F-PURE THRESHOLDS AND LOC CANONICAL THRESHOLDS

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ABSTRACT. These are notes on the author's talk given at the workshop on Integral Closure, Multiplier Ideals and Cores, AIM, December 2006.

1. INTRODUCTION

This talk is joint work with M. Mustaţă and K. Watanabe.

Let $(R,\mathfrak{m})$ be a regular local ring of equal characteristic. Let $\mathfrak{a}$ be an R–ideal and $t \geq 0$. If the characteristic is 0 then we consider the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ and if the characteristic is $p > 0$ we consider the test ideal $\tau(\mathfrak{a}^t)$.

Assume that $\sqrt{J} \supset \mathfrak{a}$ for some ideal J of R. The *jumping number of* $\mathfrak{a}$ is defined to be $\lambda^J(\mathfrak{a}) = \sup\{\, t \geq 0 \mid \mathcal{J}(\mathfrak{a}^t) \not\subset J \,\} \in \mathbb{Q}$. In particular $\mathrm{lc}(\mathfrak{a}) = \lambda^{\mathfrak{m}}(\mathfrak{a})$ the lc *threshold* is the smallest jumping number.

Definition 1.1. Let R be an F-finite , reduced and regular ring of positive characteristic. Let $\mathfrak{a}$ be an R–ideal and $t \geq 0$. Then $\tau(\mathfrak{a}^t) =$ unique smallest ideal J such that $\mathfrak{a}^{\lceil tp^e \rceil} \subset J^{[p^e]}$ for all $e \in \mathbb{N}$.

Remark 1.2. **(1)** $\mathfrak{a} \subset \tau(\mathfrak{a})$ $(t = 1)$

(2) (Hara-Yoshida) Let $R = k[x_1, \ldots, x_d]$, where k is perfect field. Let $\mathfrak{a}$ be a monomial ideal in R. Then $\tau(\mathfrak{a}^t) = \langle\, \underline{x}^{\nu} \mid \nu + \underline{1} \in \mathrm{Int}\, t \cdot \underline{P}(\mathfrak{a}) \,\rangle = \mathcal{J}(\mathfrak{a}^t)$, where $\underline{P}(\mathfrak{a})$ is the Newton polygon of $\mathfrak{a}$. The second equality is due to Howald.

Let $(R,\mathfrak{m})$ be a local ring and assume that $\sqrt{J} \supset \mathfrak{a}$. Then

$$\begin{aligned} c^J(\mathfrak{a}) &= \sup\{\, t \geq 0 \mid \tau(\mathfrak{a}^t) \not\subset J \,\} \\ &= \lim_{e \to \infty} \frac{\max\{\, r \in \mathbb{N} \mid \mathfrak{a}^r \not\subset J^{[p^e]} \,\}}{p^e} \\ &< \infty \end{aligned}$$

is called the F–jumping number of $\mathfrak{a}$ (F–threshold).

The F–pure threshold is $\mathrm{fpt}(\mathfrak{a}) = c^{\mathfrak{m}}(\mathfrak{a})$.

Remark 1.3. **(1)** $\mathcal{J}(\mathfrak{a}^{\mathrm{lct}(\mathfrak{a})})$ is radical and $\tau(\mathfrak{a}^{\mathrm{fpt}(\mathfrak{a})})$ is not necessarily radical. For example:

Let $f = x^2 + y^5 + z^5$ in characteristic 2. Then $\mathrm{fpt}(f) = 1/2$ and $\tau(f^{1/2}) = (x, y^2, z^2)$.

In characteristic 0 we have $\mathrm{lct}(f) = 9/10$ and $\mathcal{J}(f^{9/10}) = (x,y,z)$.

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(2) $t \geq 0$:

If t is the F–jumping number of $\mathfrak{a}$ then pt is the F-jumping number of $\mathfrak{a}$.(Blickle–Mustaţă–Smith)

If $\mathfrak{a} = (f)$, then t is the F–jumping number of $f \Leftrightarrow t+1$ is the F–jumping number of f.

