CONTINUOUS CLOSURE AND VARIANT NOTIONS OF INTEGRAL CLOSURE

MEL HOCHSTER

ABSTRACT. These are notes on the author's talk given at the workshop on Integral Closure, Multiplier Ideals and Cores, AIM, December 2006.

1. INTRODUCTION

Let $X \subset \mathbb{C}^n$ be a Zariski closed algebraic set and consider $\mathbb{C}[X] \hookrightarrow \mathcal{C}(X)$, where $\mathcal{C}(X)$ is the set of all continuous complex functions in the Euclidean topology. Let I be an ideal in $\mathbb{C}[X]$.

Definition 1.1. (H. Brenner) *The continuous closure* of I, denoted by I^{cont}, is $I\mathcal{C}(X) \cap \mathbb{C}[X]$.

Question 1.2. (H. Brenner) Can the continuous closure be characterized algebraically?

Theorem 1.3. (H. Brenner) *$I \subset I^{\text{cont}} \subset \bar{I}$, where $\bar{I}$ denotes the integral closure of I.*

In general $I^{\text{cont}} \subsetneq \bar{I}$. Also, if we consider a map between two rings $R \to S$, then $I^{\text{cont}}S \subset (IS)^{\text{cont}}$.

Definition 1.4. (H. Brenner) R is called a *ring of axes* if it is reduced, one–dimensional and either a smooth irreducible curve or a finite union of such curves all meeting at one point, where the singularity of the point has a completion of the form

$$\frac{\mathbb{C}[[x_1,\ldots,x_n]]}{(x_ix_j : 1 \le i < j \le n)}$$

Let I be an R–ideal and let $f \in R$. Then $f \in I^{ax} \Leftrightarrow$ for all maps $R \to S$, where S is a ring of axes, $f \in (IS)^c$, that is, the image of f is in IS.

Theorem 1.5. (H. Brenner) *In a ring of axes $I = I^{\text{cont}}$ for every ideal I. Then $I^{\text{cont}} \subset I^{ax}$.*

Conjecture 1.6. (H. Brenner)*: $I^{\text{cont}} = I^{ax}$.*

Theorem 1.7. (H. Brenner)

If I is generated by monomials in $x_1,\ldots,x_n$ and I is $\mathfrak{m}$–primary, then $I^{\text{cont}} = I^{ax} = I+$(monomials whose exponent vector is in the interior of the convex hull of the exponent vectors for I).

Date: July 13, 2007.

2. Some results

Definition 2.1. Let R be a Noetherian ring. We say that F is (I,J)–special if $F \in R[u]$, and F is a monic polynomial of the form $u^n + r_{n-1}u^{n-1} + \ldots + r_0$, where $r_{n-j} \in JI^j$ where $1 \le j \le n$. We now define the *J–special integral closure of* I. We say that $r \in \overline{I}^{\,\mathrm{sp}J}$ if there exists an (I,J)–special polynomial such that $F(r) = 0$. If $J = R$ then this is the special integral closure is simply integral closure. Theee (I,J)-special polynomials form a multiplicative system, and from this one can see that one can test whether an element is in $\overline{I}^{\,\mathrm{sp}J}$ by checking modulo every minimal prime. This reduces understanding the question to the domain case.

Theorem 2.2. (N. Epstein-M. Hochster)

Assume that R is a domain and I and J are two nonzero ideals. Then the following are equivalent:

(1) $r \in \overline{I}^{\,\mathrm{sp}J}$

(2) *There exists an n such that $r^n \in \overline{JI^n}$*

(3) *For all maps $R \hookrightarrow V$, where V is a DVR we have*

$$\mathrm{ord}(r) \ge \frac{\mathrm{ord}(J)}{n} + \mathrm{ord}(I) \quad \textit{for some } n$$

(4) *For all maps $R \hookrightarrow V$, where V is a DVR we have*

$$\mathrm{ord}(r) \ge \mathrm{ord}(I)$$

where the inequality is strict if $JV \subsetneq V$.

