THE PAVING CONJECTURE IS EQUIVALENT TO THE PAVING CONJECTURE FOR TRIANGULAR MATRICES

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Abstract. We resolve a 25 year old problem by showing that The Paving Conjecture is equivalent to The Paving Conjecture for Triangular Matrices.

1. Introduction

The Kadison-Singer Problem [14] has been one of the most intractable problems in mathematics for nearly 50 years.

Kadison-Singer Problem (KS)**.** *Does every pure state on the (abelian) von Neumann algebra $\mathbb{D}$ of bounded diagonal operators on ℓ_2 have a unique extension to a (pure) state on $B(\ell_2)$, the von Neumann algebra of all bounded linear operators on the Hilbert space ℓ_2?*

A **state** of a von Neumann algebra $\mathcal{R}$ is a linear functional f on $\mathcal{R}$ for which $f(I) = 1$ and $f(T) \geq 0$ whenever $T \geq 0$ (i.e. whenever T is a positive operator). The set of states of $\mathcal{R}$ is a convex subset of the dual space of $\mathcal{R}$ which is compact in the w^*-topology. By the Krein-Milman theorem, this convex set is the closed convex hull of its extreme points. The extremal elements in the space of states are called the **pure states** (of $\mathcal{R}$). The Kadison-Singer Problem had been dorment for many years when it was recently brought back to life in [9] and [10] where it was shown that KS is equivalent to fundamental unsolved problems in a dozen different areas of research in pure mathematics, applied mathematics and engineering.

A significant advance on the Kadison-Singer Problem was made by Anderson [2] in 1979 when he reformulated KS into what is now known as the **Paving Conjecture** (Lemma 5 of [14] shows a connection between KS and Paving). Before we state this conjecture, let us introduce some notation. For an operator T on ℓ_2^n, its matrix representation $(\langle Te_i, e_j\rangle)_{i,j\in I}$ is with respect to the natural orthonormal basis. If $A \subset \{1, 2, \cdots, n\}$, the **diagonal projection** Q_A is the matrix all of whose entries are zero except for the (i, i) entries for $i \in A$ which are all one.

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Paving Conjecture (PC)**.** *For $\epsilon > 0$, there is a natural number r so that for every natural number n and every linear operator T on l_2^n whose matrix has zero diagonal, we can find a partition (i.e. a paving) $\{A_j\}_{j=1}^r$ of $\{1, \cdots, n\}$, so that*

$$\|Q_{A_j} T Q_{A_j}\| \leq \epsilon \|T\| \quad \text{for all } j = 1, 2, \cdots, r.$$

It is important that r not depend on n in PC. We will say that an arbitrary operator T satisfies PC if $T - D(T)$ satisfies PC where $D(T)$ is the diagonal of T. It is known that the class of operators satisfying PC (the **pavable operators**) is a closed subspace of $B(\ell_2)$. Also, to verify PC we only need to verify it for any one the following classes of operators [1, 10, 8]: 1. unitary operators, 2. positive operators, 3. orthogonal projections (or just orthogonal projections with $1/2's$ on the diagonal), 4. Gram operators of the form $T^*T = (\langle f_i, f_j \rangle)_{i,j \in I}$ where $\|f_i\| = 1$ and $Te_i = f_i$ is a bounded operator. The only large classes of operators which have been shown to be pavable are "diagonally dominant" matrices [3, 4, 12], matrices with all entries real and positive [5, 13] and matrices with small entries [6].

Since the beginnings of the *paving era*, it has been a natural question whether PC is equivalent to PC for triangular operators This question was formally asked several times at meetings by Gary Weiss and Lior Tzafriri and appeared (for a short time) on the AIM website (http://www.aimath.org/The Kadison-Singer Problem) as an important question for PC. In this paper we will verify this conjecture. Given two conjectures C_1, C_2 we say that C_1 **implies** C_2 if a positive answer to C_1 implies a positive answer for C_2. They are **equivalent** if they imply each other.

