ON F-THRESHOLDS (RING THEORETIC ASPECTS)

KEI-ICHI WATANABE

Abstract. These are notes on the author's talk given at the workshop on Integral Closure, Multiplier Ideals and Cores, AIM, December 2006.

This is a joint work with M. Mustaţă and S. Takagi.

The notion of F-thresholds was introduced by Mustaţă to describe jumping numbers of test ideals (multiplier ideals) and to obtain roots of Bernstein-Sato polynomials by characteristic p methods. "Geometric" aspects will be explained by Takagi and I will talk on ring theoretic aspects of F-thresholds. Before beginning, let me explain the origin of the name. F-threshold $\leftarrow$ F-pure threshold = lc threshold

Definition 0.1. Let $(A,\mathfrak{m})$ be a d-dimensional Noetherian ring of characteristic $p>0$. Let $F: A \to A^{1/p}$ be finite, and $\mathfrak{a} \subset A$. Then we say $(A,\mathfrak{a}^t)$ is F-pure if for all $q = p^e >> 0$, there exist $c \in \mathfrak{a}^{\lfloor t(q-1)\rfloor}$ such that $A \to A^{1/q}$ defined by $1 \to c^{1/q}$ splits as A-module. Let $c(\mathfrak{a}) = \sup\{t | (A,\mathfrak{a}^t) \text{ is } F\text{-pure}\}$

Theorem 0.2 (Hara)**.** *Let $(R,\mathfrak{m})$ be a normal local ring, essentially of finite type over a field k of characteristic zero. Let $\mathfrak{a} \subset R$ be an ideal. If $(R_p,\mathfrak{m}_p)$ is obtained from $(R,\mathfrak{m})$ via reduction modulo p, then $lc(\mathfrak{a}) = \lim_{p\to\infty} c(\mathfrak{a}_p)$*

Example 0.3. If A is regular, then Hochster and Roberts showed that $A \to A^{1/p}$ splits if and only if $c \notin \mathfrak{m}^{[q]} = (a^q \mid a \in \mathfrak{m})$. For example, if $p \equiv 1 \bmod 60$ let $\mathfrak{a} = (x^3+y^4+z^5) \subset k[[x,y,z]]$. Then $(x^3+y^4+z^5)^{(20+15+12)m} \equiv (xyz)^{q-1} \bmod \mathfrak{m}^{[q]}$ and $(xyz)^{q-1} \notin \mathfrak{m}^{[q]}$. Thus, we have $c(\mathfrak{a}) = 47/60$.

Remark 0.4. Note that if $(A,\mathfrak{m})$ is regular then

$$c(\mathfrak{a}) = \lim_{q\to\infty} \frac{1}{q} \cdot \max\{r | \mathfrak{a}^r \not\subset \mathfrak{m}^{[q]}\}.$$

1. F-thresholds

Definition 1.1. Let $I \subset A$ be an ideal such that $\mathfrak{a} \subset \sqrt{I}$. Then, we let

$$\nu_{\mathfrak{a}}^I(q) = \max\{r \mid \mathfrak{a}^r \not\subset I^{[q]}\},\ c^I(\mathfrak{a}) = \lim_{q\to\infty} \nu_{\mathfrak{a}}^I(q)/q$$

Question 1.2. *Does the limit exist in definition 1.1?*

If A is F-pure and $x \in \mathfrak{a}, \notin I^{[q]}$, then $\forall q'$ we have $x^{q'} \notin I^{[qq']}$. Namely, $\nu_{\mathfrak{a}}^I(q)/q$ is increasing and there is an obvious upper bound, hence the limit exists.

Remark 1.3. Elementary properties:

1. If $I' \subset I$ then $c^{I'}(\mathfrak{a}) \le c^I(\mathfrak{a})$
2. If $\mathfrak{a} \subset \mathfrak{a}'$ then $c^I(\mathfrak{a}) \le c^I(\mathfrak{a}')$
3. $c^I(\mathfrak{a}) = c^I(\overline{\mathfrak{a}})$

Example 1.4. Let $J = (x_1, \ldots . x_d)$ be a parameter ideal. Then $(x_1, \ldots . x_d)^{q-1} \not\subset J^{[q]}$, thus $\nu_J^J(q) = (q-1)d$ and hence $c^J(J) = d$.

Example 1.5. Let I be a $\mathfrak{m}$-primary ideal. Then $\lim_{n\to\infty} c^{I^n}(I^n) = 1$.

Proposition 1.6. *Assume that $\hat{A}$ is reduced and equidimensional. Let $J \subset I \subset A$ where J is a parameter ideal. Then*

(1) *For all $\mathfrak{m}$-primary ideals $\mathfrak{a}$ we have $c^I(\mathfrak{a}) = c^J(\mathfrak{a})$ if and only if $I \subset J^*$.*
(2) *$c^J(I) = d$ if and only if $I \subset \overline{J}$.*

Remark 1.7 (Hochster-Huneke)**.** $I \subset J^*$ if and only if $\mathrm{length}_A(I^{[q]}/J^{[q]}) = o(q^d)$

2. Application to Wang's Theorem

Theorem 2.1 (Wang)**.** *Let $(A, \mathfrak{m})$ be CM, $J = (f_1, \ldots, f_d)$ a parameter ideal. If $J \subset \mathfrak{m}^s$ than $J : \mathfrak{m}^s \subset \mathfrak{m}^s$.*

When I talked it in the workshop, I knew the theorem only orally and I apology not refering his theorem correctly. I want to apply our theory to this theorem, although we have a little (?) more to go in this stage. (I wrote a more general statement in the workshop but we found a gap to fill.)

