(\mathbb{M}^0_{n-k,sa})}$. So if additionally $\|B\| < 1$ and $\alpha_3(A) = 1$, then $\alpha_2((I-P)A(I-P)) = 1$ so all 2-pavings of $(I-P)A(I-P)$ have norm ≥ 1, in which case

$$\|A\| \geq \|(I-P)A(I-P)\| \geq \frac{1}{\widetilde{\alpha}_2(\mathbb{M}^0_{n-k,sa})} > \frac{1}{\widetilde{\alpha}_3(\mathbb{M}^0_{n,sa})}$$

(the last $>$ by hypothesis), contradicting $\widetilde{\alpha}_3(\mathbb{M}^0_{n,sa}) = \frac{\alpha_3(A)}{\|A\|} = \frac{1}{\|A\|}$.

On the other hand, if every k-compression B of A has norm ≥ 1, then the displayed inequality in Corollary 7.6 becomes equality throughout:

$$\binom{n}{k} = \sum_{B\in\Pi_k} \|B\|^2 \leq \frac{k-1}{k} \sum_{B\in\Pi_k} \|B\|_{HS}^2 = \frac{k-1}{k}\binom{n-2}{k-2}\|A\|_{HS}^2 = (n \text{ or } n{-}1)\frac{k-1}{k}\binom{n-2}{k-2}\|A\|^2.$$

So each $\|B\|^2 = \frac{k-1}{k}\|B\|_{HS}^2$. Now apply Lemma 3.12 so that

$$A \equiv S \in \mathbb{M}^0_{n,sa} \cap \{\text{all zero-diagonals with } \pm 1 \text{ off diagonal entries}$$

and apply the hypothesis to S to contradict the extremality of A. □

4. Operator Norm/Hilbert-Schmidt Norm Comparisons

LEMMA 7.8. *Let $A \in \mathbb{M}_n$. Then*

$$\|A\| \leq \|A\|_{HS} \leq \sqrt{n}\|A\|.$$

Furthermore,

i. $\|A\| = \|A\|_{HS}$ *if and only if* $\operatorname{rank}(A) \leq 1$.
ii. $\|A\|_{HS} = \sqrt{n}\|A\|$ *if and only if A is a scalar multiple of a unitary.*

PROOF. The inequalities are well-known and easy to prove. Now let

$$A = U\Sigma V^*$$

be a singular value decomposition of A (i.e. U, V are unitary and $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, ..., \sigma_n)$, where $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0$). Assume $\|A\| = \|A\|_{HS}$. Then

$$\sigma_1^2 = \|A\|^2 = \|A\|_{HS}^2 = \sum_{i=1}^{n} \sigma_i^2 \Rightarrow \sigma_2 = \sigma_3 = ... = \sigma_n = 0.$$

Thus, $A = \sigma_1 u_1 v_1^*$, where u_1 and v_1 are the first columns of U and V, respectively. Hence, $\operatorname{rank}(A) \leq 1$. Conversely, if $\operatorname{rank}(A) \leq 1$, then

$$\sigma_2 = \sigma_3 = ... = \sigma_n = 0 \Rightarrow \|A\| = \|A\|_{HS}.$$

Now assume $\|A\|_{HS} = \sqrt{n}\|A\|$. Then

$$\sum_{i=1}^{n} \sigma_i^2 = \|A\|_{HS}^2 = n\|A\|^2 = n\sigma_1^2 \Rightarrow \sigma_1 = \sigma_2 = ... = \sigma_n.$$

Thus, $A = \sigma_1 UV^*$, which is a scalar multiple of a unitary. Conversely, if $A = \alpha W$, where $\alpha \in \mathbb{C}$ and W is a unitary, then

$$\|A\|_{HS}^2 = \operatorname{Tr}(A^*A) = |\alpha|^2 \operatorname{Tr}(W^*W) = |\alpha|^2 \operatorname{Tr}(I) = n|\alpha|^2 = n\|A\|^2.$$

□

COROLLARY 7.9. *If every 3-compression of $A \in \mathbb{M}_7^0$ has norm ≥ 1, then*

$$\|A\| \geq \sqrt{\frac{n-1}{k(k-1)}}.$$

Equality occurs if and only if A is a multiple of a unitary and every k-compression of A has rank one.

PROOF. Denote by Π_k the set of all k-compressions of A. Then

$$\binom{n}{k} \leq \sum_{B \in \Pi_k} \|B\|^2 \leq \sum_{B \in \Pi_k} \|B\|_{HS}^2 = \binom{n-2}{k-2} \|A\|_{HS}^2 \leq n \binom{n-2}{k-2} \|A\|^2.$$

Thus,

$$\|A\|^2 \geq \frac{\binom{n}{k}}{n\binom{n-2}{k-2}} = \frac{n-1}{k(k-1)}.$$

The stated equality condition follows immediately from Lemma 7.8. □

COROLLARY 7.10. *If every 3-compression of $A \in \mathbb{M}_7^0$ has norm ≥ 1, then* $\|A\| > 1$.

