GROUP REPORTS

ABSTRACT. These are notes from the daily group reports.

1. REPORTS FROM 12/19/06

(1) (Asymptotic regularity)

Let I be an ideal generated by forms of degree d. Then $\text{reg}(I^n) = dn + c$ for $n \gg 0$ and $R = k[\underline{x}]$.

Question: What is the least n?

The group read Römers paper where he proved:

$$\text{reg}(I^n) = dn + c \text{ for } n \geq \mathfrak{m}_y(bigin(J))$$

In other words $\dfrac{k[\,\underline{x},\,\underline{y}\,]}{J} \xrightarrow{\simeq} R[It] \to 0$.

The group agreed that they understood this paper and that they will not continue reading this paper.

(2) (Integral closure and equisingularity)

Let M and N be submodules of a free module F over a Noetherian local ring $(R, \mathfrak{m})$. We discussed how to define a notion of multiplicity for a pair of modules M and N that satisfies the following properties in the context described below.

Let $(X, 0)$ be a complex analytic space defined by the vanishing of

$$F : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^p, 0)$$

and let Y be a smooth subvariety of X. Consider the Jacobian matrix of F denoted by $JM(F) := (\partial F/\partial X_i)$. Then the column space M of $JM(F)$ is a submodule of a free module F of rank p. Let $\pi : \text{Proj}(\mathcal{R}(M)) \longrightarrow X$ and let $C = \pi^{-1}(C_M) \subseteq \text{Proj}(\mathcal{R}(M))$ where

$$C_M = \{x \in X \mid \text{rank } M(x) \leq \text{ generic rank of } M\}.$$

Desirable properties of a multiplicity are the following:

(a) Coincide with the Buchsbaum-Rim multiplicity for submodules of finite colength.
(b) Defined for a pair of modules $M \subseteq N \subseteq F$ such that if $\bar{N}/\bar{M}$ has finite length, it gives $e(M, N)$.
(c) If $O_{X,0}$ is equidimensional and $M \subseteq N \subseteq F$ then $e(M) = e(N)$ if and only if $\bar{N} = \bar{M}$.
(d) The length theoretic approach should agree with the intersection theory approach.
(e) Satisfy additivity on triples.

Date: July 15, 2007.

(f) Expansion formula: For I an ideal of R, we want formulas for $e(IM)$ in terms of mixed multiplicities.

(g) If $e(M_y)$ is independent of the parameter then we want a Principal of Specialization of Integral Dependence, i.e. we want to be able to control the dimension of fibers of C contained in $\mathrm{Proj}(\mathcal{R}(M))$ over $0 \in Y$.

(h) Given a pair of ideals I and J that may not contain each other, and may not have finite colength, should give a criterion for $\bar{I} = \bar{J}$

(3) (Reductions and colon ideals)

Let $(A, \mathfrak{m})$ is local ring, Q is a parameter ideal and $d = \dim A$. Let $I = Q : \mathfrak{m}$.

Question:

(a) When is $I \subset \overline{Q}$?

(b) When this is the case is $r_Q(I) \leq n$?

(c) Is $G(I),\ \mathcal{F}(I),\ \mathcal{R}(I)$ Cohen–Macaulay, Gorenstein , Buchsbaum?

If A is Cohen–Macaulay then Polini and Ulrich have the following conjecture:

Conjecture 1.1. *(*Polini–Ulrich*) Suppose A is Cohen–Macaulay. If* $Q \subset \mathfrak{m}^n$ *then* $I \subset \mathfrak{m}^n$.

If $\mathfrak{m}^t I = \mathfrak{m}^t Q$ then $I^2 = QI$.

Theorem 1.2. *(*Wang*)*

1. *If A is Cohen–Macaulay,* $d = \dim A \geq 3$, $n \geq 2$ *then the conjecture holds.*

2. *If A is a regular local ring,* $d = 2$ *and* $n \geq 2$ *then the conjecture holds.*

Theorem 1.3. *(*Matsolea, Takahashi (independently)*)*

Suppose A is a Gorenstein ring, $d > 0$, $e(A) \geq 3$ *and* $I = Q : \mathfrak{m}^2$. *Then* $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$ *and* $I^3 = QI^2$. *Moreover* $G(I), \mathcal{F}(I)$ *are Cohen–Macaulay.*

The group tried to understand the proofs of these papers. The next goal is to understand Wang's Paper.