Proposition 2.2. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension d whose associated graded ring $G_A(\mathfrak{m}) := \oplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an integral domain. Let $J = (f_1, \ldots, f_d)$ be a parameter ideal and we assume that there is a minimal reduction $(x_1, \ldots, x_d)$ of $\mathfrak{m}$ containing J. Write $f_i = \sum a_{ij}x_j$ and set $\xi = \det(a_{ij})$. Now, assume that $\xi \in \mathfrak{m}^a \setminus \mathfrak{m}^{a+1}$. Then if $s \geq \dfrac{a}{d} + 1$, then $J + \mathfrak{m}^s$ is integral over J.*

3. A conjecture on Multiplicity of Ideals

Theorem 3.1 (de Fernex - Ein - Mustata)**.** *Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring, essentially of finite type over a field k of characteristic zero. Let I be a $\mathfrak{m}$-primary ideal. Then*

$$e(I) \geq \left[\frac{d}{lc(I)}\right]^d$$

where $lc(I)$ is the log canonical threshold and $e(I)$ is the multiplicity of I . It becomes equality if and only if $\overline{I} = m^s$ for some s.

Remark 3.2. I believe this theorem is purely ring theoretic.

Conjecture 3.3. Let $(A, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$, and J a parameter ideal. Then for all $\mathfrak{m}$-primary ideals $\mathfrak{a}$,

$$e(\mathfrak{a}) \geq \left[\frac{d}{c^J(\mathfrak{a})}\right]^d \cdot e(J)$$

Remark 3.4. Conjecture 3.3 implies theorem 3.1 with $J = \mathfrak{m}$

Example 3.5. Let $R = k[[x_1, \ldots, x_d]]$, $J = (x_1^{a_1}, \ldots, x_d^{a_d})$, $\mathfrak{a} = (x_1^{b_1}, \ldots, x_d^{b_d})$. Then $e(\mathfrak{a}) = b_1 \cdots b_d$, $e(J) = a_1 \cdots a_d$ and $c^J(\mathfrak{a}) = a_1/b_1 + \cdots + a_d/b_d$. Thus the inequality in the conjecture holds if and only if

$$\frac{a_1/b_1 + \cdots + a_d/b_d}{d} \geq \sqrt[d]{\frac{a_1 \cdots a_d}{b_1 \cdots b_d}}$$

Example 3.6. Let $A = k[[x,y,z]]/(x^2+y^3+z^5)$, $J = (y,z)$, $\mathfrak{a} = (x,z)$. Then $c^J(\mathfrak{a}) = 5/3$ and $e(J) = 2$. Hence

$$e(\mathfrak{a}) = 3 \geq \left[\frac{2}{5/3}\right]^2 \cdot 2 = 72/25$$

Assume J and $\mathfrak{a}$ are parameter ideals and consider the minimal r such that $\mathfrak{a}^r \subset J^{[q]}$ instead of $\max\{r | \mathfrak{a}^r \not\subset J^{[q]}\}$. Then J^q and $\mathfrak{a}^r$ are quite different idals especially if q is very large. So I propose the following question. The conjecture 3.3 follows from one of the following conjectures.

Conjecture 3.7. Let J and $\mathfrak{a}$ be parameter ideals. If $\mathfrak{a}^r \subset J^{[q]}$ for $q >> 0$. Then

$$\mathrm{length}_A(A/\mathfrak{a}^r) \geq \left[\frac{d^d}{d!}\right] \cdot \mathrm{length}_A(A/J^{[q]}) - O(q^{d-1})$$

Conjecture 3.8. $q >> 0$ and $I \subset J^{[q]}$ integrally closed, then

$$\mathrm{length}_A(A/I) \geq \left[\frac{d^d}{d!}\right] \cdot \mathrm{length}_A(A/J^{[q]}) - O(q^{d-1})$$

Conjecture 3.3 is true in the following case.

Theorem 3.9 (de Fernex-M-T-W). *If $A = k[[x_1, \ldots, x_d]]$ and $J = (x_1^{a_1}, \ldots, x_d^{a_d})$ then conjecture 3.8 holds.*

4. Applications

The following is a long-standing conjecture for me.

Conjecture 4.1. Let $(A, \mathfrak{m})$ be a d-dimesional Noetherian local ring with rational singularity (or F-rational singularity) which is a complete intersection. Is it true that $e(A) \leq 2^{d-1}$?

4.1 follows from the following.

Conjecture 4.2. Let $(A, \mathfrak{m})$ be Artinian and complete intersection. Let s be a positive integer such that $\mathfrak{m}^s \neq 0$, $\mathfrak{m}^{s+1} = 0$. Then $e(A) \leq 2^s$

Remark 4.3. Conjecture 3.3 implies conjecture 4.2 via the following Proposition.

Proposition 4.4. *Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring, I an $\mathfrak{m}$-primary ideal. Then $c^I(\mathfrak{m}) = \max\{i | \mathfrak{m}^i \not\subset I\} + d$*