PROBLEMS ON FRAMES AND THE KADISON-SINGER PROBLEM

Abstract. Please send problems, which should be posted in this section, to Pete Casazza at pete@math.missouri.edu. We would like to refer to the introduction to this section and [2] for an introduction to frames and the Kadison-Singer Problem as well as the notation used in this section.

1. Frame Theory Problems Which Are Equivalent to the Kadison-Singer Problem

Problem 1.1: [Feichtinger Conjecture] [3]. Can every unit norm frame be written as a finite union of Riesz basic sequences?

Problem 1.2: [Weak Feichtinger Conjecture] [3]. Can every unit norm Bessel sequence be written as a finite union of Riesz basic sequences?

Problem 1.3: [6] Can every unit norm Bessel sequence be written as a finite union of frame sequences?

Problem 1.4: [4] Does there exist an $\epsilon > 0$ and a natural number r so that for all equal norm Parseval frames $\{f_i\}_{i=1}^{2n}$ for ℓ_2^n there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, 2n\}$ so that $\{f_i\}_{i \in A_j}$ has Bessel bound $\leq 1 - \epsilon$ for all $j = 1, 2, \cdots, r$?

Problem 1.5: [Finite Feichtinger Conjecture] [3] For every $B, C > 0$ is there a natural number $M = M(B, C)$ and an $A = A(B, C) > 0$ so that whenever $\{f_i\}_{i \in I}$ is a frame for ℓ_2^N $(N \in \mathbb{N})$ with upper frame bound B and $\|f_i\| \geq C$ for all $i \in I$, then I can be partitioned into $\{A_j\}_{j=1}^M$ so that for each $1 \leq j \leq M$, $\{f_i\}_{i \in A_j}$ is a Riesz basic sequence with lower Riesz basis bound A (and upper Riesz basis bound B)?

Problem 1.6: [7] Are there universal constants B and $\epsilon > 0$ and a natural number r so that the following holds? Let $\{f_i\}_{i=1}^M$ be elements of ℓ_2^n with $\|f_i\| \leq 1$ for $i = 1, 2, \cdots, M$ and suppose $\{f_i\}_{i=1}^M$ is a B-Bessel sequence (or a B-tight frame). There is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$ the family $\{f_i\}_{i \in A_j}$ has Bessel bound $B - \epsilon$.

Problem 1.7: [7] Are there universal constants $B \geq 4$ and $\epsilon > \sqrt{B}$ and an $r \in \mathbb{N}$ so that the following holds? Whenever $\{f_i\}_{i=1}^M$ is a unit norm B-tight frame for ℓ_2^n, there exists a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, M\}$ so that for all $j = 1, 2, \cdots, r$ the family $\{f_i\}_{i \in A_j}$ has Bessel bound $B - \epsilon$.

Problem 1.8: [5] For every unit norm B-Bessel sequence $\{f_i\}_{i=1}^M$ in $\mathbb{H}_N$ and every $\epsilon > 0$, does there exist $r = r(B, \epsilon)$ and a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, M\}$ so that for every $j = 1, 2, \cdots, r$ and all scalars $\{a_i\}_{i \in A_j}$ we have

$$\sum_{n \in A_j} |\langle f_n, \sum_{n \neq m \in A_j} a_m f_m \rangle|^2 \leq \epsilon \| \sum_{m \in A_j} a_m f_m \|^2 ?$$

R_ϵ**-Conjecture** [Casazza, Vershynin] For every $\epsilon > 0$, every unit norm Riesz basic sequence is a finite union of ϵ-Riesz basic sequences.

2. Frame Theory Problems Which May Be Easier Than the Kadison-Singer Problem

Problem 2.1: Can every unit norm Bessel sequence be written as a finite union of ω-independent sets?
[Casazza, Kutyniok, Nikolskii, Speegle, Tremain]

Problem 2.2: Can every unit norm Bessel sequence be written as a finite union of minimal systems with constant δ?
[Casazza, Tremain]

Problem 2.3: Can every unit norm Bessel sequence which is a minimal system with constant δ be written as a finite union of Riesz basic sequences? [A positive solution to KS is equivalent to having positive solutions to both Problems 2.2 and 2.3.]
[Casazza, Tremain]

3. Frame Theory Problems Related to the Kadison-Singer Problem

Problem 3.1: Can every frame of translates be written as a finite union of Riesz basic sequences?

Problem 3.2: Can every regular (or irregular) Bessel Gabor system be written as a finite union of Riesz basic sequences?
[Feichtinger]

Problem 3.3: Assume $\{x_i\}_{i \in I}$ is a sequence in $\mathbb{H}$ such that the operator

$$\sum_{i \in I} x_i \otimes x_i$$

is bounded below on the closed linear span of $\{x_i\}_{i\in I}$. Find a "nice" criterion for which $\{x_i\}_{i\in I}$ is a Riesz basis for its closed linear span.

[Larson]

Problem 3.4: [1] For every unit norm frame $\{f_i : i \in \mathbb{N}\}$ indexed by the natural numbers, does there exist a set $K \subset \mathbb{N}$ such that $\{f_i : i \in K\}$ is a Riesz basic sequence and

$$\lim_{n\to\infty} \frac{\#(K \cap [1,n])}{n} > 0? \tag{3.1}$$

[It is not known if a positive solution to KS would yield a positive solution to this problem. It is easy to see that there is a partition of the natural numbers into two sets, neither of which satisfy (3.1).]

References

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[7] N. Weaver, *The Kadison-Singer Problem in descrepancy theory*, Discrete Math. **278** (2004), 227–239.