$\overline{I}^{\,\mathrm{sp}J}$ depends only on the radical of J. If (R,m,K) is local and $J = m$ this coincides with the *special part of the integral closure* studied by N. Epstein. If $J = I$ we call $\overline{I}^{\,\mathrm{sp}I}$ the *interior integral closure of* I, and denote it by $\overline{I}^{\,\mathrm{int}}$. For monomial ideals primary to the ideal $(x_1,\ldots,x_n)$ in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, the interior integral closure is the ideal generated by monomials whose exponent vectors are in the interior of the convex hull of the exponent vectors of the monomials in the ideal. Of course, the integral closure corresponds to the convex hull in a similar way.

Also, we define $I + \overline{I}^{\,\mathrm{int}} = I^{\natural}$, which we call the *natural closure of* I. We can show that the natural closure is contained in the continuous closure. This depends on the result below.

Let $R = \mathbb{C}[X]$ with $X \subset \mathbb{C}^n$ a closed algebraic set. Consider

$$\mathbb{C}[X] \twoheadleftarrow \mathbb{C}[x_1,\ldots,x_n],$$

and that when $x_1,\ldots,x_n$ are given positive integer weights $k_1,\ldots,k_n$ the kernel is a weighted homogeneous ideal Assume $\mathrm{Rad}(F_1,\ldots,F_m) = (x_1,\ldots,x_n)$, where the F_j are weighted homogeneous and let $\deg(F_j) = d_j$. Let $\mathbb{C}$ act on $\mathbb{C}^n$ by $\lambda(c_1,\ldots,c_n) = (\lambda^{k_1}c_1,\ldots,\lambda^{k_n}c_n)$. Let F be weighted homogeneous of degree $d > \max_j d_j$.

Theorem 2.3. (H. Brenner) *With hypotheses as in the paragraph above,*

$F \in (F_1,\ldots,F_m)^{\mathrm{cont}}$.

One can see this as follows. One has that $\sum_i \overline{F}_i F_i = \sum_i |F_i|^2 = G$ vanishes only at the origin P. Therefore on $X - P$ one has $F = \sum_i g_i F_i$ where $g_i = F\overline{F_i}/G$ is continuous. For any point $y = (y_1, \ldots, y_n) \in X - \{p\}$, let $||y|| = (\sum_i |y_i|^{2/k_i})^{1/2}$. Then for any $y \in X - \{P\}$, $F(y) = ||y||^d F(y/||y||) = ||y||^d \sum_i g_i(y/||y||) F_i(y/||y||) = \sum_i h_i(y) F_i(y)$, where $h_i(y) = ||y||^{d-d_i} g_i(y/||y||)$. Each h_i extends to all of X, because $y/||y| \in \{x \in X : ||x|| = 1\}$ and this set is compact. Thus, $g_i(y/||y||)$ is bounded, while $||y||^{d-d_i} \to 0$ as $y \to P$.

One can then prove that the interior integral closure of I and, hence, the natural closure of I, is contained in the continuous closure, because a "generic" equation displaying that an element is in the interior integral closure is weighted homogeneous. We also have:

Theorem 2.4. (N. Epstein-M. Hochster) *Let R be a finitely generated reduced $\mathbb{C}$–algebra. If I is $\mathfrak{m}$–primary, where $\mathfrak{m}$ is any maximal ideal in R, then*

$$I^{\,\mathrm{ax}} = I^{\,\mathrm{cont}} = I^{\,\natural}$$

Theorem 2.5. (N. Epstein-M. Hochster) *Consider an $F \in \mathbb{C}[x_1, \ldots, x_n]$ that vanishes at a point. Then there exists a Zariski neighborhood U of the point such that $F = \sum g_i \dfrac{\partial F}{\partial x_i}$ on U, g_i is continuous.*

Question 2.6. If R is a seminormal affine reduced $\mathbb{C}$-algebra, is $I^{\,\natural} = I^{\,\mathrm{cont}} = I^{\,ax}$?

If this is true, it would prove that continuous closure and axes closure are the same in general.

This is not true when R is not seminormal: in $R = \mathbb{C}[x^2, x^3, y, xy]$ one has that if $I = yR$, then xy is in the continuous closure of R but it is easy to see that it is not in the natural closure.