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Entropy of the geodesic flow for metric spaces and Bruhat–Tits buildings

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Abstract. Let (X, d_X) be a geodesically complete Hadamard space endowed with a Borel-measure μ. Assume that there exists a group Γ of isometries of X which acts totally discontinuously and cocompactly on X and preserves μ. We show that the topological entropy of the geodesic flow on the space of (parametrized) geodesics of the compact quotient $\Gamma\backslash X$ is equal to the volume entropy of μ (if X satisfies a certain local uniformity condition). This extends a result of Manning for riemannian manifolds of nonpositive curvature to the singular case. The result in particular holds for Bruhat–Tits buildings, for which we also compute the entropy explicitly.

Key words. topological entropy, geodesic flow, Hadamard spaces, Bruhat–Tits buildings, discrete subgroups of Lie groups, p-adic groups.

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1 Introduction

The *volume growth rate* of a closed riemannian manifold (M, g) with associated measure dv_g is defined as

$$h_{\mathrm{vol}}(dv_g) := \limsup_{r\to\infty} \frac{1}{r} \log \mathrm{Vol}(B(x, r)),$$

where $B(x, r)$ is the ball of radius r around a fixed point x in the universal covering space of M. Remarkably, this rather coarse asymptotic invariant carries a lot of geometric information. In fact, it is linked via inequalities to several important geometric quantities like the simplicial volume, the bottom of the spectrum of the laplacian, the Cheeger isoperimetric constant and the growth of the fundamental group (see [14], [7] and [24], respectively). We also mention the entropy rigidity conjecture, which, roughly, posits that if M carries a locally symmetric metric g_0, then $h_{\mathrm{vol}}(dv_{g_0})$ minimizes h_{vol} among all riemannian metrics on M with the same volume (see e.g. [2], [11]). The quantity most closely related to the volume growth rate, however, is the *topological entropy of the geodesic flow*. Given a continuous flow φ on a compact metric space the topological entropy of φ, $h_{\mathrm{top}}(\varphi)$, is an asymptotic invariant which measures the orbit complexity of the flow by a single number (see Section 2.3 for the definition). Positivity of the topological entropy indicates "chaotic" behaviour.

Dinaburg and Manning proved in [12] and [22] that the topological entropy of the geodesic flow φ on (the unit tangent bundle of) a compact riemannian manifold (M,g) is greater than or equal to the volume growth rate: $h_{\mathrm{top}}(\varphi) \geqslant h_{\mathrm{vol}}(dv_g)$. Moreover, equality holds if (M,g) has nonpositive sectional curvature. The volume growth rate $h_{\mathrm{vol}}(dv_g)$ is therefore also called *volume entropy* of (M,g). The equality result, $h_{\mathrm{top}}(\varphi) = h_{\mathrm{vol}}(dv_g)$, was extended in [13] to manifolds without conjugate points. Equality in particular holds for locally symmetric spaces of noncompact type. For these manifolds it is possible to compute the volume growth rate (and hence the topological entropy) explicitly: It equals $2\|\rho\|$, where ρ is the half sum of positive roots counted with multiplicity (see [18]).

The geodesic flow (and its entropy) can also be defined in the more singular context of metric spaces admitting compact quotients. In the present paper we are mainly interested in *Hadamard spaces*, i.e., complete, simply connected geodesic metric spaces of nonpositive curvature in the sense of Alexandrov (see Section 2 and also [1], [6] for this nowadays well established notion). Even more generally we shall consider metric measure spaces, (X,d_X,μ), where (X,d_X) is a geodesic space and μ a σ-additive Borel-measure on X.

Assumptions: Throughout this text we will assume that (X,d_X) is *geodesically complete*, i.e., every geodesic segment is the restriction of a geodesic line defined on all of $\mathbb{R}$, and that X is *locally uniquely geodesic* (see Section 2). In addition we will also need the following local uniformity and global convexity conditions.

Property (U): We say that a metric measure space (X,d_X,μ) has *property* (U) if there is $0 < \delta_0 \leqslant \infty$ such that for every $0 < \delta < \delta_0$ there are positive constants $C_i(\delta)$ $(i = 1,2)$ such that

$$0 < C_1(\delta) = \inf_{x \in X} \mu(B(x,\delta)) \leqslant \sup_{x \in X} \mu(B(x,\delta)) = C_2(\delta) < \infty,$$

where $B(x,r)$ is the ball of center x and radius r in X.

Property (C): We say that a geodesically complete metric space (X,d_X) has *property* (C) if for any two geodesics $c_1 : \mathbb{R} \to X$, $c_2 : \mathbb{R} \to X$ (parametrized by arc-length) the function $t \mapsto d_X(c_1(t),c_2(t))$ is convex.