(3) (Blickle–Mustaţă–Smith)

If $(R, \mathfrak{m})$ is essentially of finite type over a field or $\mathfrak{a} = (f)$ then $c^J(\mathfrak{a}) \in \mathbb{Q}$.

Let $R = \mathbb{Z}[x_1, \dots, x_d]$ we would like to compare $\mathcal{J}(\mathfrak{a}^t)_p$ and $\tau(\mathfrak{a}_p^t)$.

Theorem 1.4. *(Hara-Yoshida)*

(1) *If $p \gg 0$ then for all $t \geq 0$ one has $\mathcal{J}(\mathfrak{a}^t)_p \supset \tau(\mathfrak{a}_p^t)$.*

(2) *Fix $t \geq 0$. If $p \gg 0$ (depending on t) then $\mathcal{J}(\mathfrak{a}^t)_p = \tau(\mathfrak{a}_p^t)$.*

Corollary 1.5. $\mathrm{lct}(\mathfrak{a}) = \lim_{p \to \infty} \mathrm{fpt}(\mathfrak{a}_p)$.

$(\mathrm{lct}(\mathfrak{a}) \geq \mathrm{fpt}(\mathfrak{a}_p)(p \gg 0))$

Example 1.6. **(1)** Let $f = x^2 + y^3$ and $p > 3$. Then $\mathrm{lct}(f) = 5/6$ and

$$\mathrm{fpt}(f) = \begin{cases} 5/6 & \text{if} \quad p \equiv 1 \bmod 3 \\ 5/6 - 1/6p & \text{if} \quad p \equiv 2 \bmod 3 \end{cases}$$

(2) Let $f \in \mathbb{Z}[x, y, z]$ be homogeneous of degree 3. Assume that f defines a nonsingular variety $Y \subset \mathbb{P}^2$ that is an elliptic curve. Then

$\mathrm{lct}(f) = 1$ and $\mathrm{fpt}(f_p) = 1 \Leftrightarrow \mathrm{Frob} : H^1(Y_p, \mathcal{O}_{Y_p}) \to H^1(Y_p, \mathcal{O}_{Y_p})$ is injective.

If Y has complex multiplication then $\mathrm{fpt}(f_p) = 1$ for half of the primes. If Y does not have complex multiplication then $\mathrm{fpt}(f_p) \neq 1$ for infinitely many primes p(Elkies). This set of infinitely many primes p has density 0 (Serre).

(3) Let $f = x^3 + y^3 + z^3$. Then $\mathrm{fpt}(f) = \begin{cases} 1 & \text{if} \quad p \equiv 1 \bmod 3 \\ 1 - 1/p & \text{if} \quad p \equiv 2 \bmod 3 \end{cases}$

(4) Let $f = y^2 z - yz^2 - x^3 + x^2 z$. Then $\mathrm{fpt}(f) = \begin{cases} 1 - 1/p & \text{if } p = 2, 19, 29, 199, 569, 809, \dots \\ 1 & \text{otherwise} \end{cases}$

Conjecture 1.7. *(*Hara–Watanabe*)*

There exists a set of primes $\mathcal{P}$ of positive density such that for each $p \in \mathcal{P}$ we have that $\mathrm{lct}(\mathfrak{a}) = \mathrm{fpt}(\mathfrak{a}_p)$.

Conjecture 1.8. *(*Mustaţă*)*

For all $n \geq 0$ *let*

$I_n = \{t \mid t = \mathrm{fpt}(\mathfrak{a}) \textit{for some } \mathfrak{a} \subsetneq R, \textit{ for some} RF - \textit{ finite regular local ring of dimension} \leq n\}.$

Then I_n *contains no infinite strictly increasing sequences.*

It is known that the Conjecture 1.8 holds for the lc threshold if $n \leq 3$.