NOTES ON THE KADISON-SINGER CONJECTURE FOR THE ARCC MEETING

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Abstract. These notes are intended solely for the ARCC meeting on the Kadison-Singer Conjecture and its audience. These notes excerpt results from an unedited "working" draft by the same set of authors and give proofs of some of the claims that we have made elsewhere. We apologize for their somewhat unpolished form.

1. Some New Equivalences of the Paving Conjecture

The following results on the paving conjectures are taken from [8].

Let me begin with the familiar.

Given $A \subseteq I$, where I is some index set, we let $Q_A \in B(\ell^2(I))$ denote the diagonal projection defined by $Q_A = (q_{i,j}), q_{i,i} = 1, i \in A, q_{i,i} = 0, i \notin A$ and $q_{i,j} = 0, i \neq j$.

Definition 1. *An operator $T \in B(\ell^2(I))$ is said to have an (r, ϵ)-paving if there is a partition of I into r subsets $\{A_j\}_{j=1}^r$ such that $\|Q_{A_j} T Q_{A_j}\| \leq \epsilon$. A collection of operators $\mathcal{C}$ is said to be (r, ϵ)-pavable if each element of $\mathcal{C}$ has an (r, ϵ)-paving.*

Note that in this definition, we do *not* require that the diagonal entries of the operator be 0.

Some classes that will play a role are:

- $\mathcal{C}_\infty = \{T = (t_{i,j}) \in B(\ell^2(\mathbb{N})) : \|T\| \leq 1, t_{i,i} = 0 \forall i \in \mathbb{N}\}$,
- $\mathcal{C} = \cup_{n=2}^\infty \{T = (t_{i,j}) \in M_n : \|T\| \leq 1, t_{i,i} = 0, i = 1, ..., n\}$,
- $\mathcal{S}_\infty = \{T \in \mathcal{C}_\infty : T = T^*\}$,
- $\mathcal{S} = \{T \in \mathcal{C} : T = T^*\}$,
- $\mathcal{R}_\infty = \{T \in \mathcal{S}_\infty : T^2 = I\}$,
- $\mathcal{R} = \{T \in \mathcal{S} : T^2 = I\}$,
- $\mathcal{P}_{1/2}^\infty = \{T = (t_{i,j}) \in B(\ell^2(\mathbb{N})) : T = T^* = T^2, t_{i,i} = 1/2, \forall i \in \mathbb{N}\}$,
- $\mathcal{P}_{1/2} = \cup_{n=2}^\infty \{T = (t_{i,j}) \in M_n : T = T^* = T^2, t_{i,i} = 1/2, i = 1, ..., n\}$.

Note that the operators satisfying, $R = R^*, R^2 = I$ are *reflections* and that for such an operator, $\sigma(R) = \{-1, +1\}$. Since, the traces of our matrices are 0, in the finite dimensional case, these types of reflections can only exist in even dimensions. If the space is $2n$-dimensional, then there exists an

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n-dimensional subspace that is fixed by R and such that for any vector x orthogonal to the subspace $Rx = -x$.

J. Anderson's[2] remarkable contribution follows.

Theorem 2 (Anderson). *The following are equivalent:*

1. *the Kadison-Singer conjecture is true,*
2. *for each $T \in \mathcal{C}_\infty$, there exists (r, ϵ)(depending on T) $\epsilon < 1$, such that T is (r, ϵ)-pavable,*
3. *there exists $(r, \epsilon), \epsilon < 1$, such that $\mathcal{C}_\infty$ is (r, ϵ)-pavablle,*
4. *there exists $(r, \epsilon), \epsilon < 1$, such that $\mathcal{C}$ is (r, ϵ)-pavable,*
5. *for each $T \in \mathcal{S}_\infty$, there exists $(r, \epsilon), \epsilon < 1$(depending on T), such that T is (r, ϵ)-pavable,*
6. *there exists $(r, \epsilon), \epsilon < 1$, such that $\mathcal{S}_\infty$ is (r, ϵ)-pavable,*
7. *there exists $(r, \epsilon), \epsilon < 1$, such that $\mathcal{S}$ is (r, ϵ)-pavable.*

Generally, when people talk about the paving conjecture they mean one of the above equivalences of the Kadison-Singer conjecture. Also, generally, when one looks at operators on an infinite dimensional space, it is enough to find (r, ϵ) depending on the operator, but for operators on finite dimensional spaces it is essential to have a uniform (r, ϵ). Finally, since $\mathcal{S}_\infty \subset \mathcal{C}_\infty$, people looking for counterexamples tend to study $\mathcal{C}_\infty$, while people trying to prove the theorem is true, study $\mathcal{S}_\infty$ or $\mathcal{S}$. However, by the above equivalences, if a counterexample exists in one set then it must exist in the other as well.

In this spirit, we showed that the following smaller sets with "more structure" are sufficient for paving.

Theorem 3. [8] *Let $\epsilon < 1$, then the following are equivalent:*

1. *the set $\mathcal{S}_\infty$ can be (r_1, ϵ)-paved,*
2. *the set $\mathcal{R}_\infty$ can be (r_1, ϵ)-paved,*
3. *the set $\mathcal{P}^\infty_{1/2}$ can be $(r_2, \frac{1+\epsilon}{2})$-paved,*
4. *the set $\mathcal{S}$ can be (r_1, ϵ)-paved,*
5. *the set $\mathcal{R}$ can be (r_1, ϵ)-paved,*
6. *the set $\mathcal{P}_{1/2}$ can be $(r_2, \frac{1+\epsilon}{2})$-paved.*

Proof. Since the reflections are a subset of the self-adjoint matrices, it is clear that (1) implies (2) and that (4) implies (5).

