

Extreme real forms of algebraic varieties

The American Institute of Mathematics

Harnack curves appear in connection with several area of mathematics. The discussion session of the workshop was devoted to possible generalizations of Harnack curves to higher dimensions. The participants proposed several conjectural answers to the question

what should be the definition of Harnack and anti-Harnack varieties ?

The present text has three sections. First, we give several equivalent definitions of Harnack curves and state some of their properties. Then, we propose several possible definitions of Harnack varieties and formulate some conjectures. The third section concerns “good” candidates to what should be called *anti-Harnack varieties*.

In the whole text, a real algebraic variety is an algebraic variety defined over \mathbb{C} endowed with an anti-holomorphic involution c . The real part $\mathbb{R}X$ of X is defined as the fixed point set of c . When dealing with toric varieties, we will consider the tautological real structure induced by the complex conjugation in $(\mathbb{C}^*)^n$.

1 Harnack curves

In this section we recall known facts about Harnack curves.

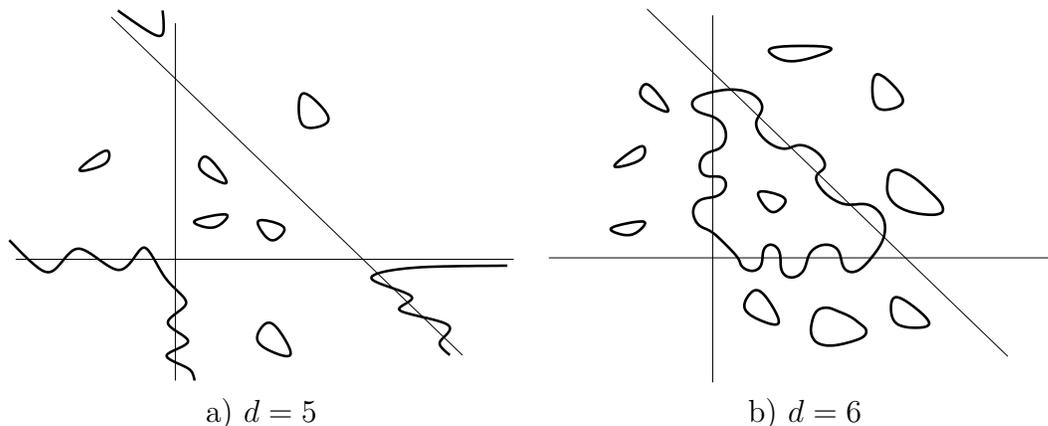


Figure 1:

Definition 1.1 *A Harnack curve of degree d is a nonsingular real algebraic curve C of degree d in $\mathbb{R}P^2$ with the maximal number (i.e., $\frac{(d-1)(d-2)}{2} + 1$) connected components and such that there exist*

3 lines l_1, l_2 and l_3 in general position in $\mathbb{R}P^2$ and 3 disjoint arcs a_1, a_2 and a_3 of $\mathbb{R}C$ lying on the same connected component of $\mathbb{R}C$ such that a_i intersects l_i in d distinct points.

Remark. This definition extends easily to curves in any toric surface.

Examples of Harnack curves of degree 5 and 6 are depicted in Figure 1 .

Being Harnack is a very strong condition. For example, one has the following properties.

Theorem 1.2 (Mikhalkin) *The topological type in $\mathbb{R}P^2$ of the union of a Harnack curve C and the lines l_i is uniquely determined by the degree of C .*

Theorem 1.3 (Mikhalkin - Rullgard) *The only possible degenerations of a Harnack curve are an oval shrinking to a point or two intersection points of C with a line l_i becoming a tangency point of C with l_i (see Figure 2).*

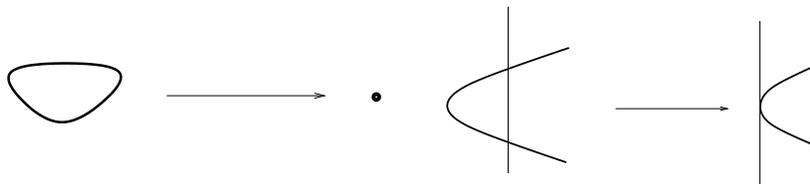


Figure 2:

One can formulate equivalent definitions for Harnack curves in terms of its *amoeba*. Let us define the Log function by

$$\begin{aligned} \text{Log} : (\mathbb{C}^*)^2 &\rightarrow \mathbb{R}^2 \\ (z, w) &\mapsto (\log(|z|), \log(|w|)) \end{aligned}$$

Given a curve C in $(\mathbb{C}^*)^2$, the image of C under the map Log is called the amoeba of C . Suppose that a coordinate system is fixed in $\mathbb{C}P^2$, then $\mathbb{C}P^2 \setminus (\{x = 0\} \cup \{y = 0\} \cup \{z = 0\}) = (\mathbb{C}^*)^2$, hence any curve in $\mathbb{C}P^2$ with a fixed coordinate system has an amoeba.

Definition 1.4 *A Harnack curve is a nonsingular real algebraic curve C such that the map $\text{Log}|_C$ is not more than 2 to 1.*

Definition 1.5 *A Harnack curve is a nonsingular real algebraic curve C such that $\text{Log}(C)$ has maximal area $\mathcal{A}(\Delta)\pi$, where $\mathcal{A}(\Delta)$ is the euclidean area of the Newton polygon of C .*

Remark. Definitions 1.4 and 1.5 extend to the case of singular curves.

Question. What is the image of a Harnack curve under the moment map of the compact toric variety?

In the case of $\mathbb{C}P^2$ this map is

$$\begin{aligned} (\mathbb{C}^*)^2 &\rightarrow \mathbb{C}P^2 \\ (z, w) &\mapsto \frac{1}{1+|z|^2+|w|^2}(|z|^2, |w|^2) \end{aligned}$$

The area of Harnack curves is not constant under this map.

Another definition of Harnack curve can be given in terms of its *co-amoeba* (also called *Alga* or *phase amoeba*). The argument map is defined as

$$\begin{aligned} \text{Arg} : (\mathbb{C}^*)^2 &\rightarrow S^1 \times S^1 \\ (z, w) &\mapsto (\arg(z), \arg(w)) \end{aligned}$$

The image of a curve C under the map Arg is called the co-amoeba of C .

The following map is 4 to 1

$$\begin{aligned} S^1 \times S^1 &\rightarrow S^1 \times S^1 \\ (s, t) &\mapsto (s^2, t^2) \end{aligned}$$

The composition of the latter map with the map Arg gives the map $2\text{Arg}(z, w) = (2\arg(z), 2\arg(w))$.

Definition 1.6 *A real algebraic curve C is Harnack if the map $2\text{Arg}|_C$ is birationally a covering of degree $2d^2$.*

Here, “birationally” means “up to blow up at intersection points of C with toric divisor and $\widetilde{S^1 \times S^1} = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ ”. All the four definitions given above are different faces of the same

object.

Theorem 1.7 (Mikhalkin) *Definitions 1.1, 1.4, 1.5 and 1.6 are equivalent.*

2 Generalisation of Harnack curves to higher dimension

Here we propose some possible answers to the following question:

what is a Harnack variety in higher dimension and codimension?

Equivalent definitions for curves could be generalized to inequivalent objects!

Here we focus on surfaces in $\mathbb{R}P^3$.

It follows from the Smith theory that for any real algebraic variety X , one has

$$\sum b_i(\mathbb{R}X, \mathbb{Z}/2\mathbb{Z}) \leq \sum b_i(X, \mathbb{Z}/2\mathbb{Z}),$$

where $b_i(\cdot, \mathbb{Z}/2\mathbb{Z})$ denotes the i^{th} Betti number with $\mathbb{Z}/2\mathbb{Z}$ coefficients. In case of equality, one says that X is an *M-variety*.

If X is a nonsingular surface of degree d in $\mathbb{R}P^3$, then $\sum b_i(X, \mathbb{Z}/2\mathbb{Z}) = d^3 - 4d^2 + 6d$.

One can reasonably expect a plane in $\mathbb{R}P^3$ to be a Harnack surface. Hence, our “hopeful definitions” of a Harnack surface are motivated by the study of the real plane defined by the equation $x + y + z = 1$. For all our hopeful definitions, the following questions are the first natural problems.

Questions.

1. What are the restrictions imposed by our hopeful definitions on the topology of a Harnack surface S ?
2. What is the maximal number of connected component of S ?
3. Is S an M -surface?
4. Is the topological type of the pair $(\mathbb{R}P^3, S)$ unique?

