FRAMES AND THE KADISON-SINGER PROBLEM

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Abstract. This is an introduction to the problems connecting frame theory and the Kadison-Singer Problem.

1. Overview

For an extensive introduction to the aspects of frame theory needed for an understanding of the Kadison-Singer Problem we refer to the survey paper [3], which is posted in this section of the web page "The Kadison-Singer Problem". In the following section we will just give the basic definitions for the concepts we will be working with. In Section 3, we will then state several conjectures in frame theory, which are equivalent to the Kadison-Singer Problem. In Section 5, we will discuss the Rado-Horn Theorem as one tool to attack in particular algorithmic aspects of the Kadison-Singer Problem.

2. Basic definitions and notations

Throughout let $\mathbb{H}$ denote a (finite or infinite dimensional) Hilbert space, and let $\mathbb{H}_N$ denote an N-dimensional Hilbert space.

2.1. Frames and Bessel sequences.

First we state the definition of a frame, which can be regarded as the most natural generalization of the concept of orthonormal bases. Frames provide robust and stable – but usually nonunique (redundant) – representations of vectors, which makes them particularly useful both in pure as well as applied mathematics.

Definition 2.1. *A family $\{f_i\}_{i\in I}$ of elements of $\mathbb{H}$ is called a* **frame** *for $\mathbb{H}$, if there are constants $0 < A \le B < \infty$ (called the* **lower** *and* **upper frame bounds***, respectively) such that, for all $f \in \mathbb{H}$,*

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2. \tag{2.1}$$

If the right hand inequality in (2.1) holds (and not necessarily the left hand inequality), we call $\{f_i\}_{i\in I}$ a **Bessel sequence with Bessel bound B**. If A and B can be chosen so that $A = B$, we call this an A-**tight frame**, and if $A = B = 1$, it is called a **Parseval frame**. If all frame elements have the same norm, this is an **equal norm** frame, and if all frame elements are of unit norm, it is called a **unit norm frame**. If the norms of the frame elements are uniformly bounded away from zero, we call this a **bounded frame**.

2.2. **Riesz basic sequences.** Next we state the basic definitions and notations related to Riesz basic sequences, which will be the building blocks of the most important equivalent conjecture in frame theory (see Subsection 3.2).

Definition 2.2. *A family of vectors* $\{f_i\}_{i\in I}$ *in* $\mathbb{H}$ *is a* **Riesz basic sequence***, if there are constants* $0 < A \le B < \infty$ *so that for all scalars* $\{a_i\}_{i\in I}$ *we have:*

$$A\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I} a_i f_i\|^2 \le B\sum_{i\in I}|a_i|^2.$$

We call A *and* B *the* **lower** *and* **upper Riesz basis bounds** *for* $\{f_i\}_{i\in I}$.

If a Riesz basic sequence $\{f_i\}_{i\in I}$ spans $\mathbb{H}$, we call it a **Riesz basis** for $\mathbb{H}$. Therefore, $\{f_i\}_{i\in I}$ being a Riesz basis for $\mathbb{H}$ is equivalent to the existence of an orthonormal basis $\{e_i\}_{i\in I}$ satisfying that the operator $T(e_i) = f_i$ is invertible. In particular, each Riesz basis is **bounded**, i.e., we have $0 < \inf_{i\in I}\|f_i\| \le \sup_{i\in I}\|f_i\| < \infty$. If the Riesz basis bounds satisfy: $A = 1-\epsilon$ and $B = 1+\epsilon$ then we call $\{f_i\}_{i\in I}$ and ϵ-Riesz basis.

R_ϵ**-Conjecture** [Casazza, Vershynin] For every $\epsilon > 0$, every unit norm Riesz basic sequence is a finite union of ϵ-Riesz basic sequences.

It was shown in [11] that the R_ϵ-Conjecture is equivalent to KS.

3. Equivalent Conjectures in Frame Theory

In the following we will present some conjectures in frame theory, which are equivalent to the Kadison-Singer Problem.

3.1. **Equivalent Conjectures for Parseval Frames.** The first conjecture we state follows from the work of Casazza, Edidin, Kalra, and Paulsen [7].

Problem 3.1. *Does there exist an* $\epsilon > 0$ *and a natural number* r *so that for all equal norm Parseval frames* $\{f_i\}_{i=1}^{2n}$ *for* ℓ_2^n *there is a partition* $\{A_j\}_{j=1}^r$ *of* $\{1, 2, \cdots, 2n\}$ *so that* $\{f_i\}_{i\in A_j}$ *has Bessel bound* $\le 1-\epsilon$ *for all* $j = 1, 2, \cdots, r$.