To see that (2) implies (1), let $A \in \mathcal{S}_\infty$, and set

$$R = \begin{pmatrix} A & \sqrt{I - A^2} \\ \sqrt{I - A^2} & -A \end{pmatrix},$$

then $R \in \mathcal{R}_\infty$ and clearly any (r, ϵ)-paving of R yields an (r, ϵ)-paving of A.

Thus, (1) and (2) are equivalent and similarly, (4) and (5) are equivalent.

To see the equivalence of (2) and (3), note that $R \in \mathcal{R}_\infty$(respectively, $\mathcal{R}$) if and only if $P = (I+R)/2 \in \mathcal{P}^\infty_{1/2}$(respectively, $\mathcal{P}_{1/2}$). Also, if $\|Q_A R Q_A\| \le \epsilon$, then $\|Q_A P Q_A\| \le (1 + \epsilon)/2$. Thus, if $\mathcal{R}_\infty$ can be (r_1, ϵ)-paved, then $\mathcal{P}^\infty_{1/2}$ can be $(r_1, \frac{1+\epsilon}{2})$-paved.

Conversely, given $R \in \mathcal{R}_\infty$, let $P = (I+R)/2$. If $\|Q_A P Q_A\| \leq (1+\epsilon)/2 = \beta$, then,

$$0 \leq Q_A P Q_A \leq \beta Q_A,$$

and since $R = 2P - I$, we have that

$$-Q_A \leq Q_A R Q_A \leq (2\beta - 1) Q_A = \epsilon Q_A.$$

Applying the same reasoning to the reflection $-R$, we get a new projection, $P_1 = (I-R)/2$, with a possibly different paving of P_1, such that $-Q_B \leq Q_B(-R)Q_B \leq \epsilon Q_B$. Thus, $-\epsilon Q_B \leq Q_B R Q_B$ and if $Q_C = Q_A Q_B$, we have that $-\epsilon Q_C \leq Q_C R Q_C \leq +\epsilon Q_C$. Therefore, we have that the set of all products of the Q_A's and Q_B's pave R. Thus, if $\mathcal{P}_{1/2}^\infty$ can be $(r_2, \frac{1+\epsilon}{2})$-paved, then $\mathcal{R}_\infty$ can be (r_2^2, ϵ)-paved.

The proof of the equivalence of (5) and (6), is identical.

Finally, (1) and (4) are equivalent by the standard limiting argument. In particular, see [6, Proposition 2.2] and the proof of [6, Theorem 2.3].

□

Corollary 4. [8] *The following are equivalent:*

1. *the Kadison-Singer conjecture is true,*
2. *for each $R \in \mathcal{R}_\infty$ there is a $(r, \epsilon), \epsilon < 1$(depending on R) such that R can be (r, ϵ)-paved,*
3. *there exists $(r, \epsilon), \epsilon < 1$, such that every $R \in \mathcal{R}$ can be (r, ϵ)-paved,*
4. *for each $P \in \mathcal{P}_{1/2}^\infty$ there is a $(r, \epsilon), \epsilon < 1$(depending on P) such that P can be (r, ϵ)-paved,*
5. *there exists $(r, \epsilon), \epsilon < 1$, such that every $P \in \mathcal{P}_{1/2}$ can be (r, ϵ)-paved.*

A sort of meta-corollary that goes with this latter result is that the frame based conjectures that are known to be equivalent to the Kadison-Singer result can be reduced to the case of uniform Parseval frames of redundancy 2. Similarly, for most harmonic analysis analogues of paving, it is enough to consider say subsets $E \subseteq [0, 1]$ of Lebesgue measure $1/2$. We state one such equivalence. Casazza and Tremain[7] have shown that the *Feichtinger conjecture* is equivalent to Kadison-Singer. The Feichtinger Conjecture states: *All bounded frames are finite unions of Riesz basic sequences.*

Theorem 5. *The Feichtinger conjecture is true if and only if for each Parseval frame $\{f_n\}_{n\in\mathbb{N}}$ for a Hilbert space with $\|f_n\|^2 = 1/2\ \forall n$ there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{N}$ into r disjoint subsets(with r depending on the frame) such that for each $j, \{f_n\}_{n\in A_j}$ is a Riesz basis for the space that it spans.*

Proof. Clearly, if the Feichtinger is true, then it is true for this special class of frames.

Conversely, assume that the above holds and let $P \in \mathcal{P}_{1/2}^\infty$. Then there exists a Parseval frame $\{f_n\}_{n\in\mathbb{N}}$ for some Hilbert space $\mathcal{H}$, such that $I - P = (\langle f_j, f_i\rangle)$ is their Grammian. Now let $\{A_k\}_{k=1}^r$ be the partition of $\mathbb{N}$ into r disjoint subsets as above and let $\mathcal{H}_k = span\{f_n : n \in A_k\}^-$ denote the closed linear span.

Since $\{f_n : n \in A_k\}$ is a Riesz basis for $\mathcal{H}_k$, there exists an orthonormal basis, $\{e_n : n \in A_k\}$ for $\mathcal{H}_k$ and a bounded invertible operator, $S_k : \mathcal{H}_k \to \mathcal{H}_k$, with $S_k(e_n) = f_n$.

We have that $Q_{A_k}(I-P)Q_{A_k} = (\langle f_j, f_j\rangle)_{i,j\in A_k} = (\langle S_k^* S_k e_j, e_i\rangle) \geq c_k Q_{A_k}$ where $S_k^* S_k \geq c_k Q_{A_k}$ for some constant $0 < c_k \leq 1$ since S_k is invertible. Hence, $Q_{A_k} P Q_{A_k} \leq (1-c_k)Q_{A_k}$ and we have that, $max\{\|Q_{A_k} P Q_{A_k}\| : 1 \leq k \leq r\} < 1$.

