EQUISINGULARITY AND INTEGRAL CLOSURE

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Abstract. These are notes on the author's talk given at the workshop on Integral Closure, Multiplier Ideals and Cores, AIM, December 2006.

1. Plane Curves

Let $C : f = 0$ be an analytic curve through the origin 0 in the complex plane $\mathbb{C}^2$. The *singularity type* of C at 0 may be defined simply as the local topological type (homeomorphism type) of the triple ($\mathbb{C}^2$, C, 0).

The topology can be translated into algebra after Brauner (1928), Burau (1932), and Zariski (1932). The topological type corresponds to a combinatorial invariant, the resolution graph, via the resolution of the singularity. We get a sequence of infinitely near points $0 =: P,\ P',\ P'', \ldots$. It produces the graph via the corresponding sequence of blowups

$$\mathbb{C}^2 =: X \leftarrow X' \leftarrow X'' \leftarrow \cdots ,$$

and the proper (strict) transforms $C^{(i)}$ of C. Continue in a minimal number of steps until the reduced preimage of C is a collection of smooth curves with normal crossings. The corresponding dual graph is the *resolution graph* $\mathbb{D}$.

Let m_i be the multiplicity of $C^{(i)}$ at $P^{(i)}$. Let E_i be the preimage of $P^{(i)}$ on the final blowup $X^{(n)}$. Set

$$I := H^0(\mathcal{O}_{X^{(n)}}(-\Sigma m_i E_i)).$$

Then I is the complete (integrally closed) ideal associated to the point P on C.

Consider all complete ideals I with the same $\mathbb{D}$. These I form a locally closed subset H of the Hilbert scheme $\mathrm{Hilb}^d_{X/\mathbb{C}}$ where d is a certain combinatorial invariant of $\mathbb{D}$, owing to the work of Nobile and Villamayor (1997).

More information about H is given by the following theorem.

Theorem 1.1 (Kleiman–Piene)**.** *The locus H of complete ideals is smooth; in fact, it is the image of a natural embedding*

$$\{\text{sequences with diagram } \mathbb{D}\} \big/ \mathrm{Aut}(\mathbb{D}) \hookrightarrow \mathrm{Hilb}^d_{X/\mathbb{C}}$$

Here the sequences with diagram $\mathbb{D}$ are parameterized by a certain smooth scheme; in fact, the scheme represents the functor of sequences. And $\mathrm{Aut}(\mathbb{D})$ denotes the automorphism group of the diagram $\mathbb{D}$; it is a finite group, and acts freely.

Problems 1.2. (1) Extend Theorem 1.1 to positive characteristic.

(2) Give a direct proof of Theorem 1.1, not involving the theory of curve singularities, which would clarify the situation in positive characteristic. In this connection, see the recent survey by Greuel, Lossen, and Shustin (arXiv:math/0612310).

(3) Develop a good deformation theory for complete ideals in arbitrary characteristic, incorporating Gustavsen's in characteristic 0. For example, over an Artin

$\mathbb{C}$-algebra, explain why a complete ideal I uniquely determines, in the corresponding total quotient ring, a minimal resolving sequence of local rings, that is, a minimal sequence of monoidal transforms rendering I invertible.

2. Higher Dimensions

Consider germs $(X, 0)$ of isolated complete intersection singularities (ICIS) of arbitrary dimension. Unfortunately, we no longer have an abstract notion of singularity type. Nevertheless, it turns out that we can say when a particular member of a family has the *same* singularity type as a general member. Indeed, using integral closure methods, we can associate numerical invariants to the members so that, if the family is Thom equisingular, resp. Whitney, equisingular, then the invariants have the same values for each member, and conversely.

To be precise, consider the following setup:

$$\mathbb{C}^a \times Y \supset X \supset \{0\} \times Y$$

where Y is a smooth parameter space; say $Y = \mathbb{C}^b$ and identify Y with $\{0\} \times Y$. Here X is the total space of the family of ambient spaces. Assume that X is a complete intersection at $(0, 0)$.

Let $f\colon X \to \mathbb{C}$ be a nonconstant analytic function, and $Z := f^{-1}(0)$ its zero locus. Assume that $X - Y$ is smooth over Y, that Z contains Y, that $Z - Y$ is smooth, and that $f^{-1}(u)$ is smooth for $u \neq 0$. Then, for each parameter value $y \in Y$, the fibers X_y and Z_y are ICIS germs at 0. So they have well-defined Milnor numbers $\mu(X_y)$ and $\mu(Z_y)$, which quantify the topological vanishing cycles.