Basic examples of spaces which satisfy property (C) are geodesically complete Hadamard spaces (see [1], I.5.4, [6], II.2.2). Note that (C) implies that X is *geodesically unique*: between any two points of X there is a unique geodesic segment.

The *geodesic flow* on a compact metric space is defined by reparametrization of arc-length parametrized geodesics (see Section 3.1). The main result of this paper is an extension of the Dinaburg–Manning result to metric measure spaces which have properties (C) and (U) and which admit compact quotients. The proof is essentially an adaption of Mannings's arguments. However in the singular case various technical details are not obvious. The main difficulty which occurs is the branching of geodesics: in contrast to the smooth case a geodesic is no longer determined by an initial point and an initial direction.

An important class of examples of singular spaces with properties (C) and (U) are the euclidean or affine buildings of Bruhat and Tits associated to semisimple algebraic groups defined over non-archimedean local fields. These metric spaces are locally finite poly-simplicial complexes which are often viewed as p-adic analogues of globally symmetric spaces. In fact, our results below support that philosophy. The existence of compact quotients is guaranteed by [4] at least in characteristic zero. For such buildings we also compute the entropy explicitely in terms of root data (for the definitions of the latter see Section 3; compare also [26]). The chief result of the present paper is the following

Main Theorem. (a) *Let (X, d_X, μ) be a geodesic metric space which is geodesically complete, locally uniquely geodesic and endowed with a Borel-measure μ. Assume that X has property (U) and that there exists a group Γ of isometries of X that acts totally discontinuously and cocompactly on X and preserves μ. Then the topological entropy of the geodesic flow φ on the space of geodesics of the compact quotient $\Gamma\backslash X$ is bounded from below by the volume entropy with respect to μ*:

$$h_{\text{top}}(\varphi) \geqslant h_{\text{vol}}(\mu).$$

(b) *Let (X, d_X, μ), Γ and φ be as in* (a) *and assume in addition that (X, d_X) has property (C), then equality holds*:

$$h_{\text{top}}(\varphi) = h_{\text{vol}}(\mu).$$

(c) *Let $(\mathscr{B}, d_{\mathscr{B}})$ be the Bruhat–Tits building associated to a connected, simply connected, semisimple linear algebraic group $\mathbf{G}$ defined over a non-archimedean local field $\mathbb{F}$ of $\mathbb{F}$-rank r. Let dv be the r-dimensional Hausdorff–Lebesgue measure associated to the metric $d_{\mathscr{B}}$ and denote by 2ρ the sum of the positive roots of $\mathbf{G}$ with respect to some Weyl chamber in a maximal $\mathbb{F}$-split torus. Let Γ be a discrete cocompact subgroup of the group $G = \mathbf{G}(\mathbb{F})$ of $\mathbb{F}$-rational points of $\mathbf{G}$. Finally let φ be the geodesic flow on the space of geodesics of the finite polyhedron $\Gamma\backslash\mathscr{B}$. Then*

$$h_{\text{top}}(\varphi) = h_{\text{vol}}(dv) = 2\|\rho\|.$$

We remark that in Part (c) of the above Main Theorem one has to normalize appropriately the logarithm used to define the entropy.

The plan of the paper is as follows. In Section 2 we prove Parts (a) and (b) of the Main Theorem. We closely follows Manning's arguments in [22] with appropriate modifications taking into account the singular geometry. In Section 3 we discuss some relevant facts about Bruhat–Tits theory and estimate the volume of balls in buildings (with respect to the Hausdorff measure induced by the building metric). These estimates rely on the structure theory of linear algebraic groups defined over local fields. As an application we then prove Part (c) of the Main Theorem. In Section 4 we discuss some connections with the growth of fundamental groups.

2 Geodesic flow and entropy for metric measure spaces

General references for this section are [1], [6], [10] and [17]. Let (X, d_X) be a complete metric space. A *geodesic segment* in X is a locally distance minimizing curve $c : I \to X$, of an intervall $I \subseteq \mathbb{R}$ into X, parametrized by arc-length. If $I = \mathbb{R}$, a geodesic segment is called a *geodesic* (*line*). The space X is *geodesically complete* if every geodesic segment is the restriction of a (arc-length parametrized) geodesic defined on $\mathbb{R}$ (see [10], 2.5). We say that X is *geodesic* if any two points $x, y \in X$ can be connected by a geodesic segment of length $d_X(x, y)$.

2.1 Volume entropy of metric measure spaces. In the following we will use the logarithm $\log_a$ with (arbitrary) basis $a > 1$ instead of the usual natural logarithm. This choice is justified by the formula in the Main Theorem (c) (and its proof) where we take $a = q$, the order of the (finite) residue class field of the local field $\mathbb{F}$. Classically, for manifolds, one takes $a = e$.