Hence, condition (5) of Corollary 4 is met and so Kadison-Singer is true and thus, by [7, Theorem 5.3], the Feichtinger conjecture is true. □

2. Some Paving Estimates

In this section we derive some estimates on paving constants that give some basic relationships between r and ϵ. In particular, we will prove that $\mathcal{P}_{1/2}$ cannot be $(2,\epsilon)$-paved for any $\epsilon < 1$.

We begin with a result on paving $\mathcal{R}$.

Theorem 6. *Assume that $\mathcal{R}$ is (r,ϵ)-pavable. Then $1 \leq r\epsilon^2$.*

Proof. Recall that an $n \times n$ matrix C is a *conference matrix* if $C = C^*, c_{i,i} = 0, c_{i,j} = \pm 1, i \neq j$ and $C^2 = (n-1)I$. Such matrices exist for infinitely many n.

Set $A = \frac{1}{\sqrt{n-1}}C$, then A is a unitary matrix with zero diagonal.

Assume that $\{1,...,n\} = B_1 \cup ... \cup B_r$ is a partition such that $\|Q_{B_i} A Q_{B_i}\| \leq \epsilon$. Let $d = max\{card(B_i)\}$ and let B_j attain this max. Note that $d \geq \frac{n}{r}$.

Set $A_j = Q_{B_j} A Q_{B_j}$, then the Schur product $A_j * A_j = \frac{1}{n-1}[J_d - I_d]$ where J_d denotes the matrix of all 1's.

Hence, $\frac{d-1}{n-1} = \|A_j * A_j\| \leq \|A_j\|^2 \leq \epsilon^2$. Thus, $\frac{n/r-1}{n-1} \leq \epsilon^2$, and the result follows by letting $n \to +\infty$ □

Proposition 7. *If every projection $P \in \mathcal{P}_{1/2}$ can be (r,ϵ)-paved then every projection Q with*

$$\frac{1}{2} - \delta \leq \langle Qe_i, e_i\rangle \leq \frac{1}{2} + \delta,$$

can be (r,β)-paved, where

$$\beta = (1+2\delta)\epsilon,$$

and so $\beta < 1$ when δ is small enough.

Proof. Let D be the diagonal of Q and let $B = Q - D$. Then

$$\|B\| \leq \frac{1+2\delta}{2}.$$

To see this note that for any vector x

$$0 \leq \langle Bx, x\rangle + \langle Dx, x\rangle \leq 1,$$

since Q is a projection. Hence,

$$-\langle Dx, x\rangle \leq \langle Bx, x\rangle \leq 1 - \langle Dx, x\rangle.$$

Hence,

$$\|B\| = \sup_{\|x\|=1} |\langle Bx, x\rangle| \leq max\{|\langle Dx, x\rangle|, |1 - \langle Dx, x\rangle|\} \leq \frac{1}{2} + \delta = \frac{1+2\delta}{2}.$$

Let $R = R^*$ be the symmetry we get by dilating

$$\frac{2}{1+2\delta}B,$$

as in the proof of Theorem 3. Let $P = \frac{1}{2}(I + R)$ be the projection with $1/2's$ on the diagonal. If we can (r, ϵ)-pave P with $\{A_j\}_{j=1}^r$ then we have (r, ϵ)-paved

$$\frac{1}{2}I + \frac{1}{1+2\delta}B.$$

Substituting $B = Q - D$ we have an (r, ϵ)-paving of

$$\frac{1}{1+2\delta}Q + \frac{1}{2}I - \frac{1}{1+2\delta}D = \frac{1}{1+2\delta}\left(Q + \frac{1+2\delta}{2}I - D\right).$$

Now, for any $j = 1, 2, \ldots, r$ since

$$\frac{1+2\delta}{2}I - D$$

is a positive operator,

$$\|Q_{A_j}QQ_{A_j}\| \leq \|Q_{A_j}(Q + \frac{1+2\delta}{2}I - D)Q_{A_j}\| \leq (1+2\delta)\epsilon < 1.$$

□

Theorem 8. *Assume that $\mathcal{P}_{1/2}$ can be (r, ϵ)-paved. Then $\frac{r}{2(r-1)} \leq \epsilon$.*

Note: When r=2 this implies that $\epsilon = 1$, and hence 2-paving is impossible.

Proof. Let $m > 2$ be an integer and consider a uniform, Parseval (n,k)-frame with $n = mr, k = m(r-1)+1$. This will give rise to a projection Q with diagonal entries, $\frac{m(r-1)+1}{mr} = \frac{1}{2} + \delta$, where $\delta = \frac{m(r-2)+3}{2mr}$. To see this, let

$$\begin{aligned} \delta &= \frac{m(r-1)+1}{mr} - \frac{1}{2} \\ &= \frac{2[m(r-1)+1] - mr}{2mr} \\ &= \frac{m(r-1)+2}{2mr}. \end{aligned}$$

By the above result, Q can be (r, β)-paved, where $\beta = (1+2\delta)\epsilon$.

However, for any r paving of Q, one of the blocks must be of size at least

$$n/r = m = n - k + 1,$$

by the choice of n and k. Since Q is a rank k projection, this block will have norm 1 by the eigenvalue inclusion principle or by the eigenvalue interlacing results. Hence $\beta \geq 1$. We solve for ϵ:

$$(1+2\delta)\epsilon \geq 1.$$

So

$$\begin{aligned} \epsilon &\geq \frac{1}{1+2\delta} \\ &= \frac{mr}{m(2r-2)+2}. \end{aligned}$$

Letting $m \to +\infty$ yields

$$\epsilon \geq \frac{r}{2(r-1)}.$$

□

Corollary 9. *The set $\mathcal{P}_{1/2}$ is not 2-pavable.*

Corollary 10. *The set $\mathcal{R}$ is not 2-pavable.*

We now generalize the results of the last theorem.