2. Preliminaries

Recall that a family of vectors $\{f_i\}_{i \in I}$ is a **Riesz basic sequence** in a Hilbert space $\mathbb{H}$ if there are constants $A, B > 0$ so that for all scalars $\{a_i\}_{i \in I}$ we have:

$$A^2 \sum_{i \in I} |a_i|^2 \leq \| \sum_{i \in I} a_i f_i \|^2 \leq B^2 \sum_{i \in I} |a_i|^2.$$

We call A, B the **lower and upper Riesz basis bounds** for $\{f_i\}_{i \in I}$. If $\epsilon > 0$ and $A = 1 - \epsilon, B = 1 + \epsilon$ we call $\{f_i\}_{i \in I}$ an ϵ-**Riesz basic sequence**. If $\|f_i\| = 1$ for all $i \in I$ this is a **unit norm** Riesz basic sequence. A natural question is whether we can improve the Riesz basis bounds for a unit norm Riesz basic sequence by partitioning the sequence into subsets.

R_ϵ**-Conjecture.** *For every $\epsilon > 0$, every unit norm Riesz basic sequence is a finite union of ϵ-Riesz basic sequences.*

The R_ϵ-Conjecture was posed by Casazza and Vershynin [11] where it was shown that KS implies this conjecture. It is now known that the R_ϵ-Conjecture is equivalent to KS [9]. We will show that PC for triangular operators implies a positive solution to the R_ϵ-Conjecture. Actually, we need the finite dimensional quantative version of this conjecture.

Finite R_ϵ-Conjecture. *Given $0 < \epsilon, A, B$, there is a natural number $r = r(\epsilon, A, B)$ so that for every $n \in \mathbb{N}$ and every unit norm Riesz basic sequence $\{f_i\}_{i=1}^n$ for ℓ_2^n with Riesz basis bounds $0 < A \leq B$, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$ the family $\{f_i\}_{i \in A_j}$ is an ϵ-Riesz basic sequence.*

There are standard methods for turning infinite dimensional results into quantative finite dimensional results so we will just outline the proof of their equivalence. We will need a proposition from [7].

Proposition 2.1. *Fix a natural number r and assume for every natural number n we have a partition $\{A_i^n\}_{i=1}^r$ of $\{1, 2, \cdots, n\}$. Then there are natural numbers $\{n_1 < n_2 < \cdots\}$ so that if $j \in A_i^{n_j}$ for some $i \in \{1, \cdots, r\}$, then $j \in A_i^{n_k}$, for all $k \geq j$. Hence, if $A_i = \{j \mid j \in A_i^{n_j}\}$ then*

(1) $\{A_i\}_{i=1}^r$ is a partition of $\mathbb{N}$.

(2) If $A_i = \{j_1^i < j_2^i < \cdots\}$ then for every natural number k we have $\{j_1^i, j_2^i, \cdots, j_k^i\} \subset A_i^{n_{j_k}}$.

Theorem 2.2. *The R_ϵ-Conjecture is equivalent to the Finite R_ϵ-Conjecture.*

Proof. Assume the Finite R_ϵ-Conjecture is true. Let $\{f_i\}_{i=1}^\infty$ be a unit norm Riesz basic sequence in $\mathbb{H}$ with bounds $0 < A, B$ and fix $\epsilon > 0$. Then there is a natural number $r \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ there is a partition $\{A_j^n\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ and for every $j = 1, 2, \cdots, r$ the family $\{f_i\}_{i \in A_j^n}$ is an ϵ-Riesz basic sequence. Choose a partition $\{A_j\}_{j=1}^r$ of $\mathbb{N}$ satisfying Proposition 2.1. By (2) of this proposition, for each $j = 1, 2, \cdots r$, the first n-elements of $\{f_i\}_{i \in A_j}$ come from one of the A_ℓ^m and hence form an ϵ-Riesz basic sequence. So $\{f_i\}_{i \in A_j}$ is an ϵ-Riesz basic sequence.