(4) (Multiplier ideals)

Questions discussed:

(a) Let $C \subset \mathbb{C}^3$ be a monomially parametrized curve where the image of t is (t^a, t^b, t^c). Let $I = I(C) \subset \mathbb{C}[x, y, z]$. What are $\mathcal{J}(I^\lambda)$ and $\mathrm{adj}(I)$?

(b) Which divisors/ valuations matter for the multiplier ideals and adjoints?

(c) Say $\mathrm{adj}(I^t) = \bigcap_v \{r \in R | v(r) \leq \llcorner t v(I) \lrcorner - v(J_v)\}$ where $t \geq 0$. If there exists a log resolution this is a multiplier ideal. If not then?

(d) Is there a finite set of valuations that define $\mathrm{adj}(I), \mathrm{adj}(I^2), \mathrm{adj}(I^3), \ldots$?

(5) (The core of ideals)

(a) Given an ideal L does there exist and ideal I such that $L = \operatorname{core}(I)$? If L is a monomial ideal then they group thinks that I must be monomial.

(b) If $\overline{\operatorname{core}(I)} = \operatorname{core}(I)$. If I is $\mathfrak{m}$–primary or I is monomial then $\operatorname{core}(I) = \operatorname{adj}(I^d)$.

(c) If J is $\mathfrak{m}$–primary and $J \subset I_1 \subsetneq \ldots \subsetneq \overline{J}$ then $\operatorname{core}(I) \subset \ldots \subset J$. How do the jumps come up. This is related to the multiplicity.

2. Reports from 12/20/06

(1) (Integral closure and equisingularity)

They started the meeting by looking at an example of computing the j-multiplicity in the case of ideals, the Whitney umbrella $z^2 - x^2y = 0$. We computed the multiplicity of its Jacobian ideal (x^2, xy, z) at the origin with the length theoretic definition and intersection theoretic approach of Gaffney. In both cases we got multiplicity 2.

Then Professor Ulrich explained a formula for computing the j-multiplicity of ideals, due to Achilles-Manaresi, Flenner-O'Carroll-Vogel, Nishida-Ulrich, in terms of length of some modules.

They also looked into the generalization of the notion of j-multiplicity of ideals to modules that may not have finite colength, by Ulrich and Validashti, and we worked out an example in this regard. Let $R = k[[y_1, \ldots, y_d]]$ be a power series ring over a field k in $d > 0$ variables $y_1, \ldots, y_d$. Let E be the free submodule of $\oplus Re_i$ generated by $y_1e_1, \ldots, y_de_d$. Then E has j-multiplicity 1.

Then Professor Gaffney talked about computing the Buchsbaum-Rim multiplicity of a module in terms of some associated Fitting ideals.

(2) (Open problems)

If $\overline{P}^{\text{int}} = P^{(2)}$ then $r \in \overline{I}^{int}$ if $\exists n$ such that $r^n \in \overline{I^{n+1}}$. Equivalently $r^n \in \overline{I^{n+1}}$ if $\exists n$ such that $r^n \in I^{n+1}$.

The group considered different examples for Hübl's conjecture. The also worked on the details of the argument for depth ≥ 2.

(3) (The core of ideals)

Given an ideal L is $L = \operatorname{core}(I)$ for some I? The group worked on the example of $L = (x^3, x^2y, y^3)$ over the ring $R = k[x,y]_{x,y}$, where k is an infinite field. They determined that $L \neq \operatorname{core}(I)$ for any I. They also proved that if such an I would exist it would have to be a monomial ideal. They also worked on different examples and reported that:

- $(x,y) \neq \operatorname{core}(I)$,
- $(x^2, y^2) \neq \operatorname{core}(I)$,
- $(x^3, y^3) \neq \operatorname{core}(I)$,
- $(x^2, xy, y^2) \neq \operatorname{core}(I)$,
- $(x,y)^3 = \operatorname{core}((x,y)^2)$,

- $(x,y)^4 \neq \text{core}(I)$, for all ideals I in the ring R. They also concluded that there exists an ideal I in R such that $\text{core}(I) = (x,y)^{2n+1}$, where $n \in \mathbb{N}$. Also $\text{core}(I) \neq (x,y)^{2n}$, for all $n \in \mathbb{N}$ and for all ideals I in R.