Proposition 1. *Let* (X, d_X, μ) *be a complete* (*noncompact*) *geodesic metric measure space which is locally uniquely geodesic and satisfies property* (U) *of Section* 1 *and assume that there exists a group* Γ *of isometries that acts totally discontinuously and cocompactly on* X *and preserves* μ. *For any* $x \in X$ *let* $B(x, r)$ *be the ball of center* x *and radius* r *in* X. *Then the limit*

$$h_{\mathrm{vol}}(\mu) := \lim_{r \to \infty} \frac{1}{r} \log_a \mu(B(x, r))$$

exists and is independent of x.

Proof. Since X is locally uniquely geodesic, $X \backslash \Gamma$ has positive injectivity radius (see [6], I.7.53). In view of property (U) and since Γ is cocompact, the proof is then word-for-word the same as Manning's proof in the case of manifolds (see [22]).

The limit $h_{\mathrm{vol}}(\mu)$ is called the *volume entropy of the metric measure space* (X, d_X, μ) (or of any of its compact quotients $\Gamma \backslash X$).

2.2 The geodesic flow for metric spaces and topological entropy. Let (X, d_X) be a geodesic metric space which is geodesically complete and locally uniquely geodesic. Let Γ be a group which acts totally discontinuously and isometrically on X with compact quotient $\Gamma \backslash X$. Let d be the metric on $\Gamma \backslash X$ induced by d_X. Since X is locally isometric to $\Gamma \backslash X$ the latter is also geodesically complete. By $\mathscr{G}(X)$ and $\mathscr{G}(\Gamma \backslash X)$ we denote the set of all geodesics (defined on $\mathbb{R}$) of X and of the compact quotient $\Gamma\backslash X$, respectively. We then define the *geodesic flow* $\tilde{\varphi} := \{\tilde{\varphi}_s \mid s \in \mathbb{R}\}$ on $\mathscr{G}(X)$ by reparametrization, i.e. by,

$$\tilde{\varphi}_s : \mathscr{G}(X) \to \mathscr{G}(X); \quad c \mapsto \tilde{\varphi}_s(c) \quad (s \in \mathbb{R})$$

where the parametrized geodesic $\tilde{\varphi}_s(c) : \mathbb{R} \to X$ is given by $\tilde{\varphi}_s(c)(t) := c(s+t)$. A geodesic in $\Gamma\backslash X$ is the image of a geodesic in X under the canonical projection $\pi : X \to \Gamma\backslash X$ and the geodesic flow on $\Gamma\backslash X$ is given by $\varphi = \pi \circ \tilde{\varphi}$. On the set $\mathscr{G}(\Gamma\backslash X)$ we define a metric $d_{\mathscr{G}}$ by

$$d_{\mathscr{G}}(c_1, c_2) := \int_{-\infty}^{\infty} d(c_1(t), c_2(t)) \frac{e^{-|t|}}{2} dt$$

(compare [15], 8.3 B).

The space of geodesics of a compact manifold is compact. This also holds in the present situation.

Lemma 1. *The metric space $(\mathscr{G}(\Gamma\backslash X), d_{\mathscr{G}})$ is compact.*

Proof. Let $c_1, c_2 \in \mathscr{G}(\Gamma\backslash X)$. A simple computation using twice the triangle inequality and the definition of $d_{\mathscr{G}}$ yields for any $T \geqslant 0$:

$$d_{\mathscr{G}}(c_1, c_2) \leqslant e^{-T} + \sup_{t \in [-T, T]} d(c_1(t), c_2(t)).$$

The claim then follows from Arzelà–Ascoli (see [6]).

The *topological entropy* h_{top} is defined for a continuous flow on a compact metric space (see [17] and, for the geodesic flow on manifolds, also [25]). We are going to define that notion for the geodesic flow φ on the metric space $(\mathscr{G}(\Gamma\backslash X), d_{\mathscr{G}})$ (which is compact by Lemma 1).