Now assume the the Finite R_ϵ-Conjecture fails. Then there is some $0 < \epsilon, A, B$, natural numbers $n_1 < n_2 < \cdots$ and unit norm Riesz basic sequences $\{f_i^r\}_{i=1}^{n_r}$ for $\ell_2^{n_r}$ so that whenever $\{A_j\}_{j=1}^r$ is a partition of $\{1, 2, \cdots, n_r\}$ one of the sets $\{f_i^r\}_{i \in A_j}$ is not an ϵ-Riesz basic sequence. Considering

$$\{f_i\}_{i=1}^\infty = \{f_i^r\}_{i=1, r=1}^{n_r, \infty} \in \left(\sum_{r=1}^\infty \oplus \ell_2^{n_r} \right)^{1/2},$$

we see that this family of vectors forms a unit norm Riesz basic sequence with bounds $0 < A, B$ but for any natural number r and any partition $\{A_j\}_{j=1}^r$ of $\mathbb{N}$ one of the sets $\{f_i\}_{i \in A_j}$ is not an ϵ-Riesz basic sequence. □

3. The Main Theorem

Our main theorem is:

Theorem 3.1. *The Paving Conjecture is equivalent to the Paving Conjecture for Triangular matrices.*

Proof. Since a paving of T is also a paving of T^*, we only need to show that The Paving Conjecture for Lower Triangular Operators implies the Finite R_ϵ-Conjecture. Fix $0 < \epsilon, A, B$, fix $n \in \mathbb{N}$ and let $\{f_i\}_{i=1}^n$ be a unit norm Riesz basis for ℓ_2^n with bounds A, B. We choose a natural number $r \in \mathbb{N}$ satisfying:

$$1 - \frac{B^4}{A^4 r} \geq 1 - \frac{\epsilon}{2}.$$

We will do the proof in 5 steps.

Step 1: There is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for every $j = 1, 2, \cdots, r$ and every $i \in A_j$ and every $1 \leq k \neq j \leq r$ we have:

$$\sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2 \leq \sum_{\ell \in A_k} |\langle f_i, f_\ell \rangle|^2.$$

The argument for this is due to Halpern, Kaftal and Weiss ([13], Proposition 3.1) so we will outline it for our case. Out of all ways of partitioning $\{1, 2, \cdots, n\}$ into r-sets, choose one, say $\{A_j\}_{j=1}^r$, which minimizes

$$\sum_{j=1}^r \sum_{i \in A_j} \sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2. \tag{3.1}$$

We now observe that for each $1 \leq j \leq r$, each $i \in A_j$ and all $1 \leq k \neq j \leq r$ we have

$$\sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2 \leq \sum_{\ell \in A_k} |\langle f_i, f_\ell \rangle|^2.$$

To see this, assume this inequality fails. That is, for some j_0, i_0, k_0 as above we have

$$\sum_{i_0 \neq \ell \in A_{j_0}} |\langle f_{i_0}, f_\ell \rangle|^2 > \sum_{\ell \in A_{k_0}} |\langle f_{i_0}, f_\ell \rangle|^2.$$

We define a new partition $\{B_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ by: $B_j = A_j$ if $j \neq j_0, k_0$; $B_{j_0} = A_{j_0} - \{i_0\}$; $B_{k_0} = A_{k_0} \cup \{i_0\}$. It easily follows that

$$\sum_{j=1}^r \sum_{i \in B_j} \sum_{i \neq \ell \in B_j} |\langle f_i, f_\ell \rangle|^2 < \sum_{j=1}^r \sum_{i \in A_j} \sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2,$$

which contradicts the minimality of Equation 3.1.

