LOCAL SYZYGIES OF MULTIPLIER IDEALS

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Abstract. These are notes on the author's talk given at the workshop on Integral Closure, Multiplier Ideals and Cores, AIM, December 2006.

1. Multiplier ideals

Definition 1.1. Let R be a local ring of a smooth complex variety of dimension d at a point. Let $\mathfrak{b} \subset R$. Given $c > 0$, we define $\mathcal{J}(\mathfrak{b}^c) \subset R$. Choose generators $f_1, \ldots, f_r \in \mathfrak{b}$. Then

$$\mathcal{J}(\mathfrak{b}^c) = \{h | \frac{|h|^2}{(|f_i|^2 + \ldots + |f_r|^2)^c} \text{ locally integrable}\}.$$

Equivalently, let $\mu : X' \to X = \mathrm{Spec}(R)$ be a log resolution of $\mathfrak{b}$, therefore $\mathfrak{b} \cdot \mathcal{O}_{X'} = \mathcal{O}(-F)$. Then

$$\mathcal{J}(\mathfrak{b}^c) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [cF]).$$

Example 1.2. Let $\mathfrak{b} = (x_1^{e_1}, \ldots, x_d^{e_d}) \subset \mathbb{C}[x_1, \ldots, x_d]$. Then

$$\mathcal{J}(\mathfrak{b}^c) = (x_1^{m_1}, \ldots, x_d^{m_d} | \sum_i \frac{m_i + 1}{e_i} > c).$$

Problem 1.3. Which ideals can arise as multiplier ideals? That is, given $\mathfrak{a}$, is $\mathfrak{a} = \mathcal{J}(\mathfrak{b}^c)$ for some $\mathfrak{b}$ and c?

2. Integral Closure

Definition 2.1. An Ideal $\mathfrak{a}$ is integrally closed if it satisfies either of the following:

(1) If $f_1, \ldots, f_r \in \mathfrak{a}$ are local generators, and $g \in R$ satisfies $|g(z)| \leq_{loc} C \cdot \sum |f_i(z)|$, then $g \in \mathfrak{a}$. This, by the way, indicates that the continuous closure is contained in the integral closure.

(2) There is a birational proper map $\nu : X^+ \to X$, X^+ normal, plus an effective divisor F such that $\mathfrak{a} = \nu_* \mathcal{O}_{X'}(-F)$.

Corollary 2.2. *From second definition it's clear that multiplier ideals are integrally closed.*

Problem 2.3. Is every integrally closed ideal a multiplier ideal?

Example 2.4 (Mustaţă, Howald, Teitler)**.** Every integrally closed monomial ideal is a multiplier ideal.

Example 2.5 (Lipman-Watanabe, Favre-Jonsson)**.** If the ring has dimension $d = 2$, every integrally closed ideal in a regular local ring is a multiplier ideal.

3. Local Syzygies of Multiplier Ideals

Theorem 3.1 (Lazarsfeld, Lee)**.** *Let $(R, \mathfrak{m})$ be a local ring of a smooth complex variety of dimension d at a point and let $\mathcal{J} = \mathcal{J}(\mathfrak{b}^c) \subset R$ be any multiplier ideal. If $p \geq 1$, then no minimal p^{th} syzygy of $\mathcal{J}$ can vanish modulo $\mathfrak{m}^{d+1-p}$.*

If $p = 1$, choose minimal generators $h_1, \ldots, h_e \in \mathcal{J}$. Consider minimal syzygy $\sum g_i h_i = 0$. Then $ord(g_i) \leq d-1$ for at least one i. In general, consider a minimal free resolution of the multiplier ideal:

$$\ldots \to R^{e_2} \xrightarrow{u_2} R^{e_1} \xrightarrow{u_1} R^{e_0} \to \mathcal{J} \to 0$$

with domain $u_i = R^{e_i}$. Set $\mathrm{Syz}_p(\mathcal{J}) = \mathrm{Im}(u_p) \subset R^{e_{p-1}}$. Then no minimal generator of $\mathrm{Syz}_p(\mathcal{J})$ can lie in $\mathfrak{m}^{d+1-p} R^{e_{p-1}}$.

Remark 3.2. There is no restrictions on the order of vanishing of the generators of multiplier ideals. For example, $\mathfrak{m}^k = \mathcal{J}(\mathfrak{m}^{d-1+k})$. The theorem implies that if $d \geq 3$, multiplier ideals are very special among integrally closed ideals.

Example 3.3. Choose m general functions $h_1, \ldots, h_m \in R$ with orders $> d$. Then the ideal $\mathfrak{a}$ generated by them is reduced, hence integrally closed. But Koszul syzygies violate the theorem, so it is not a multiplier ideal. When $p = 1$, take (f, g), with orders $d-1$. Then this can be realized as a multiplier ideal.

Remark 3.4. The above theorem follows from the fact that if $R/\mathfrak{m} = \mathbb{C}$, then for all $0 \leq p \leq d$, the map $\mathrm{Tor}_p(\mathfrak{m}^{d-p}\mathcal{J}, \mathbb{C}) \to \mathrm{Tor}_p(\mathcal{J}, \mathbb{C})$ vanishes.