In order to formulate the Thom and the Whitney equisingular conditions, we need some notation. Let $C(X, f)$ denote the closure of the set of pairs (x, H) where $x \in X - Y$ and H is a hyperplane tangent at x to $f^{-1}fx$. Let $C(X, f) \mid Y$ denote the preimage of Y. Finally, let $C(Y)$ denote the set of pairs (x, H) where $x \in Y$ and H is a hyperplane containing Y.

The Thom condition A_f. As sets, $C(X, f)|Y = C(X, f) \cap C(Y)$.

The Whitney condition W_f (Henry–Merle–Sabbah, 1984). When $C(X, f)|Y$ and $C(X, f) \cap C(Y)$ are viewed as subvarieties of $C(X, f)$, their ideals have the same integral closure.

Theorem 2.1 (Thom–Mather–$\cdots$–Gaffney)**.** *The Whitney condition W_f implies the topological triviality of the triple (X, f, Y); namely, there is a homeomorphism $h\colon X_0 \times Y \to X$ such that $f \circ h = (f|_{X_0}) \times 1_Y$.*

Theorem 2.2 (Gaffney–Kleiman)**.** *Given a nested sequence of linear-space sections*

$$X = X^0 \supset X^1 \supset \cdots \supset X^n = Y$$

in general position, the following conditions are equivalent:

(i) *the Whitney condition W_f holds;*
(ii) *each triple $(X^i,\ f|X^i,\ Y)$ is topologically trivial;*
(iii) *each function $y \mapsto (\mu(X^i_y), \mu(Z^i_y))$ is constant.*

Theorem 2.3 (Gaffney–Kleiman–Massey)**.** *The Thom condition A_f holds iff the function $y \mapsto (\mu(X_y), \mu(Z_y))$ is constant.*

Remark 2.4. Theorem 2.3 was proved in the case $X = \mathbb{C}^a \times Y$ by Lê and Saito in 1973 using methods of differential topology. The theorem was reproved a few months later using integral closure methods by Teissier, and his work has inspired all the subsequent work involving integral closure methods in equisingularity theory.

3. Multiplicities

The theory of multiplicities is, of course, centuries old. However, the first rigorous general algebraic treatment was given by Chevalley (1945). Samuel (1951) used the Hilbert polynomial to give the first definition valid for an ideal of finite colength in an arbitrary Noetherian local ring. His definition was generalized by Buchsbaum and Rim (1964) to submodules of finite colength in a free module as follows.

Let A be a Noetherian local ring of dimension d. Let M be a submodule of finite colength in a free A-module F of rank r. Let SF denote the symmetric algebra of F, and $RM \subset SF$ the Rees algebra of M, which is the subalgebra of SF generated by M placed in degree 1. For example, if $r = 1$, then $M \subset A$ is an ideal, and

$$SF = A \oplus A \oplus A \oplus \cdots ,$$
$$RM = A \oplus M \oplus M^2 \oplus \cdots .$$

Set $f := d + r - 1$. Then, for all sufficiently large m,

$$\operatorname{length}_A(S_m F/R_m M) = e \cdot m^f/f\,! + \cdots ,$$

and e is, by definition, the *Buchsbaum–Rim multiplicity* of M in F.

To use the Buchsbaum–Rim multiplicity in equisingularity theory, choose coordinates $x_1, \ldots, x_a$ for $\mathbb{C}^a$ and $y_1, \ldots, y_b$ for $\mathbb{C}^b$. Say $X : f_1, \ldots, f_p = 0$, and say $f = f_0|X$. Take M to be the column space of the Jacobian matrix

$$\begin{pmatrix} \partial f_0/\partial x_1 & \cdots & \partial f_0/\partial x_n \\ \vdots & & \vdots \\ \partial f_p/\partial x_1 & \cdots & \partial f_p/\partial x_n \end{pmatrix}.$$

So M is a submodule of a free module of rank $p+1$ over the local ring of the germ $(X, 0)$. Set $M_y := M|X_y$, and set $\mathfrak{m}_y := \langle x_1, \ldots, x_a \rangle$, the maximal ideal of $(X_y, 0)$.