For any real number $T > 0$ define a new metric $d_{\mathscr{G},T}$ on $\mathscr{G}(\Gamma\backslash X)$ by

$$d_{\mathscr{G},T}(c_1, c_2) := \max_{0 \leqslant s \leqslant T} d_{\mathscr{G}}(\varphi_s(c_1), \varphi_s(c_2)).$$

A subset $A \subset \mathscr{G}(\Gamma\backslash X)$ is called (r, δ)-*separated* if for any two different points $c, c' \in A$ one has $d_{\mathscr{G},r}(c, c') \geqslant \delta$. Let $\max(r, \delta)$ be the *maximal* cardinality of an (r, δ)-separated subset of $\mathscr{G}(\Gamma\backslash X)$. Then the toplogical entropy of the geodesic flow φ is the following limit:

$$h_{\text{top}}(\varphi) := \lim_{\delta \to 0} \limsup_{r \to \infty} \frac{1}{r} \log_a \max(r, \delta).$$

Equivalently, $h_{\text{top}}(\varphi)$ can also be defined as follows (see [17], 3.1). A subset $B \subset \mathscr{G}(\Gamma\backslash X)$ is called (r, δ)-*spanning* if for all $b \in \mathscr{G}(\Gamma\backslash X)$ there is $b' \in B$ such that

$d_{\mathscr{G},r}(b,b') \leqslant \delta$. Let $\min(r,\delta)$ be the *minimal* cardinality of an (r,δ)-spanning subset of $\mathscr{G}(\Gamma\backslash X)$. Then the toplogical entropy of the geodesic flow φ can also be obtained as

$$h_{\mathrm{top}}(\varphi) = \lim_{\delta\to 0} \limsup_{r\to\infty} \frac{1}{r} \log_a \min(r,\delta).$$

2.3 Proof of the Main Theorem, Part (a) and (b). *Proof of Part* (a): We wish to estimate the cardinality of certain separated sets. To that end we pick $\varepsilon > 0$ and a basepoint $x \in X$. By Proposition 1 there exists $r_0 = r_0(\varepsilon)$ such that for (large) $r > r_0$ we have $\mu(B(x,r)) \geqslant a^{(h_{\mathrm{vol}}-\varepsilon)r}$. Since X is locally uniquely geodesic the injectivity radius $\mathrm{Inj}(\Gamma\backslash X)$ of the compact space $\Gamma\backslash X$ is positive (see [6], I.7.53, I.7.55). Take $0 < \delta < \mathrm{Inj}(\Gamma\backslash X)$ and assume also that $2\delta < \delta_0$ of property (U). The above measure estimate then implies that there exists an increasing and divergent sequence of radii $(r_k)_{k\in\mathbb{N}}$ such that

$$\mu(B(x, r_k + \delta/2) - B(x, r_k)) \geqslant a^{(h_{\mathrm{vol}}-\varepsilon)r_k}.$$

Let Y_{r_k} be a maximal subset of the shell $B(x, r_k+\delta/2) - B(x, r_k) \subset X$ whose points are pairwise 2δ apart. Next note that by property (U) we have

$$\sup_{p\in X} \mu(B(p,2\delta)) = C_2(2\delta).$$

Hence the cardinality of Y_{r_k} satisfies

$$|Y_{r_k}| \geqslant C_2^{-1} a^{(h_{\mathrm{vol}}-\varepsilon)r_k}. \tag{1}$$

We now set $\delta^* := \frac{1}{4}\delta(1 - e^{-\delta/4})$ (note that δ^* is a strictly monotone function of δ). We fix some r_k and wish to construct an (r_k, δ^*) separated set in $\mathscr{G}(\Gamma\backslash X)$. For simplicity we write r for r_k. By assumption X is geodesic, i.e., any two points in X can be joined by a minimal geodesic segment. For each point in Y_{r_k} we can therefore choose a geodesic segment joining it to x. Since X is assumed to be geodesically complete there is, for each such segment, a complete geodesic line, i.e., an element of $\mathscr{G}(X)$, extending it. Let $\tilde{A}(r) \subset \mathscr{G}(X)$ be the set of these geodesics and, for the projection $p : \mathscr{G}(X) \to \mathscr{G}(\Gamma\backslash X)$, set $A(r) := p(\tilde{A}(r))$. We want to show that $A(r)$ is (r,δ^*)-separated. Pick $\tilde{c}_{xq}, \tilde{c}_{xq'} \in \tilde{A}(r)$. We then have

$$\begin{aligned} d_X(\tilde{c}_{xq}(r), \tilde{c}_{xq'}(r)) &\geqslant d_X((q,q') - d_X(\tilde{c}_{xq}(r), q) - d_X(\tilde{c}_{xq'}(r), q') \\ &> 2\delta - \delta/2 - \delta/2 = \delta. \end{aligned} \tag{2}$$