Let us fix a coordinate system in $\mathbb{R}P^3$. It defines four coordinate planes and four affine charts isomorphic to \mathbb{R}^3 . A real algebraic surface S in $\mathbb{R}P^3$ defines a real algebraic surface \tilde{S} in any of these affine charts, as well as a (reducible) real algebraic curve $F\tilde{S}$ defined as the intersection of \tilde{S} with the three coordinate planes of the affine chart.

Put $\tilde{\beta}_i = \dim H_i(\tilde{S}, F\tilde{S}, \mathbb{Z}/2\mathbb{Z})$.

Hopeful Definition 1.1 *A Harnack surface is a nonsingular real algebraic surface S of degree d in $\mathbb{R}P^3$ such that for any affine chart of $\mathbb{R}P^3$ one has*

$$\sum \tilde{\beta}_i = d^3.$$

One can be more cautious in Hopeful Definition 1.1, by removing two planes from $\mathbb{R}P^3$ instead of a single one.

Let S be a nonsingular real algebraic surface S of degree d in $\mathbb{R}P^3$. Choose two coordinate planes H_1 and H_2 and denote by \hat{S} the intersection of S with $\mathbb{R}P^3 \setminus (H_1 \cup H_2)$, and by $F\hat{S}$ the intersection of S with the two remained coordinate planes (*i.e.*, the two coordinate planes which are not H_1 or H_2).

Put $\hat{\beta}_i = \dim H_i(\hat{S}, F\hat{S}, \mathbb{Z}/2\mathbb{Z})$

Cautious Hopeful Definition 1.1 *The surface S is a Harnack surface if*

$$\sum \hat{\beta}_i = d^3$$

for any choice of H_1 and H_2 .

With this more cautious Hopeful Definition 1.1, the topology of a Harnack surface is fixed.

Proposition 2.1 (Mikhalkin) *If S is a Harnack surface of degree d in the sense of Cautious Hopeful Definition 1.1, then S is an M -surface whose real part is the disjoint union of $\frac{d^3}{6} - d^2 + \frac{11d}{6} - 1$ spheres and a component with an appropriate number of handles.*

Questions. Consider the map $\phi(x, y, z) = (x^d, y^d, z^d)$. What is the image by ϕ of the plane with equation $\{x + y + z = 1\}$?

In the case of curves, one obtains a maximally singular Harnack curve, *i.e.*, a rational Harnack curve with only one intersection point with any of the three lines.

In the case of surfaces, is it possible to deform the result into a “non-singular smooth Harnack surface”?

Now, let us consider Definition 1.4. The example of the plane $\{x + y + z = 1\}$ suggests the following hopeful definition.

Hopeful Definition 1.4 *A Harnack surface is a nonsingular real algebraic surface S such that each fiber of the map $\text{Log}|_S$ consists of at most 2 circles.*

Definition 1.5 does not admit a straightforward generalization. Indeed, as soon as the dimension of the hypersurface is greater than 2, the volume of the amoeba is infinite. Hence we cannot naively generalize the formula “ $\text{Area}(\text{amoeba}) = \pi^2 \text{Area}(\Delta)$ ”.

Question. Is there some other invariant (measure, characteristic, ... ?) that we can use “to measure” $\text{Log}(S)$?

Can we find such an invariant via integral geometry?

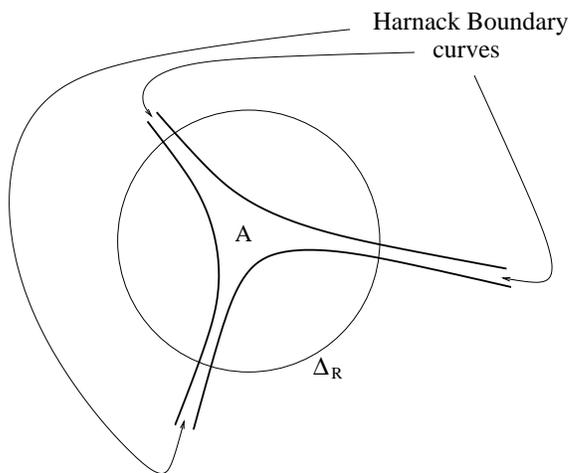


Figure 3:

Idea. Denote by Δ_R the R -scaled Newton polytope of S (see Figure 3 for a schematic picture). Require in addition that all the boundary curves of S are Harnack curves. Then, compute an asymptotic expansion of $\text{Vol}(\Delta_R \cap \text{Log}(S))$.