PROOF. By Lemma 7.9,

$$\|A\|^2 \geq \frac{7-1}{3(3-1)} = 1.$$

Suppose $\|A\| = 1$. Again by Lemma 7.9, A is unitary and every 3-compression of A has rank one. It follows that every 3-compression of A has exactly two zero columns or exactly two zero rows. Consider A_{123}, the $\{1, 2, 3\}$-compression of A. Without loss of generality, we may assume that the second and third columns of A_{123} are zero. It follows that the first column of A_{123} has norm 1. Thus,

$$A = \begin{bmatrix} 0 & 0 & 0 & * & * & * & * \\ a_{21} & 0 & 0 & * & * & * & * \\ a_{31} & 0 & 0 & * & * & * & * \\ 0 & * & * & 0 & * & * & * \\ 0 & * & * & * & 0 & * & * \\ 0 & * & * & * & * & 0 & * \\ 0 & * & * & * & * & * & 0 \end{bmatrix},$$

where $|a_{21}|^2 + |a_{31}|^2 = 1$. Conjugating by $U_{(23)}$, if necessary, we may assume that $a_{21} \neq 0$. Case 1: Suppose $|a_{21}| = 1$. By considering, in order, A_{123}, A_{124}, A_{125}, A_{126}, and A_{127}, we have that

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & * & 0 & * & * & * \\ 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & * & * & * & * & 0 \end{bmatrix}.$$

Considering A_{234}, we have that either $|a_{34}| = 1$ or $|a_{43}| = 1$. Conjugating by $U_{(34)}$, if necessary, we may assume the former. Considering, in order, A_{234}, A_{345}, A_{346}, and A_{347}, we have that

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & * & * & 0 \end{bmatrix}.$$

But then $\|A_{235}\| = 0$, a contradiction.
Case 2: Suppose $|a_{21}| < 1$. By considering, in order, A_{124}, A_{234}, and A_{345}, we have that

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & * & * & * \\ a_{21} & 0 & 0 & a_{24} & 0 & 0 & 0 \\ a_{31} & 0 & 0 & a_{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & * & 0 & 0 & 0 & * & * \\ 0 & * & * & 0 & * & 0 & * \\ 0 & * & * & 0 & * & * & 0 \end{bmatrix},$$

where $|a_{21}|^2 + |a_{24}|^2 = 1$ and $|a_{24}|^2 + |a_{34}|^2 = 1$. But then $\|A_{345}\| < 1$, a contradiction. □

Lemma 7.11. *Let $A \in \mathbb{M}_4^0$. If every 2-2 paving of A has norm ≥ 1, then either $\|A\| > 1$ or, up to permutation,*

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & c & 0 & 0 \end{bmatrix},$$

where $|a| = |b| = |c| = 1$.

Proof. Assume $\|A\| = 1$. Create a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if $|a_{ij}|, |a_{ji}| < 1$. We may assume the following axioms:

(1) $G11$ is not a subgraph of G. Otherwise, A has a 2-2 paving of norm < 1.
(2) For all i, $deg(i) > 0$. Otherwise, either row i or column i of A has at least two entries of modulus $\geq 1 \Rightarrow \|A\| \geq \sqrt{2}$.

This leaves $G13$, which proves the result. □

5. Averaging and Constrained Averaging

Let $A^* = A = (a_{ij}),\ E(A) = 0$, with the reduction assumption for $\mathbb{M}^0_{7,sa}$ that the B's range over all the 3×3 zero-diagonal matrices with norm at least 1 (in which case each Hilbert-Schmidt norm is at least $\frac{3}{2}$) or in the case of constrained averaging, all the B's with diagonal projection not containing prescribed i, j pairs.

The following weighted formulas for the Hilbert-Schmidt norm of a 7×7 zero-diagonal selfadjoint matrix in terms of the Hilbert-Schmidt norms of some or all of its 3-diagonal compressions PAP for averaging and constrained averaging are obtained by careful groupings of triplet integer subsets of $[1, 7]$ to compensate for overcounting due to multiple occurrences, analogous to the elementary counting formula for finite sets: $|A \cup B| = |A| + |B| - |A \cap B|$.