Theorem 11. *For each $r, n \in \mathbb{N}$ with $r > 1$ there is an $\epsilon_n > 0$ so that whenever P is a projection on ℓ_2^n with $\frac{1}{r} \leq \langle Pe_i, e_i \rangle \leq 1 - \frac{1}{r}$ for all $i = 1, 2, \ldots, n$ then P is $(r, 1-\epsilon_n)$-pavable.*

Moreover, for any $\delta > 0$ there is an $n \in \mathbb{N}$ and a projection P on ℓ_2^{2n} of rank n so that $\frac{1}{r} - \delta \leq \langle Pe_i, e_i \rangle \leq 1 - \frac{1}{r} + \delta$ for all $i = 1, 2, \ldots, 2n$ while P is not (r, ϵ)-pavable for any $\epsilon < 1$.

Proof. Given our assumptions, we will check the Rado-Horn Theorem (See [6, 7]) to see that the row vectors of our projection can be divided into r linearly independent sets. Then the rest of the first part of the theorem follows by the same argument (adjusted for r) as in the last theorem. For any $J \subset \{1, 2, \ldots, 2n\}$ let P_J be the orthogonal projection of ℓ_2^{2n} onto the span $\{Pe_i\}_{i \in J}$. Now,

$$dim\ span\ \{Pe_i\}_{i\in J} = \sum_{i=1}^{2n} \|P_J P e_i\|^2 \geq \sum_{i \in J} \|Pe_i\|^2 \geq |J| \frac{1}{r}.$$

By the Rado-Horn Theorem we can now write $\{Pe_i\}_{i=1}^{2n}$ as a union of r-linearly independent sets.

For the moreover part, choose a $k \in \mathbb{N}$ so that

$$\frac{1}{r} - \delta < \frac{k}{rk+1} \leq \frac{1}{r} \leq 1 - \frac{1}{r}.$$

Now, choose an n so that

$$\frac{1}{r} - \delta \leq \frac{n - rk}{2n - (rk+1)} \leq 1 - \frac{1}{r} + \delta.$$

With $\{e_i\}_{i=1}^n$ the unit vectors in ℓ_2^n we can choose an equal norm Parseval frame $\{f_i\}_{i=1}^{rk+1}$ for $\{e_i\}_{i=1}^k$. Next, choose an equal norm Parseval frame $\{f_i\}_{i=rk+2} 2n$ for $\{e_i\}_{i=k+1}^n$. Now,

$$\frac{1}{r} - \delta \leq \|f_i\|^2 = \frac{k}{rk+1} \leq \frac{1}{r} \leq 1 - \frac{1}{r},$$

and

$$\frac{1}{r} - \delta \le \frac{n - rk}{2n - (rk+1)} \le 1 - \frac{1}{r} + \delta.$$

Taking the embedding of this Parseval frame with $2n$-elements for ℓ_2^n we get a projection P on ℓ_2^{2n} which has rank n and looks like

$$\begin{bmatrix} \|f_1\|^2 & b_{1,2} & \dots & b_{1,(rk+1)} & 0 & 0 & \dots & 0 \\ b_{21} & \|f_1\|^2 & \dots & b_{2,(rk+1)} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ b_{(rk+1),1} & b_{(rk+1),2} & \dots & \|f_1\|^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \|f_{rk+2}\|^2 & a_{(rk+2),(rk+3)} & \dots & a_{(rk+2),2n} \\ 0 & 0 & \dots & 0 & a_{(rk+3),(rk+2)} & \|f_{rk+2}\|^2 & \dots & a_{(rk+3),2n} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & a_{(2n),(rk+2)} & a_{(2n),(rk+3)} & \dots & \|f_{rk+2}\|^2 \end{bmatrix}$$

For this projection, for any $J \subset \{1, 2, \dots n\}$ with $|J| > r$ the family $\{e_i\}_{i \in J}$ is linealy dependent and so PQ_A has a zero eigenvalue. Hence, $(I - P)Q_A$ has one as an eigenvalue and hence is not ϵ-pavable for any $\epsilon > 0$. □

3. Counterexamples to the Akemann-Anderson Conjectures

In [1] Akemann and Anderson introduce two paving conjectures, denoted *Conjecture A* and *Conjecture B.* They prove that Conjecture A implies Conjecture B and that Conjecture B implies Kadison-Singer, but it is not known if either of these implications can be reversed. Weaver[14] provides a set of counterexamples to Conjecture A. Thus, if these three statements were all equivalent then Weaver's counterexample would be the end of the story. However, it is generally believed that Conjecture A is strictly stronger than the Kadison-Singer conjecture.

In this section, we show that the Grammian projection matrices of *any* uniform, equiangular (n,k)-frame, with $n > 5k$ yield counterexamples to Conjecture A. It is known that infinitely many such frames exist for arbitrarily large n and k. The significance of our new set of counterexamples is that by the results of J. Bourgain and L. Tzafriri [5], there exists $\epsilon < 1$, such that the family of self-adjoint, norm one, 0 diagonal matrices obtained from these frames is $(2, \epsilon)$-pavable.

Thus, these new examples drive an additional wedge between Conjecture A and Kadison-Singer.

We then turn our techniques to Conjecture B and derive some results that could lead to a counterexample to Conjecture B.

