STATE EXTENSIONS AND THE KADISON-SINGER PROBLEM

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Abstract. These notes outline some ideas for approaching the Kadison-Singer problem.

1. Introduction

The unsolved case of the Kadison-Singer problem is concerned with the uniqueness of extensions of pure states defined on a discrete MASA to the C*-algebra of all of the operators on the corresponding Hilbert space. However, this problem was fairly quickly shown to be equivalent to various problems about "paving" operators with projections from the MASA. Indeed, in Lemma 4 of their original paper [7], they proved that the extension version was equivalent to a paving problem. The remarkable work of J. Anderson [2, 3, 4] deepened this connection between paving and extensions and since that time almost all research on the Kadison-Singer problem has focused on various equivalent paving conjectures. Even the many new equivalences that have arisen from the work of P. Casazza and his collaborators, such as the Feichtinger conjecture, are essentially at heart paving conjectures.

Yet if the Kadison-Singer problem has a negative answer, then there must exist pure states on the discrete MASA which have non-unique extensions. In their original paper, Kadison and Singer proved that the pure states corresponding to vector states all have unique extensions. Thus, the only pure states that could have non-unique extensions are the ones that correspond to "other" points in the space of pure states on the MASA. Since the MASA is discrete, it can be identified with the continuous functions on the Stone-Cech compactification of a countable discrete set and these "other" pure states correspond to point evaluations at points in the Stone-Cech compactification that are outside the original set. This suggests that one approach to the Kadison-Singer problem involves a study of the Stone-Cech compactification.

The goal of these notes is to provide an introduction to this approach to the Kadison-Singer problem, to give an overview of what is known and state some potential questions for further investigation.

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2. The Stone-Cech Compactification

Since the MASA that we are interested in is a discrete MASA on a separable, infinite dimensional Hilbert space, up to unitary equivalence, the Hilbert space can be taken to be $\ell^2(\mathbb{N})$, where $\mathbb{N}$ denotes the natural numbers and the MASA can be taken to be all bounded, diagonal operators on this space, which can be identified with $\ell^\infty(\mathbb{N})$.

However, for the purposes of these notes, we prefer to identify our discrete set as all the integers, $\mathbb{Z}$, our Hilbert space as $\ell^2(\mathbb{Z})$ so that the diagonal MASA, $\mathcal{D}$, with respect to this representation is identified with $\ell^\infty(\mathbb{Z})$. In this setting, the bilateral shift is a unitary and conjugation by the bilateral shift implements the automorphism of $\mathcal{D}$ corresponding to translation by 1. Thus, we have a dynamical system that, some feel, might play a role. See [8].

Let $\beta\mathbb{Z}$ denote the Stone-Cech compactification of $\mathbb{Z}$. Recall that this space is the "maximal" compactification of $\mathbb{Z}$. Thus, for every compact, Hausdorff space X, and countable set $\{x_n\}_{n\in\mathbb{Z}} \subset X$, there exists a unique continuous function, $f : \beta\mathbb{Z} \to X$ satisfying, $f(n) = x_n$ for every $n \in \mathbb{Z}$. It is often convenient to identify points in $\beta\mathbb{Z}$ with *ultrafilters* on $\mathbb{Z}$. Recall that an ultrafilter on $\mathbb{Z}$ is a collection of subsets of $\mathbb{Z}$ that is closed under finite intersections, does not contain the empty set and is maximal among all such collections satisfying these properties. To recall the correspondence between points and ultrafilters, note that since the integers are dense in $\mathbb{Z}$, every open set is uniquely defined by the integers that belong to the set. Thus, given a point, $\omega \in \mathbb{Z}$, the neighborhood base for ω can be identified with a collection of subsets of $\mathbb{Z}$, which is, in fact, an ultrafilter. Thus, each point in $\beta\mathbb{Z}$ yields an ultrafilter and this correspondence is in fact a bijection between $\beta\mathbb{Z}$ and the collection of all ultrafilters.

Points in $\beta\mathbb{Z}$ can be either described by their topological properties or in terms of the ultrafilter that defines them. Often it is difficult reconciling these two defintions.