Next consider the two geodesics $c := p \circ \tilde{c}_{xq}$ and $c' := p \circ \tilde{c}_{xq'}$ in $\mathscr{G}(\Gamma\backslash X)$. As $\delta < \mathrm{Inj}(\Gamma\backslash X)$, the estimate (2) implies that there is $s \in [0,r]$ such that $d(c(s), c'(s)) > \delta$. But then

$$\begin{aligned} d_{\mathscr{G}}(\varphi_s(c), \varphi_s(c')) &= \int_{-\infty}^{\infty} d(c(s+t), c'(s+t)) \frac{1}{2} e^{-|t|}\, dt \\ &\geqslant \int_0^{\delta/4} [d(c(s), c'(s)) - 2t] \frac{1}{2} e^{-t}\, dt \\ &> \int_0^{\delta/4} \frac{1}{4} \delta e^{-t}\, dt = \frac{1}{4}\delta(1 - e^{-\delta/4}) = \delta^*. \end{aligned}$$

This shows that $d_{\mathscr{G},r}(c,c') := \max_{0 \leqslant t \leqslant r} d_{\mathscr{G}}(\varphi_t(c), \varphi_t(c')) > \delta^*$, i.e., that $A(r) \subset \mathscr{G}(\Gamma\backslash X)$ is an (r, δ^*)-separated set for the geodesic flow on the compact metric space $(\mathscr{G}(\Gamma\backslash X), d_{\mathscr{G}})$.

By (2) the cardinality of $A(r)$ is equal to the cardinality of $\tilde{A}(r)$. The estimate (1) then yields that the maximal cardinality of an (r_k, δ^*)-separated subset is bounded from below by $C_2(2\delta)^{-1} a^{(h_{\mathrm{vol}} - \varepsilon) r_k}$ (recall that we set $r = r_k$). Hence, the definition of h_{top} yields $h_{\mathrm{top}} \geqslant h_{\mathrm{vol}} - \varepsilon$. Finally, as ε was arbitrarily chosen, we obtain $h_{\mathrm{top}}(\varphi) \geqslant h_{\mathrm{vol}}(\mu)$, which proves (a).

Proof of Part (b): In view of (a) we need to prove the converse inequality $h_{\mathrm{top}}(\varphi) \leqslant h_{\mathrm{vol}}(\mu)$. We again fix a basepoint $x \in X$ and consider a compact fundamental domain $\mathscr{F}$ of Γ in X which contains x. Let D be the diameter of $\mathscr{F}$ and as before let $\delta < \mathrm{Inj}(\Gamma\backslash X)$ and also $\delta < \delta_0$ of property (U). We also fix some integer $n \geqslant 3$ and set $r_1 := -n \ln \delta$. Proposition 1 asserts that for given $\varepsilon > 0$ there is $r_2 = r_2(\varepsilon)$ such that for all $r > r_2$ we have

$$\mu(B(x,r)) \leqslant a^{(h_{\mathrm{vol}} + \varepsilon) r}. \tag{3}$$

Let $Z(r)$ be a maximal δ-separated subset of $S(r) := \{z \in X \mid r - D \leqslant d_X(z, \mathscr{F}) \leqslant r\}$ (i.e., for any u, v with $u \neq v$ holds $d_X(u,v) > \delta$). By property (U) and (3) there is a constant $0 < C_1(\delta/2)$ such that for all (large) $r \geqslant r_0 := \max\{r_1, r_2\}$ the following estimate for the cardinality of $Z(r)$ holds

$$|Z(r)| \leqslant C_1^{-1} \mu(B(x, r + D + \delta/2)) \leqslant C_1^{-1} a^{(h_{\mathrm{vol}} + \varepsilon)(r + D + \delta/2)}. \tag{4}$$

Let F be a maximal δ-separated subset of the fundamental domain $\mathscr{F}$. Since X is geodesic, property (C) implies that between any $y \in F$ and $z \in Z(r)$ there exists a unique geodesic segment (parametrized by arc-length) of length between $r - D$ and $r + D$. Since X is assumed to be geodesically complete, there exists a (possibly non-unique) extension of such a segment to a geodesic line. For each $y \in F$ and $z \in Z(r)$ we can thus choose a geodesic line $\tilde{c}_{yz} : \mathbb{R} \to X$ with $y = \tilde{c}_{yz}\left(-\frac{r}{n}\right)$. Set $\tilde{B}(r) := \{\tilde{c}_{yz} \mid y \in F, z \in Z(r)\}$ and let $B(r) := p(\tilde{B}(r))$ where $p : \mathscr{G}(X) \to \mathscr{G}(\Gamma\backslash X)$ is the canonical projection mapping geodesics in X to geodesics in $\Gamma\backslash X$. We want to show that $B(r)$ is a $\left(r\left(1 - \frac{2}{n}\right), 6\delta\right)$-spanning set for the geodesic flow on $\mathscr{G}(\Gamma\backslash X)$.

