

PROPOSED PROBLEMS

ABSTRACT. These are the problems proposed at the workshop on Integral Closure, Multiplier Ideals and Cores, AIM, December 2006.

Problem 0.1 (D. Eisenbud). Find a good bound for the stabilization for the regularity of the powers of an ideal, i.e. find n_0 such that for $\forall n \geq n_0$ we have $\text{reg}(I^n) = \rho n + e$ for some positive integers ρ and e . Also describe such e . Suggested project: Reading Römer's paper.

Problem 0.2 (N. V. Trung). If we replace I^n by $I^{(n)}$ or $\text{In}(I^n)$ in above question, then is there a linear bound for the regularity? For monomial ideals there is such a bound for the regularity of $I^{(n)}$. If $\dim A/I \leq 1$ then there is a linear bound for the regularity of $\text{In}(I^n)$. See also a paper by Herzog-Hoa-Trung.

Problem 0.3 (Z. Teitler). Which prime exceptional divisors impose restrictions for multiplier ideals? Compute examples of multiplier ideals. To compute adjoint ideals we need to get a handle on what valuations/divisor are needed. Rees valuations are always needed (I. Swanson). See also a paper by K. Smith and Thompson.

Problem 0.4 (J. Lipman). Take all true statements about integrally closed ideals in regular local rings of dimension 2 and generalize them, if possible, for all finitely supported ideals in a d -dimensional regular local ring.

Problem 0.5 (S. Kleiman). Relativize Lipman's question! In particular, develop a deformation theory for complete ideals.

Problem 0.6 (M. Hochster). Prove Lech's Conjecture: If $R \rightarrow S$ be a flat map of Noetherian local rings, then $e(R) \leq e(S)$. We may assume R and S are complete and of the same dimension. We may also assume that R is a domain with infinite residue class field. The conjecture is known for $\dim \leq 2$ (Lech). The conjecture is still open even if R is Cohen–Macaulay.

Problem 0.7 (Z. Teitler). There are very few computed examples of asymptotic multiplier ideals. Compute more examples.

Problem 0.8 (J. Lipman). Does subadditivity of adjoints of ideals hold in all regular local rings?

Problem 0.9 (J. Lipman). Prove or disprove vanishing conjecture from Lipman's talk, for non finitely supported ideals. Notes are available at www.math.purdue.edu/~lipman

Problem 0.10 (J. Lipman). Let $A \rightarrow B$ be an injective map of regular local rings of characteristic zero such that B is finite over A . Let \mathfrak{p} be a prime ideal in A and suppose $B/\sqrt{\mathfrak{p}B}$ is regular. Is A/\mathfrak{p} regular?

Problem 0.11 (N. Epstein). Let (R, \mathfrak{m}) be a regular local ring and let ℓ be the analytic spread of the ideal I . Assume that $x^t \in \overline{\mathfrak{m}I^{\ell t}}$ for some positive integer t . Then is it true that $x \in \mathfrak{m}I$? In the case of equicharacteristic zero this is known to be true. What about mixed characteristic?

Problem 0.12 (T. Gaffney). For a pair of submodules of a free module that may not have finite colength how can we define a multiplicity such that it preserves the Principal of Specialization, Triples property, and Integral Dependence?

Problem 0.13 (C. Polini). Let $J \subset I$ be a minimal reduction of I , let $g = \text{ht } I$, and let j be a positive integer. When is $\text{adj}(I^j) = J^{n+j-g+1} : \overline{I}^n$ for n large enough? It is true when characteristic is zero and $\text{Proj}(\overline{R[It]})$ has rational singularities. When I is \mathfrak{m} -primary and $R[It]$ is Cohen–Macaulay and normal then the statement is equivalent to $R[It]$ having rational singularities.

Problem 0.14 (D. Eisenbud). Let R be a regular local ring of characteristic zero and let \mathfrak{p} be a prime ideal in R . When $\mathfrak{p}^{(2)} \subset \mathfrak{m}\mathfrak{p}$? This is true if $\mathfrak{m}\mathfrak{p}$ is integrally closed and false in positive characteristic. When is $\mathfrak{m}\mathfrak{p}$ integrally closed? See Eisenbud and Mazur’s paper.

Problem 0.15 (K. Watanabe). Let $\mathfrak{Q} = (x_1, \dots, x_d)$ be a parameter ideal and $q = p^e \gg 1$, where p is the characteristic of A . If $\mathfrak{Q}^{[q]} \supset I$, I integrally closed, is it true that

$$\ell_R(R/I) \geq \frac{d^d}{d!} \cdot \ell_R(R/\mathfrak{Q}^{[q]}) - O(q^{d-1})?$$

This is true if $A = k[[X_1, \dots, X_d]]$ and $\mathfrak{Q} = (X_1^{a_1}, \dots, X_d^{a_d})$.

Problem 0.16 (K. Watanabe). Let (A, \mathfrak{m}) be a Noetherian local ring, x_1, \dots, x_d a system of parameters and $I = \overline{I} \subset (x_1, \dots, x_d)$ with minimal colength. Find lower bounds on the colengths; also for (x_1^q, \dots, x_d^q) .

Problem 0.17 (S. Goto). Let (R, \mathfrak{m}) be a Noetherian local ring and let \mathfrak{Q} be a parameter ideal. When is $\mathfrak{Q} : \mathfrak{m}^n$ integral over \mathfrak{Q} ? The statement is true in the following cases:

- (1) R is Cohen-Macaulay of dimension at least 3 and \mathfrak{Q} is contained in \mathfrak{m}^n where n is at least 2 (Wang).
- (2) R is Cohen-Macaulay of dimension 2, but not regular and \mathfrak{Q} is contained in \mathfrak{m}^n where n is at least 2 (Wang).
- (3) R is a Gorenstein local ring of positive dimension and $e(R) \geq 3$.

Problem 0.18 (F. Enescu). Find a generalization of Briançon–Skoda theorem on integral closure in the context of Noetherian filtration of ideals (Kuronya and Wolfe have obtained extensions of the Briançon–Skoda theorem for filtrations that they call stable and are defined via some geometric conditions).

Problem 0.19 (S. Goto). Let (A, \mathfrak{m}) be a Noetherian local ring and I an ideal of A . Then the length of $(I^n : \mathfrak{m})/I^n$ is a polynomial in n for n large enough. What is its degree? If I is of maximal analytic spread, then the degree is equal to $\dim A - 1$ (due to Katz).