Since $\mathcal{D} = \ell^\infty(\mathbb{Z}) = C(\beta\mathbb{Z})$, we know that the pure states on $\mathcal{D}$ are given by point evaluations at arbitrary points in $\beta\mathbb{Z}$. In Kadison and Singer's original paper [7], they remark that the pure states corresponding to points in $\mathbb{Z}$, i.e., the vector inner products with the canonical basis vectors, all have unique pure state extensions. So the problem is with other pure states, i.e., the points in the so-called Corona set, $\mathbb{Z}^* \equiv \beta\mathbb{Z} \setminus \mathbb{Z}$.

There is a tendency to treat all of the corona points as the "same", but if the Kadison-Singer problem has a negative answer, then necessarily some corona points will have non-unique extensions and some will have unique extensions. This follows from a result of Reid[10].

Reid[10] showed that the points corresponding to *rare* ultrafilters have unique pure state extensions. An ultrafilter on a countable set, say $\mathbb{Z}$, is rare[5], if for every partition of $\mathbb{Z}$ into finite subsets, $\mathbb{Z} = \cup_k A_k$ there exists a set B in the ultrafilter such that for every $k, B \cap A_k$ has at most one element. It can be seen that *rare* is the same as requiring that there is an integer m such that for every $k, B \cap A_k$ has at most m elements, which is often used as the definition of rare. To see the equivalence of the two definitions, note that if $B \cap A_k$ has at most m elements, then we can write B as a disjoint union, $B = B_1 \cup \cdots \cup B_m$ such that for all i and $k, B_i \cap A_k$ has at most 1 element and using the fact that for each i, either B_i or its complement is in the ultrafilter, we get that B_i is in the ultrafilter for some i.

Assuming the continuum hypothesis, rare points exist and the set of points corresponding to rare ultrafilters is dense in the corona[5]. The uniqueness of pure state extensions for such ultrafilters implies easily that the image of an atomic MASA in the Calkin algebra has only one conditional expectation from the Calkin algebra onto it. This mirrors the uniqueness result from [7].

Anderson [2] proves that if $A \in B(\ell^2(\mathbb{Z}))$ and ω is a rare ultrafilter, then there exists a set B in ω such that $P_B A P_B$ is diagonal plus compact, where P_B is the diagonal projection corresponding to the characteristic function of B. This result together with Anderson's various paving equivalences to uniqueness, yields another proof of Reid's result, see [2].

Motivated by Reid's result, we ask:

Problem 1. *If a dense set of corona points has a unique pure state extension to B(H), then must every corona point have a unique pure state extension?*

By Reid's result and assuming the continuum hypothesis, if the Kadison-Singer problem has a negative answer, then some points in the corona will have non-unique extensions while others will have unique extensions. If one is interested in finding a counterexample, it is important to try and identify properties of points that make them potential candidates for non-unique extension. Alternatively, one can make partial progress on the problem, by finding other types of ultrafilters which might force uniqueness of the extension. A quick glance through any book on the Stone-Cech compactifaction shows that the corona is very non-homogeneous. Thus, it is natural to look at various properties that points in the corona are known to have and try to guess which one's are likely to be "bad" or "good" for uniqueness of extension.

This idea of looking for other types of ultrafilters that may or may not have unique extensions is implicit in Section 8 of Anderson's work [2] and in Reid's work[10]. One type of point looked at by Reid and Anderson, are the *P-points* in the corona. A point in a topological space is a P-point if every G_δ containing the point contains a neighborhood of the point. **Caution**-this is *not* equivalent to the notion of p-point from function theory! Choquet[5] defines an ultrafilter to be *δ-stable* if it has the property that for every partition, $\mathbb{Z} = \cup_k A_k$ of $\mathbb{Z}$, either A_k is in the ultrafilter for some k or there exists a set B in the ultrafilter such that for every $k, B \cap A_k$ is finite. Choquet[5] proves that an ultrafilter is δ-stable if and only if it is a P-point in the corona[5, Proposition 1].

It follows from the topological definition that if a continuous function f vanishes at a P-point ω, then the zero set of f contains a clopen neighborhood of the point ω. Hence there is a clopen set of pure states that assigns the same value to f as ω.

Like rare points, it is not known if P-points necessarily exist, but if we assume either the continuum hypothesis or Martin's axiom, then they not only exist, but the P-points and the non-P-points both form dense subsets of the corona. See, for example, [11].

There is some feeling that the P-points share sufficiently many properties with rare points that they should also possess unique extensions.