To that end consider a geodesic segment in $\Gamma\backslash X$ of length r and choose a segment from $u \in \mathscr{F}$ to v in X which covers it (compare [10], Lemma 3.4.17). Then there are points $y \in F$ and $z \in Z(r)$ with $d_X(u, y) \leqslant \delta$ and $d_X(v, z) \leqslant \delta$. We also have $r - 2\delta \leqslant d_X(y, z) \leqslant r + 2\delta$. Let $\tilde{c}_1$ be a geodesic in X which extends the segment between u and

v and set $\tilde{c}_2 := \tilde{c}_{yz} \in \tilde{B}(r)$ as above, such that $u = \tilde{c}_1\left(-\frac{r}{n}\right)$ and $y = \tilde{c}_2\left(-\frac{r}{n}\right)$. Let $z' := \tilde{c}_2\left(r - \frac{r}{n}\right)$ be the point at distance r from y along the segment of $\tilde{c}_2$ from y to z. By assumption (property (C)) the function $t \mapsto d_X(\tilde{c}_1(t), \tilde{c}_2(t))$ is convex. Thus we have for all $t \in \left[-\frac{r}{n}, r - \frac{r}{n}\right]$,

$$\begin{aligned} d_X(\tilde{c}_1(t), \tilde{c}_2(t)) &\leqslant d_X\left(\tilde{c}_1\left(-\frac{r}{n}\right), \tilde{c}_2\left(-\frac{r}{n}\right)\right) + d_X\left(\tilde{c}_1\left(r - \frac{r}{n}\right), \tilde{c}_2\left(r - \frac{r}{n}\right)\right) \\ &= d_X(u, y) + d_X(v, z'). \end{aligned}$$

Since

$$d_X(v, z') \leqslant d_X(v, z) + d_X(z, z') \leqslant \delta + 2\delta = 3\delta,$$

we find that the projections $c_i = p \circ \tilde{c}_i$ $(i = 1, 2)$ in $\Gamma \backslash X$ satisfy

$$d(c_1(t), c_2(t)) \leqslant d_X(\tilde{c}_1(t), \tilde{c}_2(t)) \leqslant 4\delta \quad \text{for all } t \in \left[-\frac{r}{n}, r - \frac{r}{n}\right]. \tag{5}$$

We next estimate $d_{\mathscr{G}}(\varphi_s(c_1), \varphi_s(c_2))$. Using (5) and—for arbitrary t_1, t_2—the triangle inequality

$$d(c_1(t_1), c_2(t_1)) \leqslant d(c_1(t_2), c_2(t_2)) + 2|t_2 - t_1|,$$

we compute for $s \in \left[0, r - \frac{2}{n} r\right]$:

$$\begin{aligned} &d_{\mathscr{G}}(\varphi_s(c_1), \varphi_s(c_2)) \\ &\quad = \int_{-\infty}^{\infty} d(c_1(s+t), c_2(s+t)) \frac{1}{2} e^{-|t|}\, dt \\ &\quad \leqslant \int_{-\infty}^{-r/n-s} \left[d\left(c_1\left(-\frac{r}{n}\right), c_2\left(-\frac{r}{n}\right)\right) + 2\left| t + s + \frac{r}{n} \right| \right] \frac{1}{2} e^{-|t|}\, dt \\ &\qquad + \int_{-r/n-s}^{0} 4\delta \frac{1}{2} e^{-|t|}\, dt + \int_{0}^{r-r/n-s} 4\delta \frac{1}{2} e^{-t}\, dt \\ &\qquad + \int_{r-r/n-s}^{\infty} \left[d\left(c_1\left(r - \frac{r}{n}\right), c_2\left(r - \frac{r}{n}\right)\right) + 2\left| s + t - r + \frac{r}{n} \right| \right] \frac{1}{2} e^{-t}\, dt \\ &\quad \leqslant 2\delta \int_{r/n+s}^{\infty} e^{-t}\, dt + \int_{r/n+s}^{\infty} \left(t - s - \frac{r}{n}\right) e^{-t}\, dt + 2\delta \int_{0}^{r/n+s} e^{-t}\, dt \\ &\qquad + 2\delta \int_{0}^{r-r/n-s} e^{-t}\, dt + 2\delta \int_{r-r/n-s}^{\infty} e^{-t}\, dt + \int_{r-r/n-s}^{\infty} \left(t + s - r + \frac{r}{n}\right) e^{-t}\, dt \\ &\quad = 4\delta + e^{-r/n-s} + e^{-r+r/n+s} \leqslant 4\delta + 2e^{-r/n} \leqslant 6\delta, \end{aligned}$$

where, for the last two inequalities, we used that $0 \leqslant s \leqslant r - \frac{2}{n} r$ and that $r \geqslant r_0(\delta, n) = -n \ln \delta$. In particular we have

$$d_{\mathscr{G}, r(1-2/n)}(c_1, c_2) = \sup_{s \in [0, r-(2/n)r]} d_{\mathscr{G}}(\varphi_s(c_1), \varphi_s(c_2)) \leqslant 6\delta.$$