We now describe the Akemann-Anderson conjectures. Let $P = (p_{i,j}) \in M_n$ be the matrix of a projection and set $\delta_P = max\{p_{i,i} : 1 \le i \le n\}$. By a *diagonal symmetry* we mean a diagonal matrix whose diagonal entries are ± 1, that is, S is a diagonal self-adjoint unitary.

Conjecture A [1, 7.1.1]**.** For any projection, P there exists a diagonal symmetry, S, such that $\|PSP\| \le 2\delta_P$.

Conjecture B [1, 7.1.3]**.** There exists $\gamma, \epsilon > 0$(and independent of n) such that for any P with $\delta_P < \gamma$ there exists a diagonal symmetry, S, such that $\|PSP\| < 1 - \epsilon$.

Weaver[14] states that a counterexample to Conjecture B would probably lead to a negative solution to Kadison-Singer. We believe that these two conjectures are really more closely related to 2-pavings and this is why we believe that counterexamples to Conjecture B should be close at hand.

Finally, note that Conjecture B is about paving projections with small diagonal. But our results show that Kadison-Singer is equivalent to paving projections with diagonal 1/2. This would also seem to put further distance between these Akemann-Anderson conjectures and the Kadison-Singer conjecture.

Proposition 12. *Let* $P = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *be a projection, written in block-form with* A $m \times m, B$ $m \times (m+l), C$ $(m+l) \times (m+l)$*, where* $l \geq 0$*. Then there exists a* $m \times m$ *unitary* U_1 *and an* $(m+l) \times (m+l)$ *unitary* U_2 *such that,* $U_1^* A U_1 = D_1, U_1^* B U_2 = (D_2, 0), U_2^* C U_2 = \begin{pmatrix} D_3 & 0 \\ 0 & D_4 \end{pmatrix}$ *where each of the* D_i*'s is a diagonal matrix with non-negative entries,* D_1, D_2, D_3 *are all* $m \times m, D_4$ *is* $l \times l$ *with* 1*'s and* 0*'s for its diagonal entries and the* 0*'s represent matrices of all zeroes that are either* $m \times l$ *or* $l \times m$*.*

Proof. First note that since P is a projection we have that $A^2 + BB^* = A, B^*B + C^2 = C$ and $AB + BC = B$. Also, since the rank of B is at most m, the matrix B^*B must have a kernel of dimension at least l.

Conjugating P by a unitary of the form $U = \begin{pmatrix} I_m & 0 \\ 0 & U_2 \end{pmatrix}$, we may diagonalize C and the new matrix, P_1, will still be a projection. Since $U_2^* B^* B U_2 = U_2^*(C - C^2)U_2$, we see that both sides of this equation are in diagonal form. Since at least l of the diagonal entries of $U_2^* B^* B U_2$ are zeroes, after applying a permutation if necessary, we may assume that,

$$U_2^* B^* B U_2 = \begin{pmatrix} D_2^2 & 0 \\ 0 & 0 \end{pmatrix}, U_2^* C U_2 = \begin{pmatrix} D_3 & 0 \\ 0 & D_4 \end{pmatrix},$$

where D_2, D_3, D_4 are as claimed.

Now we may polar decompose the $m \times (m+l)$ matrix $BU_2 = W|BU_2| = W \begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix}$, where W is a $m \times (m+l)$ partial isometry whose initial space is the range of $|BU_2|$. Thus, $W = (W_1, 0)$ where W_1 is an $m \times m$ partial isometry. Hence, we may extend W_1 to an $m \times m$ unitary U_1 with $W_1 D_2 = U_1 D_2$ and $BU_2 = (U_1, 0) \begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix} = (U_1 D_2, 0)$.

Conjugating P_1 by the unitary $\begin{pmatrix} U_1 & 0 \\ 0 & I_{m+l} \end{pmatrix}$ we arrive at a new projection of the form,

$$\begin{pmatrix} U_1^* A U_1 & D_2 & 0 \\ D_2 & D_3 & 0 \\ 0 & 0 & D_4 \end{pmatrix}.$$

Note that since this last matrix is a projection, $U_1^* A U_1 D_2 + D_2 D_3 = D_2$ and so, $U_1^* A U_1 D_2$ is diagonal. If all of the entries of D_2 were non-zero, then this would imply that $U_2^* A U_2$ is diagonal. In general, this implies that $U_1^* A U_1$(which is self-adjoint) is of the form diagonal direct sum with another matrix corresponding to the block where D_2 is 0. Conjugating $U_1^* A U_1$ by another unitary to diagonalize this lower block, yields the desired form.

Finally, note that since D_4 is a diagonal projection, all of its entries must be 0's or 1's. □

Lemma 13. *Let* $P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *be a non-zero projection with real entries and let* $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Then* $\|PSP\| = |1 - 2c|$.

Proof. If P is rank 2 then $P = I$ and the result is trivial. So assume that P is rank one. We have that $PSP = \begin{pmatrix} a^2 - b^2 & ab - bc \\ ab - bc & b^2 - c^2 \end{pmatrix}$ and since P is a rank one projection, $a + c = 1, b^2 + c^2 = c$. A little calculation shows that the characteristic polynomial of PSP is $x^2 - Tr(PSP)x + Det(PSP) = x^2 - (1 - 2c)x$, and hence the eigenvalues are 0 and 1-2c, from which the result follows. □

Note that when S is a diagonal symmetry, then $\|PSP\| = \|P(-S)P\|$ and so we may and do assume in what follows that the number of -1's is greater than or equal to the number of $+1$'s. Also, given a matrix A, we let $\sigma(A)$ denote the spectrum of A and set $\sigma'(A) \equiv \sigma(A) \backslash \{0\}$.