Problem 2. *If $\omega \in \beta\mathbb{Z} \setminus \mathbb{Z}$ is a P-point, then does the state given by evaluation at ω have a unique, pure state extension to B(H)?*

Anderson[2] proves that if ω is a P-point, then it fails to have the *continuous restriction property* which is about non-unique extensions for continuous MASA's. If a point is a P-point and rare, then he proves that it has the discrete restriction property but not the continuous restriction property.

Another closely related type of ultrafilter that we should mention are the *selective*[6] and *absolute*[5] ultrafilters. An ultrafilter is called absolute provided that it is δ-stable and rare. An ultrafilter is called *selective* provided that for every partition $\mathbb{Z} = \cup_k A_k$, either A_k is in the ultrafilter for some k or there is a set B in the ultrafilter such that for every $k, B \cap A_k$ has at most one element. Choquet[5] shows that an ultrafilter is selective if and only if it is absolute.

The final type of point that we would like to bring attention to are the *idempotent* points. One doesn't need any extra hypotheses to guarantee that these points exist and some believe that these are strong candidates for counterexamples. These points are more closely tied to the dynamical system.

We outline the definition of idempotent points below. For an excellent source on these ideas, see [6].

These are best understood from the viewpoint of $\mathbb{Z}$ actions. Given a compact, Hausdorff space X and a homeomorphism, $h : X \to X$ we define an action of $\mathbb{Z}$ on X by setting, $n+x = h^{(n)}(x)$, so that $0+x = x, n+(m+x) = (n+m)+x$. By the properties of the Stone-Cech compactifaction, for each fixed $x \in X$, the map $\mathbb{Z} \to X$ defined by $n \to n+x$ extends uniquely to a continuous map, $\beta\mathbb{Z} \to X$, which we write as $\omega \to \omega + x$. Note that given two points $\omega, \gamma \in \beta\mathbb{Z}$, we can write, $\omega + (\gamma + x)$.

This idea can be used to define a binary operation on $\beta\mathbb{Z}$. First, note that translation by one, is a homeomorphism of $\mathbb{Z}$ and hence extends uniquely to a continuous function, $h : \beta\mathbb{Z} \to \beta\mathbb{Z}$. Applying the above ideas to this map h, we see that for every pair of points $\omega, \gamma \in \beta\mathbb{Z}$ we can define a binary operation by $(\omega, \gamma) \to \omega + \gamma$ which agrees with ordinary addition on $\mathbb{Z}$.

The notation is deceptive, because this binary pairing is *not* commutative. However, it does satisfy that for any X and h, we have $\omega + (\gamma + x) = (\omega + \gamma) + x$.

Finally, a point ω is called *idempotent,* if $\omega + \omega = \omega$. Remarkably, using the compactness of the space $\beta\mathbb{Z} \setminus \mathbb{Z}$ it can be shown that idempotent points exist [6] in the corona.

Intuitively, if we think of points in $\beta\mathbb{Z}$ as "limits" then ω idempotent, implies that for any homeomorphism, h, we have

$$\lim_{n\to\omega} \lim_{k\to\omega} h^{(n+k)}(x) = \lim_{j\to\omega} h^{(j)}(x).$$

Another vital property of idempotent points is that if ω is idempotent, then the closure of the orbit of ω under the homeomorphism, $h : \beta\mathbb{Z} \to \beta\mathbb{Z}$ induced by translation is a proper closed invariant subset of the corona. Moreover, there exists an h-equivariant retract of $\beta\mathbb{Z}$ onto this subset, see [9].

Problem 3. *If $\omega \in \mathbb{Z}$ is an idempotent point, then does the pure state extension given by evaluation at ω possess non-unique extensions?*

A somewhat related problem is the following:

Problem 4 (Betul Tanbay)**.** *If ω_1 and ω_2 are points such that the pure state extensions given by evaluation at ω_1 and ω_2 each have unique extensions, then does evaluation at $\omega_1 + \omega_2$, necesarily have a unique extension?*

There is a reason that an approach involving the Stone-Cech compactification and ultrafilters might be fruitful. Many good mathematicians have struggled to construct an operator that fails paving to no avail. This approach is somewhat "transcendental". It could give a counterexample to the Kadison-Singer problem without giving any clues about how a non-pavable operator could be obtained. In fact, if a counterexample was shown for some ultrafilter whose existence was only guaranteed with some stronger axioms, then we would be able to assert that non-pavable operators exist, assuming, say, the continuum hypothesis! In which case the actual construction of such an operator could still be quite far off.

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