(4) (Multiplier ideals)

Consider the ideal $I = (t^a, t^b, t^c)$. The group discussed whether

$$\text{adj}(I^2) = \bigcap_v \{r \in R | v(r) \geq \llcorner \lambda v(I) \lrcorner - v(\mathcal{J}_r/R)\}.$$

(5) (Reductions and colon ideals)

Let Q be a parameter ideal in a Cohen–Macaulay local ring $(A, \mathfrak{m})$ and suppose $Q\mathfrak{m}^t$. By the result of Wang we have that $Q : \mathfrak{m}^t \subset \mathfrak{m}^t$. This is known in the case A is Cohen–Macaulay and $\dim A \geq 3$ and also in the case A is Cohen–Macaulay, t is nonnegative and $\dim A = 2$. The group read Wang's paper and K. Watanabe gave an approach in positive characteristic using F–thresholds.

They also discussed the following question: Is $\overline{Q : \mathfrak{m}} = Q : \mathfrak{m} = I$? If $\dim A = 2$ and A is a regular local ring does one have a positive answer? Usually $I^2 = QI$ and $I \subset Q : I = \mathfrak{m}$. The group plans to discuss the one dimensional case.

Another question they are interested in is the following:

Let $(R, \mathfrak{m})$ be a regular local ring, I an R–ideal. Suppose $\sqrt{I} = \mathfrak{m}$ and $\mu(I) = d + 1$, where $d = \dim R$. If R contains a field then the Rees algebra, $\mathcal{R}(I)$, of I is Cohen–Macaulay and normal. They would like to understand Ciupercă's proof.

3. Reports from 12/21/06

(1) (Integral closure and equisingularity)

Professor Gaffney proposed a notion of multiplicity of a module E that is a submodule of a free module, with an intersection theoretic approach. He denoted it by $k(E)$ and suggested that $k(E)$ and $j(E)$ might be the same.

Then using polar methods he computed the multiplicity in some examples, including the multiplicity of the free submodule E of R^d generated by $y_1 e_1, \ldots, y_d e_d$ over the power series $R = k[[y_1, \ldots, y_d]]$. It turned out to be 1, which was equal to $j(E)$ in the sense of Ulrich-Validashti.

Professor Kleiman discussed his approach in his paper with Thorup on mixed Buchsbaum-Rim multiplicities and we tried to compare the grading that Kleiman and Thorup had used with the grading that Ulrich and Validashti had used to define the j-multiplicity of a module.

At the end, we talked about a formula relating the Buchsbaum-Rim multiplicity of a module and the Hilbert-Samuel multiplicity of the associated Fitting ideals over a regular local ring of dimension 2, based on a paper by Chen-Liu-Ulrich. And professor Gaffney mentioned another possible variation of the formula.

(2) (Reductions and colon ideals)

The group found a counterexample to Goto's conjecture. Then they discussed a problem suggested by Huneke: Find

$$\sup\{\frac{s}{t} \mid t > 0,\ s \geq 0,\ \text{such that } \exists\ Q \text{ a parameter ideal, with } Q \subset \mathfrak{m}^t \text{ and } Q : \mathfrak{m}^s \subset \overline{Q}\}.$$

If A is a regular local ring and $d = \dim A = 2$ then this number is 1. In the case of $d = 1$ they tested numerical semigroups.

(3) (The core of ideals)

The group continued with their computations as before.

(4) (Multiplier ideals) The group addressed the following questions:

- **(i)** Is the set of valuations needed to define $\mathrm{adj}(I^n)$ sufficient to determine $\mathrm{adj}(I^\lambda)$? The answer is no. They have a case where $\mathcal{J}(I^r)$ has a singular divisor.
- **(ii)** Let $I \subset R$, where R is a regular local ring of dimension d or $R = \mathbb{C}[[x_1, \ldots, x_d]]$. Assume that I is $\mathfrak{m}$–primary. Then $\mathrm{ord}(\mathrm{adj}(I)) \geq \mathrm{ord}(I) - d + 1$. Do we have equality? The answer is no in general. It is true for a large class of monomial ideals. It is false however for $I = (x^3, y^{10}, z^{10})$. Then $v = (1/3, 1/10, 1/10)$. Hence $x^\lambda \in \mathrm{adj}(x^3, y^{10}, z^{10})$ if and only if $(\lambda + \underline{1}).v > 1$. But no degree one polynomial satisfies this.
  Outstanding Questions:
- **(iii)** Is there a finite set of valuations defining $\mathrm{adj}(I^n)$ for all n?
- **(iv)** Does there exist a (canonical) minimal set of valuations that we can say suffice for adjoint ideals?