Conjecture. From this asymptotic expansion of this increasing volume, one should be able to extract a constant term that determines Harnack property.

Optimistic Conjecture. Subtracting linear term in the asymptotic expansion yields the constant term, *i.e.*, there are no terms in \sqrt{R} , $\log(R)$, ... and one has

$$\text{Vol}(\Delta_R \cap \text{Log}(S)) = (2\pi^2 d^2)R + \text{const. terms} + \text{trailing terms}$$

(the linear coefficient comes from the condition that the boundary curves are Harnack).

Moreover, being Harnack determines the topology of $\mathbb{R}S$.

We give a generalization of Definition 1.6 in the form of a conjecture.

Brave Conjecture. In the surface case, Hopeful Definition 1.6 should be that 2Arg is a circle fibration over $S^1 \times S^1 \times S^1$.

3 Around anti-Harnack varieties

Hyperbolic hypersurfaces seem to be good candidates for anti-Harnack varieties.

Definition 3.1 A hypersurface of degree d in $\mathbb{R}P^n$ is hyperbolic if there exists a point p in $\mathbb{R}P^n$ such that all lines through p meet the hypersurface in d real points.

For example, hyperbolic curves of degree 5 and 6 are depicted in Figure 4.

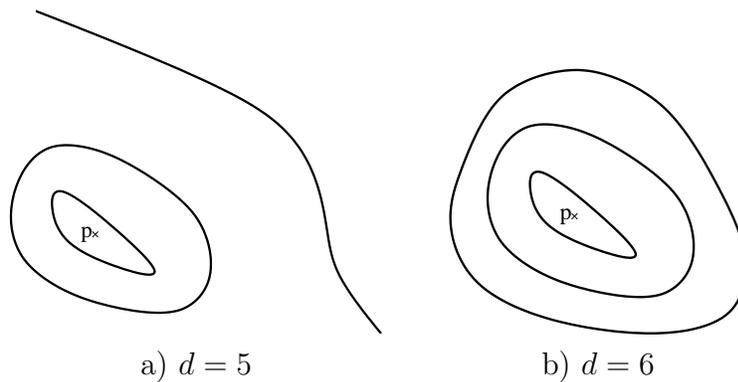


Figure 4:

Theorem 3.2 (Nuy) *The hyperbolic hypersurfaces of a given degree are rigidly isotopic.*

In particular, this implies that the topological type of a hyperbolic hypersurface is determined by its degree. Moreover, it comes out in the proof that the moduli space of such hypersurfaces is contractible.

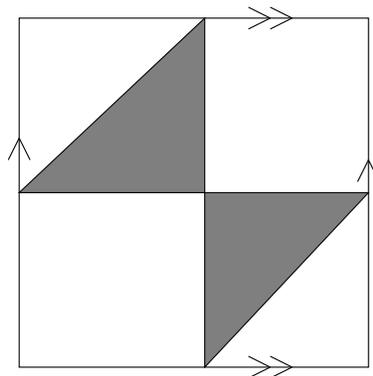


Figure 5:

The interior of an oval of a curve in $\mathbb{R}P^2$ is the orientable component of the complement of this oval in $\mathbb{R}P^2$. A *nest* of a real algebraic curve is a totally ordered (with respect to the inclusion) set of ovals of the curve. A nest is maximal, if it contains $\lfloor \frac{d}{2} \rfloor$ ovals, where d is the degree of the curve. It is easy to see that a real algebraic curve in $\mathbb{R}P^2$ is hyperbolic if and only if it contains a (unique) maximal nest.

Theorem 3.3 (Dubrovin-Vinnikov) *Let C be a hyperbolic curve in $\mathbb{R}P^2$. Then $C = \det(xA + yB + zC)$ where A, B and C are hermitian, and for x, y and z in the maximal nest, $xA + yB + zC$ is positive definite.*

Theorem 3.4 (Mikhalkin-Okounkov, work in progress) *A real algebraic curve in $\mathbb{R}P^2$ is hyperbolic and contains the positive quadrant inside its maximal nest if and only if its co-amoeba is a union of two triangles as depicted in Figure 5 (the rectangle represents $S^1 \times S^1$).*