(0)
$$6||A||^2 \geq ||A||^2_{HS} = \frac{1}{5}\sum_{all}^{35} ||B||^2_{HS} \qquad \text{(Averaging)}$$

(12)
$$6||A||^2 \geq ||A||^2_{HS} = 2|a_{12}|^2 + \left(\frac{1}{4}\sum_{134-267}^{20} + \frac{1}{6}\sum_{345-567}^{10}\right) ||B||^2_{HS} \qquad \text{(Constrained Averaging here and below)}$$

(row)
$$6||A||^2 \geq ||A||^2_{HS} = 2||Ae_1||^2 + \frac{1}{4}\sum_{1 \notin B}^{20} ||B||^2_{HS}$$

(12,23)
$$6||A||^2 \geq ||A||^2_{HS} = 2|a_{12}|^2 + 2|a_{23}|^2 + \left(\frac{1}{4}\sum_{1\in B, 2\notin B}^{10} + \frac{1}{3}\sum_{1\notin B, 2\in B}^{6} + \frac{1}{4}\sum_{1,2\notin B, 3\in B}^{6} + \frac{1}{12}\sum_{1,2,3\notin B}^{4}\right) ||B||^2_{HS}$$

(12,13)
$$6||A||^2 \geq ||A||^2_{HS} = 2|a_{12}|^2 + 2|a_{13}|^2 + \left(\frac{1}{3}\sum_{1\in B, 2,3\notin B}^{6} + \frac{1}{4}\sum_{1\notin B, 2\in B}^{10} + \frac{1}{6}\sum_{1,2\notin B}^{10}\right) ||B||^2_{HS}$$

(12,23,34)
$$6||A||^2 \geq ||A||^2_{HS} = 2|a_{12}|^2 + 2|a_{23}|^2 + 2|a_{34}|^2 + \left(\frac{1}{3}\sum_{135-147, all2's, 356-367}^{15} + \frac{1}{6}\sum_{156-167, 456-467}^{6} + (0)\sum_{567}^{1}\right) ||B||^2_{HS}$$

Application of constrained averaging:

If $|a_{ij}| \geq 1$ (wlog $i, j = 1, 2$) and A satisfies the 3-compression reduction given above, then by (12),

$$
\begin{aligned}
6||A||^2 \geq ||A||_{HS}^2 &= 2|a_{12}|^2 + \left(\frac{1}{4}\sum_{134-267}^{20} + \frac{1}{6}\sum_{345-567}^{10}\right)||B||_{HS}^2 \\
&\geq 2 + \left(\frac{1}{4}\sum_{134-267}^{20} + \frac{1}{6}\sum_{345-567}^{10}\right)\frac{3}{2}||B|| \\
&\geq 2 + \left(\frac{20}{4} + \frac{10}{6}\right)\frac{3}{2} = 2 + \left(5 + \frac{5}{3}\right)\frac{3}{2} = 12
\end{aligned}
$$

So $6||A||^2 \geq 12$, $||A|| \geq \sqrt{2}$, $\widetilde{\alpha}_3(A) \leq \frac{1}{\sqrt{2}} \approx .7071$, smaller than the $\widetilde{\alpha}_3(\mathbb{M}_{7,sa}^0)$-table upper range in $[\frac{2}{3}, \frac{2}{\sqrt{7}}) = [.6667,\ .7559)$. This then rules out entries with larger than 1 modulus for an extremal bad paver in case one succeeds in proving $\widetilde{\alpha}_3(\mathbb{M}_{7,sa}^0) \in (\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{7}})$.

Moreover, since $\frac{1}{||A||} = \widetilde{\alpha}_3(\mathbb{M}_{7,sa}^0)$, if A were extremal,
and wlog $|a_{12}| = \max_{i,j} |a_{ij}|$, then $||A||^2 = \frac{1}{\widetilde{\alpha}_3(\mathbb{M}_{7,sa}^0)^2} \in (\frac{7}{4}, \frac{9}{4}]$ and

$$
\begin{aligned}
||A||^2 \geq \frac{1}{6}||A||_{HS}^2 &= \frac{1}{3}|a_{12}|^2 + \frac{1}{6}\left(\frac{1}{4}\sum_{134-267}^{20} + \frac{1}{6}\sum_{345-567}^{10}\right)||B||_{HS}^2 \\
&\geq \frac{|a_{12}|^2}{3} + \left(\frac{1}{4}\sum_{134-267}^{20} + \frac{1}{6}\sum_{345-567}^{10}\right)\frac{3}{2}||B|| \\
&\geq \frac{|a_{12}|^2}{3} + \frac{1}{6}\left(\frac{20}{4} + \frac{10}{6}\right)\frac{3}{2} = \frac{|a_{12}|^2}{3} + \frac{5}{3} > \frac{9}{4}
\end{aligned}
$$

leads to the contradiction: $\widetilde{\alpha}_3(\mathbb{M}_{7,sa}^0) = \frac{1}{\|A\|} < \frac{2}{3}$. Hence

$$|a_{12}|^2 \leq \frac{27}{4} - 5 = \frac{7}{4}\ \text{, i.e., } \max_{i,j} |a_{ij}| \leq \frac{\sqrt{7}}{2} < ||A||.$$

Bibliography

[1] Read and Wilson, *An Atlas of Graphs*, Clarendon Press, Oxford, 1998.