Theorem 14. *Let* $P = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *be an* $n \times n$ *projection and let* $S = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ *be a diagonal symmetry. Then* $\|PSP\| \geq \max\{|1 - 2\lambda| : \lambda \in \sigma'(C) \cup \sigma'(A)\}$.

Proof. Given any unitary of the type in the above Proposition, we have that $\|PSP\| = \|U^* PSPU\| = \|(U^* PU)(U^* SU)(U^* PU)\| = \|(U^* PU)S(U^* PU)\|$. Thus, we may and do assume that P has been replaced by $U^* PU$. But this reduces the norm calculation to the direct sum of a set of 2×2 matrices of the form of the lemma together with the diagonal projection D_4. Now if $\lambda \in \sigma'(C)$, then this 2×2 matrix is necessarily rank one and so the lemma applies. Note also that in this case the corresponding eigenvalue of D_1 is $1 - \lambda$ and that $|1 - 2(1 - \lambda)| = | - 1 + 2\lambda| = |1 - 2\lambda|$ so the values of this function agree. When $\lambda = 0$, then this 2×2 matrix is either the 0 matrix or it is rank 1 and the corresponding eigenvalue of D_1 is 1. □

We now provide a counterexample to Conjecture A.

Theorem 15. *Let $\{f_1, ..., f_n\}$ be a uniform equiangular Parseval frame for $\mathbb{C}^k$ with $n > 2k$ and let $P = (\langle f_i, f_j \rangle)$ be the correlation matrix. If there exists a diagonal symmetry, S, such that, $\|PSP\| \leq 2\delta_P = \frac{2k}{n}$, then $(k-1)n^2 \leq 4k^2(n-1)$.*

Proof. Without loss of generality we may assume that S is a diagonal symmetry with m diagonal entries that are $+1$ and $n-m$ diagonal entries that are -1 and, $m \leq n-m$. Putting P into the form of the Proposition, we see that since D_4 is a projection, if it is non-zero, then $\|PSP\| = 1$. So we may assume that $D_4 = 0$.

Similarly, if any of the diagonal entries of D_1 or D_3 are 1, then $\|PSP\| = 1$. Thus, when we put P into the form of the above Proposition, we obtain a direct sum of 2×2 rank 1 projections, together with some matrices of all 0's.

Let $0 < \lambda_1 \leq ... \leq \lambda_t < 1$, denote the non-zero diagonal entries of D_1, so that the corresponding diagonal entries of D_3 are $1-\lambda_1, ..., 1-\lambda_t$, and the remaining entries of D_3 are 0's. By the above Theorem, we have that $\|PSP\| = \max\{|1-2\lambda_1|, |1-2\lambda_k|\} = \max\{1-2\lambda_1, 2\lambda_k - 1\} \leq \frac{2k}{n}$. Hence, $\frac{n-2k}{2n} \leq \lambda_1$ and $\lambda_k \leq \frac{n+2k}{2n}$.

Since P is a rank k projection, we have that $k = Tr(P) = Tr(D_1) + Tr(D_3) = t$.

Since $Tr(D_1) = Tr(A) = mk/n$, we have that $0 < \lambda_1 \leq m/n \leq \lambda_k$. Hence, $\frac{n-2k}{2n} \leq m/n \leq \frac{n+2k}{2n}$ yielding $n \leq 2k + 2m$. Note also, that by the choice of m we have that $2m \leq n$, so that the other inequality is automatically satisfied.

If we let, $\mu_1, ..., \mu_k$ be the corresponding entries of D_2, then since each matrix, $\begin{pmatrix} \lambda_i & \mu_i \\ \mu_i & 1-\lambda_i \end{pmatrix}$ is a rank one projection and since $\mu_i \geq 0$, we have that $\mu_i^2 = \lambda_i(1-\lambda_i)$.

Since P is the correlation matrix of a uniform equiangular (n,k)-frame, by [9], we have that every off-diagonal entry of P is of constant modulus, $c = \sqrt{\frac{k(n-k)}{n^2(n-1)}}$. This yields,

$$\sum_{i=1}^{k} \mu_i^2 = Tr(B^*B) = m(n-m)c^2 \leq \frac{n^2c^2}{4} = \frac{k(n-k)}{4(n-1)}.$$

Now observe that the function $t(1-t)$ is increasing on [0,1/2] and decreasing on [1/2,1]. Thus, we have that $\min\{\lambda_1(1-\lambda_1), \lambda_k(1-\lambda_k)\} = \min\{\mu_1^2, ..., \mu_k^2\} \leq Tr(B^*B)/k \leq \frac{n-k}{4(n-1)}$.

However, since $\frac{n-2k}{2n} \leq \lambda_1$, we have $\frac{n-2k}{2n}(1-\frac{n-2k}{2n}) = \frac{n^2-4k^2}{4n^2} \leq \lambda_1(1-\lambda_1)$. Similarly, using the fact that $1/2 < \frac{n+2k}{2n}$, one sees that $\frac{n+2k}{2n}(1-\frac{n+2k}{2n}) = \frac{n^2-4k^2}{4n^2} \leq \lambda_k(1-\lambda_k)$.

Combining these inequalities, yields $\frac{n^2-4k^2}{4n^2} \leq \frac{n-k}{4(n-1)}$. Cross-multiplying and canceling like terms yields the result. □

Note that the above inequality, for n and k large becomes asymptotically, $n \leq 4k$. Thus, any uniform, equiangular (n,k)-frame with $n/k >> 4$, and n sufficiently large will yield a counterexample.

