Report by Roman Sauer on the discussion group about simplicial volume

The following two questions, or conjectures, were the main topic of our discussion.

Question (Gromov). *Let M be an oriented closed connected aspherical manifold whose simplicial volume is zero. Do all L^2-Betti numbers of M vanish then?*

Question. *Let $n \in \mathbb{N}$. Is there a constant $C_n > 0$ such that the L^2-Betti numbers of a closed, aspherical, smooth, n-dimensional manifold M are bounded by $C_n \operatorname{MinEnt}(M)^n$? Here $\operatorname{MinEnt}(M)$ is the minimal entropy of M.*

By work of Gromov, a positive answer to the first question implies a positive answer to the second question. We discussed a modified version of the simplicial volume (introduced by Gromov in another language [2, p. 305ff], studied further by Schmidt [4], and used in the context of minimal volume by Sauer [3]), the so-called *integral foliated simplicial volume* $\|M\|_{fol}$, which is defined in terms of a measure preserving action of the fundamental group of the manifold on a probability space X and $L^\infty(X, \mathbb{Z})$-valued cycles on the universal covering of a manifold M. More precisely, $\|M\|_{fol}$ is the infimum of the l^1-norms of cocycles representing the image of the fundamental class under the chain homomorphism

$$C_*(M) = \mathbb{Z} \otimes_{\mathbb{Z}\pi_1(M)} C_*(\widetilde{M}) \to L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\pi_1(M)} C_*(\widetilde{M})$$

The L^2-Betti numbers of M can be bounded from above by $\|M\|_{fol}$ [4] (see also [3]) but it is not yet clear how to relate $\|M\|_{fol}$ to the simplical volume of M. One should try to attack the minimal entropy question above directly using the foliated simplicial volume. This was successfully done for the corresponding question about the minimal volume and L^2-Betti numbers [3]. We also briefly discussed the role of the spherical volume of Besson-Courtois-Gallot [1] in this context.

References

[1] G. Besson, G. Courtois, and S. Gallot, *Volume et entropie minimale des espaces localement symétriques*, Invent. Math. **103** (1991), no. 2, 417–445 (French).

[2] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Reprint of the 2001 English edition, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2007. Based on the 1981 French original; With appendices by M. Katz, P. Pansu and S. Semmes; Translated from the French by Sean Michael Bates.

[3] Roman Sauer, *Amenable covers, volume, and L^2-Betti numbers of aspherical manifolds* (2007). to appear in Crelle's Journal.

[4] Marco Schmidt, *L^2-Betti numbers of $\mathcal{R}$-spaces and the Integral Foliated Simplicial Volume*, Universität Münster, 2005. doctoral thesis.