(5) At the end there was another group formed to discuss characteristic p criteria for integral dependance. The group on the Core of ideals did not continue and neither did the group on Multiplier ideals.

4. Final Reports from 12/21/06

(1) (Reductions and colon ideals)

Let $(R, \mathfrak{m})$ be a Noetherian local ring and Q a parameter ideal. Define

$$\mathrm{goto}_Q(R) = \sup\{\frac{s}{t} \mid t > 0,\ s \geq 0,\ \text{such that } \exists\ Q \text{ a parameter ideal, with } Q \subset \mathfrak{m}^t \text{ and } Q : \mathfrak{m}^s \subset \overline{Q}\}.$$

If $d = \dim R = 1$ then:

- **(i)** If $k \subset K$ is a finite field extension then $R = k[[Kt]] \subset K[[t]] = S$ and $\mathrm{goto}(R) = 1$
- **(ii)** Let R be a numerical semigroup of the form $k[x^{a_1}, \ldots, x^{a_n}]$, where $a_1 < a_2 < \ldots < a_n$. The Frobenius number F is the largest number that is not in the semigroup. They group thinks that

$$\mathrm{goto}(R) < \lceil \frac{a_n + F - (\text{largest number in the semigroup } < a_n)}{a_1} + \varepsilon \rceil.$$

  It is sufficient to take parameter ideals that are generated by pure powers.

(2) (Integral closure and equisingularity)

Professor Ulrich started the meeting by talking on specialization of integral closure, based on a joint work with Jooyoun Hong.

Then we worked out an example of computing the j-multiplicity of a family of modules. We considered the example in Section 4 of Gaffney's paper on polar methods, invariants of pair of modules and equisingularity. Let $R = k[x,y]_{(x,y)}$ with maximal ideal $\mathfrak{m} = (x,y)$ and let E_λ be the submodule of the free module $F = R^2$ generated by the column space of the matrix

$$\begin{pmatrix} 0 & x^2y^2 & 0 & 0 \\ x^2y+\lambda xy & \lambda(x+y^6) & y^2x+\lambda x^2y & \lambda(xy)^b \end{pmatrix}$$

where λ is a parameter and b is a positive integer. In both, intersection theoretic and length theoretic approach, the multiplicity of the family is 7, independent of the parameter λ.

(3) (Characteristic p criteria for integral dependance)

Theorem 4.1. *Let $(A,\mathfrak{m})$ be a Cohen–Macaulay ring and let I be an $\mathfrak{m}$–primary ideal. Let $J \subset I$ be a parameter ideal. Then $c^J(I) = d \Leftrightarrow I \subset \overline{J}$.*

They were able to prove the theorem without the Cohen–Macaulay assumption on the ring. They still have nice conditions on the ring. They show that

$$I \subset \overline{J} \Leftrightarrow I \not\subset J \text{ and } \exists a_1 \text{ such that } I^{dQ} \subset J^{(1-1/a_1)dQ},$$

where $Q \gg 0$. They also show that $c^J(I) = d \Leftrightarrow \forall q_0$ and $\forall Q \gg 0$, $I^{dQ+Q/q_0} \subset J^{[Q]}$. They also believe that the theorem might be true even if J is not a parameter ideal.

Definition 4.2. Let $\mathfrak{a} \subset \sqrt{I}$ and define $\nu_{\mathfrak{a}}^I(q) = \max\{r \mid \mathfrak{a}^r \not\subset I^{[q]}\}$ and $c^I(\mathfrak{a}) = \lim \dfrac{\nu_{\mathfrak{a}}^I(q)}{q}$.

This sequence is bounded. If we have F–purity then the limit exists. If we do not have F–purity they would like to show that it is still increasing.