Corollary 16. *There exist uniform, equiangular Parseval frames whose projection matrices are counterexamples to Conjecture A.*

Proof. In [4, Example 6.4] a real uniform, equiangular (276, 23)-frame is exhibited and these values satisfy $(k-1)n^2 > 4k^2(n-1)$. In [11], uniform, equiangular (n,k)-frames are constructed using Singer difference sets of size,

$$n = \frac{q^{m+1}-1}{q-1}, k = \frac{q^m-1}{q-1},$$

where $q = p^r$ with p a prime. Note that $n/k > q-1$. Since Singer difference sets are known to exist for infinitely large q, these frames give a whole family of counterexamples. □

We now turn our attention to Conjecture B. We let $\gamma, \epsilon > 0$ be as in the statement of the conjecture.

Theorem 17. *Let $\gamma, \epsilon > 0$ be fixed, let $\{f_1, ..., f_n\}$ be a uniform Parseval frame for $\mathbb{R}^k$ with $k/n < \min\{\gamma, \epsilon/2, 1/2\}$ and let $P = (\langle f_i, f_j \rangle)$ be the correlation matrix. For each partition of $\{1, ..., n\} = R \cup T$ into two disjoint sets, R, T, let Q_R, Q_T denote the corresponding diagonal projections. If $\frac{Tr(Q_S P Q_T P Q_S)}{k} < \epsilon/2(1-\epsilon/2)$ for all such partitions, then P is a counterexample to Conjecture B, that is, $\delta_P < \gamma$ and $\|PSP\| > 1-\epsilon$, for every diagonal symmetry, S.*

Proof. Each such partition defines a diagonal symmetry as before and corresponding to such a partition we write $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. Note that except for the 0's, $B = Q_R P Q_T$.

We have that $\delta_P = k/n < \gamma$. We repeat the proof above, with $m = \min\{|R|, |T|\}$.

Letting λ_1 be the minimum non-zero eigenvalue and λ_k the largest eigenvalue of A as before, we have $1-\epsilon \geq \|PSP\| \geq \max\{|1-2\lambda_1|, |1-2\lambda_k|\}$ and, hence, $\lambda_1 \geq \epsilon/2$ and $1-\lambda_k \geq \epsilon/2$.

Using the properties of the function $t \to t(1-t)$ and the fact that $\sum_{i=1}^k \lambda_i(1-\lambda_i) = Tr(B^*B)$, we have that $\epsilon/2(1-\epsilon/2) \leq \min\{\lambda_1(1-\lambda_1), \lambda_k(1-\lambda_k)\} \leq 1/k Tr(B^*B)$, which yields the result. □

Using equiangular frames we can obtain a relation between γ and ϵ in Conjecture B.

Theorem 18. *Assume that Conjecture B is true for a pair $\gamma, \epsilon > 0$ and let $\{f_1, ..., f_n\}$ be a uniform, equiangular (n,k)-frame with $k/n \le \gamma$. Then $\epsilon/2(1-\epsilon/2) \le \frac{n-k}{4(n-1)}$.*

Proof. By the above theorem, we have that there exists a partition with $|R| = m$, such that $\epsilon/2(1-\epsilon/2) \le 1/kTr(Q_RPQ_TPQ_R) = 1/km(n-m)c^2 \le n^2/(4k)c^2 = \frac{n-k}{4(n-1)}$. □

If we have that infinitely many uniform, equiangular (n,k)-frames exist for which $k/n \to \gamma$, then

$$\frac{n-k}{4(n-1)} = \frac{1-k/n}{4(1-1/n)} \to \frac{1-\gamma}{4},$$

and hence, $\epsilon/2(1-\epsilon/2) \le \frac{1-\gamma}{4}$. If it is correct that for each prime p, there are infinitely many Singer difference sets, with $q = p^r$, then we may choose, $1/q \le \gamma$ and we get that $\epsilon/2(1-\epsilon/2) < \frac{1-1/q}{4}$.

Unfortunately, there are no uniform Parseval (n,k) frames which satisfy the sufficient conditions in Theorem 17 for giving a counter-example to Conjecture B. We will show this below.

First, let us check the notation so we know we are all on the same page. If $\{f_i\}_{i=1}^n$ is a Parseval frame for l_2^k with analysis operator T then the frame operator is $S = T^*T = I$ and $P = TT^*$ is a projection on l_2^n onto the image of the analysis operator (which is now an isometry). Let $\{R, S\}$ be a partition of $\{1, 2, \ldots, n\}$. If $x = \sum_{i=1}^n a_i e_i$ then

$$Q_R x = \sum_{i \in R} a_i e_i.$$

Next,

$$PQ_R x = \sum_{j=1}^n \langle \sum_{i \in R} a_i f_i, f_j \rangle e_j.$$

Finally,

$$Q_T PQ_R x = \sum_{j \in T} \langle \sum_{i \in R} a_i f_i, f_j \rangle e_j.$$

It follows that

$$Q_T PQ_R e_i = \sum_{i \in R} \sum_{j \in T} \langle f_i, f_j \rangle e_j.$$

Now we have:

Lemma 19. *Given the conditions above we have*

$$Trace\ Q_R PQ_T PQ_R = \sum_{i \in R} \sum_{j \in T} |\langle f_i, f_j \rangle|^2.$$

Proof. We compute:

$$\begin{aligned}\sum_{i=1}^{n}\langle Q_R P Q_T P Q_R e_i, e_i\rangle &= \sum_{i=1}^{n}\langle Q_T P Q_R e_i, P Q_R e_i\rangle \\ &= \sum_{i=1}^{n}\langle Q_T P Q_R e_i, Q_T P Q_R e_i\rangle \\ &= \sum_{i=1}^{n}\|Q_T P Q_R e_i\|^2 \\ &= \sum_{i\in R}\sum_{j\in T}|\langle f_i, f_j\rangle|^2.\end{aligned}$$

□

Now we need to recall a result of Berman, Halpern, Kaftal and Weiss [3].