Since $c_2 = p(\tilde{c}_2) \in B(r)$, we have shown that $B(r)$ is a $\left(r\left(1 - \frac{2}{n}\right), 6\delta\right)$-spanning set in $\mathscr{G}(\Gamma \backslash \mathscr{B})$. The estimate (4) now yields

$$\min\left(r\left(1 - \frac{2}{n}\right), 6\delta\right) \leqslant |B(r)| \leqslant |\tilde{B}(r)| \leqslant |Z(r)| \leqslant C_1(\delta)^{-1} a^{(h_{\mathrm{vol}}+\varepsilon)(r+D+\delta/2)}.$$

Hence we get from the (second) definition of h_{top} that

$$h_{\mathrm{top}}(\varphi) \leqslant \left(1 - \frac{2}{n}\right)^{-1} (h_{\mathrm{vol}} + \varepsilon).$$

Finally we use that ε and the integer $n \geqslant 3$ are arbitrary and get $h_{\mathrm{top}}(\varphi) \leqslant h_{\mathrm{vol}}(\mu)$. Together with Part (a) this completes the proof of Part (b) of the Main Theorem.

3 Basic examples: Bruhat–Tits buildings

The most important examples of metric spaces for which the results of Section 2 hold are Bruhat–Tits buildings. The purpose of the present section is to review the relevant details in order to verify that these buildings in fact have all the required properties. We also compute the volume entropy explicitly.

3.1 Bruhat–Tits theory. General references are the paper of Bruhat–Tits [9] and the books of Borel [3], Brown [8], Macdonald [21], Margulis [23] and Ronan [27].

Let $\mathbb{F}$ be a complete, locally compact, non-archimedean local field, i.e., a finite extension either of the the p-adic numbers $\mathbb{Q}_p$ or of a formal power series field over a finite field ([30], Theorems I.5, I.8). Let $v : \mathbb{F}^\times \to \mathbb{Z}$ be the discrete valuation of $\mathbb{F}^\times$, where $\mathbb{F}^\times$ is the multiplicative group of non-zero elements of $\mathbb{F}$. Let $\mathscr{O} = \{\xi \in \mathbb{F} \mid v(\xi) \geqslant 0\}$ be the ring of integers. Then $\mathscr{P} = \{\xi \in \mathbb{F} \mid v(\xi) \geqslant 1\}$ is a maximal ideal in $\mathscr{O}$, and since both $\mathscr{O}$ and $\mathscr{P}$ are open and compact, the quotient $\mathscr{O}/\mathscr{P}$ is a finite field, the residue class field of $\mathbb{F}$. Let q denote its cardinality, $q := |\mathscr{O}/\mathscr{P}|$. The ultrametric absolute value of $\xi \in \mathbb{F}$ is $|\xi| := q^{-v(\xi)}$.

Let $\mathbf{G}$ be a connected, simply connected, semisimple linear algebraic group defined over $\mathbb{F}$ and let $G = \mathbf{G}(\mathbb{F})$ be the group of $\mathbb{F}$-rational points of $\mathbf{G}$. Then G is a locally compact group. Let $\mathbf{S} \subset \mathbf{G}$ be a maximal $\mathbb{F}$-split torus, i.e., an algebraic subgroup which is $\mathbb{F}$-isomorphic to $(\mathbb{F}^\times)^r$ for some $r \in \mathbb{N}$. Any two such tori are conjugate and r is called the $\mathbb{F}$-*rank* of $\mathbf{G}$ (or G). We will always assume that $\mathbb{F}$-rank $\geqslant 1$. Write $\mathbf{N}$ and $\mathbf{Z}$ respectively for the normalizer and centralizer of $\mathbf{S}$ in $\mathbf{G}$, and set $S := \mathbf{S}(\mathbb{F})$, $N := \mathbf{N}(\mathbb{F})$, $Z := \mathbf{Z}(\mathbb{F})$ for the respective groups of $\mathbb{F}$-rational points. Finally define $H := \{z \in Z \mid |\chi(z)| = 1 \text{ for all characters } \chi \text{ of } Z\}$.