Step 2: For every $j = 1, 2, \cdots, r$ and every $i \in A_j$ we have

$$\sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2 \leq \frac{B^2}{r}.$$

Define an operator $Sf = \sum_{i=1}^{n} \langle f, f_i \rangle f_i$. Then,

$$\langle Sf, f \rangle = \sum_{i=1}^{r} |\langle f, f_i \rangle|^2,$$

and since $\{f_i\}_{i=1}^n$ is a Riesz basis with bounds A, B we have

$$A^2 I \leq S \leq B^2 I.$$

Now, by Step 1,

$$\begin{aligned}
\sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2 &\leq \frac{1}{r} \left[\sum_{i \neq \ell \in A_j} |\langle f_i, f_\ell \rangle|^2 + \sum_{j \neq k = 1}^{r} \sum_{\ell \in A_k} |\langle f_i, f_\ell \rangle|^2 \right] \\
&\leq \frac{1}{r} \sum_{i=1}^{n} |\langle f_i, f_\ell \rangle|^2 \\
&\leq \frac{1}{r} \|S\| \|f_i\|^2 \leq \frac{B^2}{r}.
\end{aligned}$$

Step 3: For each $j = 1, 2, \cdots, r$ and all $i \in A_j$, if P_{ij} is the orthogonal projection of span $\{f_\ell\}_{\ell \in A_j}$ onto span $\{f_\ell\}_{i \neq \ell \in A_j}$ then

$$\|P_{ij} f_i\|^2 \leq \frac{B^4}{A^4 r}.$$

Define the operator S_{ij} on span $\{f_\ell\}_{\ell \in A_j}$ by

$$S_{ij}(f) = \sum_{i \neq \ell \in A_j} \langle f, f_\ell \rangle f_\ell.$$

Then $A^2 I \leq S_{ij} \leq B^2 I$ and $\{S_{ij}^{-1} f_\ell\}_{i \neq \ell \in A_j}$ are the dual functionals for the Riesz basic sequence $\{f_\ell\}_{i \neq \ell \in A_j}$. Also, as in Step 1, $A^2 I \leq S_{ij} \leq B^2 I$. So by

Step 2,

$$\begin{aligned}\|P_{ij}f_i\|^2 &= \|\sum_{i\neq\ell\in A_j}\langle f_i,f_\ell\rangle S_{ij}^{-1}f_\ell\|^2\\ &\le \|S_{ij}^{-1}\|^2\|\sum_{i\neq\ell\in A_j}\langle f_i,f_\ell\rangle f_\ell\|^2\\ &\le \frac{B^2}{A^4}\sum_{i\neq\ell\in A_j}|\langle f_i,f_\ell\rangle|^2 \le \frac{B^4}{A^4 r}.\end{aligned}$$

Step 4: Fix $1\le j\le r$ and let $A_j=\{i_1,i_2,\cdots,i_k\}$. If we Gram-Schmidt $\{f_{i_\ell}\}_{\ell=1}^k$ to produce an orthonormal basis $\{e_{i_\ell}\}_{\ell=1}^k$ then for all $1\le m\le k$ we have

$$|\langle f_{i_m},e_{i_m}\rangle|^2 \ge 1-\frac{\epsilon}{2}.$$

Fix $1\le m\le k$ and let Q_m be the orthogonal projection of span $\{e_{i_\ell}\}_{\ell=1}^k$ onto span $\{e_{i_\ell}\}_{\ell=1}^m=$ span $\{f_{i_\ell}\}_{\ell=1}^m$. By Step 3,

$$\|Q_m f_{i_m}\|^2 \le \|P_{mj}f_{i_m}\|^2 \le \frac{B^4}{A^4 r}.$$

Since

$$f_{i_m}=\sum_{\ell=1}^m \langle f_{i_\ell},e_{i_\ell}\rangle e_{i_\ell},$$

we have

$$\begin{aligned}|\langle f_{i_m},e_{i_m}\rangle|^2 &= \|f_{i_m}\|^2-\|Q_{m-1}f_{i_m}\|^2\\ &\ge 1-\frac{B^4}{A^4 r}\ge 1-\frac{\epsilon}{2},\end{aligned}$$

where the last inequality follows from our choice of r.