Theorem 3.1 (Gaffney–Kleiman–Massey)**.** *The Thom condition A_f holds iff the function $y \mapsto e(M_y)$ is constant. The Whitney condition W_f holds iff the function $y \mapsto e(\mathfrak{m}_y M_y)$ is constant.*

Theorem 3.1 is equivalent to Theorems 2.3 and 2.2. Indeed,

$$e(M_y) = \mu(X_y) + \mu(Z_y)$$

owing to formulas of Buchsbaum and Rim and of Lê and Greuel. Furthermore, $e(\mathfrak{m}_y M_y)$ is a linear combination of the sums $\mu(X_y^i) + \mu(Z_y^i)$ owing to an expansion formula for $e(\mathfrak{m}_y M_y)$ in terms of mixed multiplicities proved by Thorup and the author (1994). The equivalence now follows from the upper semi-continuity of the Milnor number μ. Hence, to prove all three theorems, we need only prove Theorem 3.1. Its proof involves integral dependence, and is discussed in the next section.

Problem. With an eye toward generalizing Theorem 3.1, generalize the notion of multiplicity to the case of arbitrary submodules of a free module, so that the new multiplicity is upper semi-continuous and satisfies an expansion formula in terms of mixed multiplicities.

4. Integral Dependence

Again, let M be a submodule of a free module F over a ring A. Recall that an element $g \in F$ is said to be *integral*, or integrally dependent, over M if

$$g^n + r_1 g^{n-1} + \cdots + r_n = 0 \text{ in } SF \text{ for some } n \text{ and } r_i \in RM.$$

To use this notion in equisingularity theory, take A to be local ring of the germ $(X, 0)$. The key is the following lemma, which was named and proved by Teissier (1973) in the case that M is an ideal. The lemma was generalized from ideals to modules by Gaffney and the author (1999). The basis is a corresponding generalization of the Böger–Rees theorem. It was proved by Thorup and the author (1994), but the proof was complicated. It was simplified by Gaffney and Massey (1999), by Simis, Ulrich and Vasconcelos (2001), and by Thorup and the author (2000).

Lemma 4.1 (Principle of Specialization of Integral Dependence)**.** *Assume that the function $y \mapsto e(M_y)$ is constant on Y. If $g|X_y$ is integral over M_y for all y in a Zariski open subset of Y, then g is integral over M.*

The principle is applied with g taken, for $1 \le j \le b$, to be the vector

$$g_j := \begin{pmatrix} \partial f_0/\partial y_j \\ \vdots \\ \partial f_p/\partial y_j \end{pmatrix}.$$

Integral dependence is connected to the Thom and Whitney conditions by the next lemma, where $\mathfrak{m}_Y := \langle x_1, \ldots, x_a \rangle A$.

Lemma 4.2 (Gaffney–Kleiman)**.** *The Thom condition A_f holds iff the g_j are integral over M. The Whitney condition W_f holds iff the g_j are integral over $\mathfrak{m}_Y M$.*

The final ingredient in the proof of Theorem 3.1 is the following genericity lemma.

Lemma 4.3 (Hironaka, Henry–Merle–Sabbah)**.** *The Thom condition A_f and the Whitney condition W_f both hold on some nonempty Zariski open subset of Y.*

These lemmas are used as follows. Assume that the function $y \mapsto e(M_y)$ is constant on Y. By Lemma 4.3, the Thom condition A_f holds on some nonempty Zariski open subset U of Y. Let M be the column space of the Jacobian matrix. Then by Lemma 4.2, for all $y \in U$, the $g_j|X_y$ are integral over M_y. So by Lemma 4.1, the g_j are integral over M. Hence, by Lemma 4.2, the Thom condition A_f holds over Y. Similarly, if the function $y \mapsto e(\mathfrak{m}_y M_y)$ is constant on Y, then the Whitney condition W_f holds.

For more information about the proof of Theorem 3.1, see the author's paper, "Equisingularity, multiplicity, and dependence," Commutative algebra and algebraic geometry (Ferrara), Lecture Notes in Pure and Appl. Math., 206, Dekker, New York, 1999, pp. 211–225.

Problem. With an eye toward generalizing Theorem 3.1, generalize the notion of multiplicity to the case of arbitrary submodules of a free module, so that the Principle of Specialization of Integral Dependence continues to hold.