Theorem 20. *Let $(a_{ij})_{i,j=1}^n$ be a self-adjoint matrix with non-negative entries and with zero diagonal so that*

$$\sum_{m=1}^{n} a_{im} \le B, \quad \textit{for all } i = 1, 2, \ldots, n.$$

Then for every $r \in \mathbb{N}$ there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that for every $j = 1, 2, \ldots, r$,

$$\sum_{m\in A_j} a_{im} \le \sum_{m\in A_\ell} a_{im}, \quad \textit{for every } i \in A_j \textit{ and } \ell \ne j. \tag{1}$$

Now we are ready for our result.

Proposition 21. *If $\{f_i\}_{i=1}^n$ is a uniform (n,k)-Parseval frame, then there is a partition $\{R, T\}$ of $\{1, 2, \ldots, n\}$ so that*

$$Trace\ Q_R P Q_T P Q_R \ge \frac{k}{4}(1 - \frac{k}{n}).$$

In particular, if $\frac{k}{n}$ is small then the conditions of Theorem 17 do not hold.

Proof. Applying 20 to the matrix of values $(a_{ij})_{i,j=1}^n$ where $a_{ii} = 0$ and $a_{ij} = |\langle f_i, f_j\rangle|^2$ for $i \ne j$ we can find a partition $\{R, T\}$ of $\{1, 2, \ldots, n\}$ (and without loss of generality we may assume that $|R| \ge \frac{n}{2}$) satisfying for all $i \in R$:

$$\sum_{i\ne j\in R}|\langle f_i, f_j\rangle|^2 \le \sum_{j\in T}|\langle f_i, f_j\rangle|^2.$$

It follows that for all $i \in R$:

$$\begin{aligned}\frac{k}{n} &= \sum_{j=1}^{n} |\langle f_i, f_j\rangle|^2 \\ &= \frac{k^2}{n^2} + \sum_{i\neq j\in R} |\langle f_i, f_j\rangle|^2 + \sum_{j\in T} |\langle f_i, f_j\rangle|^2 \\ &\leq \frac{k^2}{n^2} + 2\sum_{j\in T} |\langle f_i, f_j\rangle|^2.\end{aligned}$$

It follows that for all $i \in R$

$$\sum_{j\in T} |\langle f_i, f_j\rangle|^2 \geq \frac{1}{2}\left(\frac{k}{n} - \frac{k^2}{n^2}\right).$$

Now,

$$\begin{aligned}\sum_{i\in R}\sum_{j\in T} |\langle f_i, f_j\rangle|^2 &\geq |R|\frac{1}{2}\left(\frac{k}{n} - \frac{k^2}{n^2}\right) \\ &\geq \frac{n}{2}\frac{1}{2}\left(\frac{k}{n} - \frac{k^2}{n^2}\right) \\ &= \frac{k}{4}\left(1 - \frac{k}{n}\right).\end{aligned}$$

Now, given $0 < \epsilon < 1$,

$$\frac{\epsilon}{2}(1 - \frac{\epsilon}{2}) < \frac{1}{4}$$

So for the assumptions of Theorem 17 to hold we would need

$$\frac{k}{4}\left(1 - \frac{k}{n}\right) \leq Trace\ Q_R P Q_T P Q_R \leq k\frac{\epsilon}{2}(1 - \frac{\epsilon}{2}) < \frac{1}{4}$$

But for k/n small enough, this is a contradiction. □

References

[1] C.A. Akemann and J. Anderson, *Lyapunov theorems for Operator Algebras,* Mem. Amer. Math. Soc. **94** 458(1991).

[2] J. Anderson, *Extreme points in sets of positive linear maps on B(H),* J. Func. Anal. **31**(1979), 195-217.

[3] K. Berman, H. Halpern, V. Kaftal and G. Weiss, *Matrix norm inequalities and the relative Dixmier property*, Integ. Eqns. and Operator Theory **11** (1988) 28–48.

[4] B.G. Bodmann and V.I. Paulsen, *Frames, graphs and erasures,* Lin. Alg. and Appl. **404**(2005), 118-146.

[5] J. Bourgain and L. Tzafriri, *On a problem of Kadison and Singer,* J. Reine Angew. Math. **420**(1991), 1-43.

[6] P.G. Casazza, M. Fickus, J.C. Tremain and E. Weber, *The Kadison-Singer Problem in Mathematics and Engineering: A Detailed Account,* preprint.

[7] P.G. Casazza and J.C. Tremain, *The Kadison-Singer Problem in Mathematics and Engineering,* preprint.

[8] P.G. Casazza, D Edidin, D. Kalra, and V. Paulsen, *Kadison-Singer*, preprint.

[9] R.B. Holmes and V.I. Paulsen, *Optimal frames for erasures,* Lin. Alg. and Appl. **377**(2004), 31-51.

[10] R. Kadison and I. Singer, *Extensions of pure states,* Amer. J. Math. **81**(1959), 547-564.

[11] D. Kalra, *Cyclic Equiangular Frames,* Lin. Alg. and Appl., to appear.

[12] V.I. Paulsen, *Equivariant Maps and Bimodule Projections,* J. Functional Analysis, to appear.

[13] R.C. Walker, *The Stone-Cech Compactification,* Springer-Verlag, 1974.

[14] N. Weaver, *A counterexample to a conjecture of Akemann and Anderson,* Bull. London Math. Soc. **34**(2002), 1-7.

[15] N. Weaver, *The Kadison-Singer Problem in discrepancy theory,* Discrete Math. *278*(2004), 227-239.

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