Step 5: We complete the proof.

Let

$$M=\left(\langle f_{i_s},e_{i_t}\rangle\right)_{s\neq t=1}^k,$$

where by this notation we mean the $k\times k$-matrix with zero diagonal and the given values off the diagonal. By the Gram-Schmidt Process, M is a lower triangular matrix with zero diagonal. Define an operator $T:\ell_2^k\to$ *span* $\{e_{i_\ell}\}_{\ell=1}^k$ by

$$T\left((a_{i_\ell})_{\ell=1}^k\right)=\sum_{\ell=1}^k a_{i_\ell}f_{i_\ell}.$$

If K is the matrix of T with respect to the orthonormal basis $\{e_{i_\ell}\}_{\ell=1}^{k}$ and $D = D(K)$ is the diagonal of K then $M = (K - D)^*$ and so

$$\|M\| \leq \|K\| + \|D\| = \|T\| + 1 \leq B + 1.$$

By The Paving Conjecture for lower triangular matrices, there is a natural number L_j (which is a function of $0 < \epsilon$ and B only) and a partition $\{B_\ell^j\}_{\ell=1}^{L_j}$ of $\{i_1, i_2 \cdots, i_k\}$ so that

$$\|Q_{B_\ell^j} M Q_{B_\ell^j}\| \leq \frac{\epsilon}{2},$$

for all $\ell = 1, 2, \cdots, L_j$ ($Q_{B_\ell^j}$ was defined in the introduction). Now, for all scalars $(a_{i_s})_{i_s \in B_\ell^j}$, if

$$f = \sum_{i_s \in B_\ell^j} a_{i_s} f_{i_s},$$

then

$$\begin{aligned}
\|\sum_{i_s \in B_\ell^j} a_{i_s} f_{i_s}\| &= \|D(f) + Q_{B_\ell^j} M^* Q_{B_\ell^j}(f)\| \\
&\geq \|Df\| - \|Q_{B_\ell^j} M^* Q_{B_\ell^j}(f)\| \\
&\geq (1 - \frac{\epsilon}{2})\|f\| - \frac{\epsilon}{2}\|f\| \\
&\geq (1 - \epsilon)\|f\|.
\end{aligned}$$

Similarly,

$$\|\sum_{i_s \in B_\ell^j} a_{i_s} f_{i_s}\| \leq (1 + \epsilon)\|f\|.$$

It follows that $\{f_i\}_{i \in B_\ell^j}$ is an ϵ-Riesz basic sequence for all $j = 1, 2, \cdots, r$ and all $\ell = 1, 2, \cdots, L_j$. Hence, the Finite R_ϵ-Conjecture holds which completes the proof of the theorem. □

Let us make an observation concerning the proof of the main theorem.

Definition 3.2. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of vectors in a Hilbert space $\mathbb{H}$. For each $i = 1, 2, \cdots$ let P_i be the orthogonal projection of $\mathbb{H}$ onto span $\{f_\ell\}_{i \neq \ell \in \mathbb{N}}$. Our sequence is said to be* ϵ**-minimal** *if $\|P_i\| \leq \epsilon$ for all $i = 1, 2, \cdots$.*

The first three steps of the proof of Theorem 3.1 yields:

Corollary 3.3. *If $\{f_i\}_{i=1}^{\infty}$ is a unit norm Riesz basic sequence in a Hilbert space $\mathbb{H}$ then for every $\epsilon > 0$ there is a partition $\{A_j\}_{j=1}^{r}$ of $\mathbb{N}$ so that for all $j = 1, 2, \cdots, r$, the family $\{f_i\}_{i \in A_j}$ is ϵ-minimal.*

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