An equivalent statement to that of Problem 3.1 is: "For all $j = 1, 2, \cdots, r$, $\{f_i\}_{i\in A_j}$ is a Riesz basic sequence with Riesz basis bounds ϵ and $1-\epsilon$."

3.2. **The Feichtinger Conjecture and some Variations.** An intensively studied problem in frame theory [5, 10, 11] is the Feichtinger Conjecture - which we now know is equivalent to the Kadison-Singer Problem [11]. It first arose in 2003 and appeared in print for the first time in [5].

Feichtinger Conjecture. *Every bounded frame (equivalently, every unit norm frame) can be partitioned into a finite union of Riesz basic sequences.*

There are many variations of this conjecture – which are also known to be equivalent to the Kadison-Singer Problem.

Conjecture 3.2 (Weak Feichtinger Conjecture)**.** *Every bounded Bessel sequence can be partitioned into a finite union of Riesz basic sequences.*

Another equivalent form due to Casazza, Kutyniok, and Speegle [8] is

Conjecture 3.3. *Every bounded Bessel sequence can be partitioned into a finite union of frame sequences.*

There are also finite forms of the Feichtinger Conjecture [5].

Conjecture 3.4 (Finite Feichtinger Conjecture)**.** *For every $B, C > 0$ there is a natural number $M = M(B, C)$ and an $A = A(B, C) > 0$ so that whenever $\{f_i\}_{i\in I}$ is a frame for ℓ_2^N ($N \in \mathbb{N}$) with upper frame bound B and $\|f_i\| \geq C$ for all $i \in I$, then I can be partitioned into $\{I_j\}_{j=1}^M$ so that for each $1 \leq j \leq M$, $\{f_i\}_{i\in I_j}$ is a Riesz basic sequence with lower Riesz basis bound A and upper Riesz basis bound B.*

3.3. **Equivalent Conjectures for Bessel Sequences and Frames.** Weaver [16] has given some equivalent forms of the Kadison-Singer Problem in terms of Bessel sequences and frames.

Conjecture 3.5 (KS_r)**.** *There is a natural number r and universal constants B and $\epsilon > 0$ so that the following holds. Let $\{f_i\}_{i=1}^M$ be elements of ℓ_2^n with $\|f_i\| \leq 1$ for $i = 1, 2, \cdots, M$ and suppose $\{f_i\}_{i=1}^M$ is a B-Bessel sequence (or a B-tight frame). There is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$ the family $\{f_i\}_{i\in A_j}$ has Bessel bound $B - \epsilon$.*

Weaver further showed that Conjecture 3.5 is equivalent to the Kadison-Singer Problem even if we strengthen its assumptions so as to require that the vectors $\{f_i\}_{i=1}^M$ are of equal norm and that our frame is tight, but at great cost to our $\epsilon > 0$.

Conjecture 3.6. *There are universal constants $B \geq 4$ and $\epsilon > \sqrt{B}$ and an $r \in \mathbb{N}$ so that the following holds: Whenever $\{f_i\}_{i=1}^M$ is a unit norm B-tight frame for ℓ_2^n, there exists a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, M\}$ so that for all $j = 1, 2, \cdots, r$ the family $\{f_i\}_{i\in A_j}$ has Bessel bound $B - \epsilon$.*

We give one final conjecture in frame theory which is equivalent to the Kadison-Singer Problem [10].

Conjecture 3.7. *For every unit norm B-Bessel sequence $\{f_i\}_{i=1}^M$ in $\mathbb{H}_N$ and every $\epsilon > 0$, there exists $r = f(B, \epsilon)$ and a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, M\}$*

so that for every $j = 1, 2, \cdots, r$ and all scalars $\{a_i\}_{i\in A_j}$ we have

$$\sum_{n\in A_j} |\langle f_n, \sum_{n\neq m\in A_j} a_m f_m\rangle|^2 \leq \epsilon \| \sum_{m\in A_j} a_m f_m\|^2.$$

4. Frames of Translates and Gabor Frames

We begin with stating the necessary definitions. As usual, for some $g \in L^2(\mathbb{R})$, the **modulation of** g **by** $y \in \mathbb{R}$ is defined by

$$M_y(f)(x) = e^{2\pi iyx} f(x)$$

and the **translation of** g **by** $z \in \mathbb{R}$ is defined by

$$T_z(f)(x) = f(x-z).$$

These operator lead to the definition of Gabor frames and – as a special case – to frames of translates.

Definition 4.1. *Let $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}^+$. If the family $\{M_{am}T_{bn}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, we call this a* **Gabor frame***. If it is just a Bessel sequence, we call it a* **Bessel Gabor system***.*

If $\Lambda \subset \mathbb{R}^2$ and $\{M_yT_zg\}_{(y,z)\in\Lambda}$ is a frame (respectively, a Bessel sequence) for $L^2(\mathbb{R})$ we call this an **irregular Gabor frame** *(respectively, an* **irregular Bessel Sequence***). In the special case that $\Lambda = \{1\} \times \Lambda_2$ ($\Lambda_2 \subset \mathbb{R}$) and $\{T_zg\}_{z\in\Lambda_2}$ is a frame for its closed linear span, we call this a* **frame of translates***.*

It is known [5] that Gabor frames with a, b rational can be written as finite unions of Riesz basic sequences. Also, Gabor frames which are sufficiently **localized** can be written as finite unions of Riesz basic sequences [1, 2, 13]. But the general question for Gabor frames (and irregular Bessel Gabor systems) remains open.

Problem 4.2. *Can every Gabor frame be written as a finite union of Riesz basic sequences?*

Problem 4.3. *Can every Bessel Gabor system be written as a finite union of Riesz basic sequences?*

Problem 4.4. *Can every irregular Bessel Gabor system be written as a finite union of Riesz basic sequences?*

It is shown in [4] that every Bessel sequence of translates can be divided into two sets such that the following holds: for each of these sets, any subset is a frame for its closed linear span if and only if it is a Riesz basic sequence. Therefore we ask:

Problem 4.5. *Can every frame of translates be written as a finite union of Riesz basic sequences?*

5. The Rado-Horn Theorem

The Rado-Horn Theorem [14, 15] (see Edmonds and Fulkerson [12] for the matroid case or Casazza, Kutyniok, and Speegle [9] for information on the redundant case) is a particularly useful tool for relating frame theory to the Kadison-Singer Problem. This theorem gives a characterization of those sets of vectors which can be written as the finite union of M linearly independent sets.

Theorem 5.1 (Rado-Horn). *Let I be a countable index set and let $\{f_i : i \in I\}$ be a collection of vectors in a real vector space. There is a partition $\{I_j : j = 1, \cdots, M\}$ such that for each $1 \leq j \leq M$, $\{f_i : i \in I_j\}$ is linearly independent if and only if for all finite $J \subset I$*

$$\frac{|J|}{\dim \operatorname{span} \{f_i : i \in J\}} \leq M. \tag{5.1}$$

A standard application of the Rado-Horn Theorem to frame theory is the following (cf. [11, 3]).

Proposition 5.2. *Every equal norm Parseval frame $\{f_i\}_{i=1}^{KN}$ for $\mathbb{H}_N$ can be partitioned into K linearly independent spanning sets.*

In light of this relation we can interpret the Kadison-Singer Problem as just a quantitative version of the Rado-Horn Theorem. However, there are two difficulties involved. First, in light of recent results by Casazza, Edidin, Kalra, and Paulsen [7], the Kadison-Singer Problem requires both a quantitative version of Rado-Horn as well as a division into a larger number of subsets. That is, in [7] it is shown that Problem 3.1 fails for $r = 2$ while by Proposition 3.1 the Rado-Horn Theorem does decompose all such frames into two linearly independent sets. The advantage of this approach is that it could lead to an **algorithm** for solving the Kadison-Singer Problem which will most likely have broad applications. Second, the known proofs of Theorem 5.1 share a similar flavor. They all take a careful accounting of where vectors are, and where they can be moved in order to increase the linear independence of the system. As such, the techniques do not seem amenable for partitioning into Riesz sequences.

6. Problems weaker than the Kadison-Singer Problem

Definition 6.1. *A sequence of vectors $\{f_i\}_{i\in I}$ in $\mathbb{H}$ is ω-independent if*

$$\sum_{i\in I} a_i f_i = 0 \Rightarrow a_i = 0, \quad \text{for all } i \in I.$$

Using the Rado-Horn Theorem it was shown in [5] that every bounded Bessel sequence is a finite union of linearly independent sets. The next level of this problem is:

Problem 6.2. *Can every unit norm Bessel sequence be written as a finite union of ω-independent sequences?*

Unfortunately, we are not aware of a useful criterion (such as in the Rado-Horn Theorem) for determining when a sequence can be partitioned into ω-independent sequences.

Definition 6.3. *A sequence of vectors $\{f_i\}_{i\in I}$ in $\mathbb{H}$ is a minimal system with constant δ if for all $i \in I$ we have*

$$dist\left(f_i, \overline{span}_{i\neq j\in I} f_j\right) \geq \delta.$$

So a problem stronger than Problem 3.1 is:

Problem 6.4. *Can every unit norm Bessel sequence be written as a finite union of minimal systems with constant δ?*

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