Let $X(\mathbf{S}) = \mathrm{Hom}_{\mathbb{F}}(\mathbf{S}, \mathbb{F}^{\times})$ be the set of characters of $\mathbf{S}$. To a (multiplicative) root $\tilde{\alpha} \in X(\mathbf{S})$ of the pair $(\mathbf{G}, \mathbf{S})$ (see [23], 0.27) is associated a unique (additive) root α defined by $\alpha(s) := \log_q|\tilde{\alpha}(s)| \in \mathbb{Z}$ $(s \in S)$. Any such α is an element of the real vector space $\mathfrak{a}^* := X(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ of dimension $r := \mathrm{rank}_{\mathbb{F}} \mathbf{G}$, and thus defines a linear form on the dual $\mathbb{R}$-vector space $\mathfrak{a}$. There is a unique continuous homomorphism $v : Z \to \mathfrak{a}$ which satisfies $\alpha(v(s)) = -v(\tilde{\alpha}(s))$, for each $s \in S$ and $\tilde{\alpha} \in X(\mathbf{S})$. Moreover, one has $\mathrm{Ker}\, v = H$ and hence $v(Z) \cong Z/H$ (see [29], 1.2).

The group $W := N/H$ is the *affine* Weyl group associated to a reduced root system Σ of rank $r = \mathrm{rank}_{\mathbb{F}}\, G$ (see [5], Ch. 6.2 for the definitions). The corresponding Weyl group W_0 is the Weyl group of the pair $(\mathbf{S}, \mathbf{G})$, i.e., $W_0 \cong N/Z$. However, in general, Σ is different from the relative root system $\Phi = \Phi(\mathbf{S}, \mathbf{G})$. Nevertheless every root of Σ is proportional to some root of Φ and vice versa (see [29], 1.7). We now endow $\mathfrak{a}$ with a W_0-invariant inner product $\langle\, , \rangle$. Then the affine Weyl group W acts as an affine Coxeter group, i.e., a group of isometries of the euclidean space $\mathfrak{a}$ generated by reflections in affine hyperplanes belonging to a W-invariant, locally finite set $\mathscr{H}$ of affine hyperplanes. The complex obtained by the partition of the space by the elements of $\mathscr{H}$ is an *affine Coxeter complex*. Its maximal cells are called *chambers*. Every codimension 1 cell is contained in exactly two chambers.

In a group G as above there further exists a so-called *Iwahori subgroup* B for which $B \cap N = H$ and such that (B, N) is a BN-pair (or Tits system) for G (see e.g. [8] or [21]). Associated to such a pair is an *affine Bruhat–Tits building* $\mathscr{B}$ (see [9]). This is a locally finite polysimplicial complex given as the product of the Bruhat–Tits buildings associated to the almost $\mathbb{F}$-simple factors of G. If one fixes a basic chamber $\mathscr{C} \subset \mathscr{B}$ then one can choose B as the stabilizer of $\mathscr{C}$ in G.

We recall some basic properties of such buildings (see e.g. [6], [8], [9] and [21] for details).

(a) The building $\mathscr{B}$ has a collection of subcomplexes called *apartments*. Every apartment is an affine Coxeter complex of dimension $r = \mathrm{rank}_{\mathbb{F}}\, G$ whose chambers are also maximal cells of $\mathscr{B}$. The building is thick, i.e., every codimension 1 face is contained in at least three chambers.

(b) One can identify a basic apartment $\mathscr{A} \subset \mathscr{B}$ with the euclidean space $(\mathfrak{a}, \langle\, , \rangle)$ ([21], 2.4.1.) and thus obtains a well-defined structure of euclidean space on each apartment. The euclidean distance $d_{\mathscr{A}}(x, y)$ defined for $x, y \in \mathscr{A}$ can be extended to a complete metric $d_{\mathscr{B}}$ on $\mathscr{B}$ ([21], 2.4.8). Moreover $(\mathscr{B}, d_{\mathscr{B}})$ is a geodesically complete Hadamard space (see [6], II.5.10, II.10.A.4 and [8], VI.3). In particular such a building $\mathscr{B}$ has the convexity property (C) defined in Section 1.

(c) The group G acts isometrically and strongly transitively on $\mathscr{B}$ (see [8], V.3.) The stabilizers of chambers are conjugates of B and stabilizers of vertices are maximal compact and open subgroups of G (see [9]).

Under the identification of the vector space $\mathfrak{a}$ with the apartment $\mathscr{A}$ the origin $0 \in \mathfrak{a}$ corresponds to a base point $x_0 \in \mathscr{A}$ (which is also a vertex of $\mathscr{B}$). Let K be the maximal parahoric subgroup $K := \mathrm{stab}(x_0)$ of G. Then K is open and maximal compact and can be written as $K = BW_0B$ ([21], 2.6.8).

For each root $\alpha \in \mathfrak{a}^*$ there is a unique $\vec{\alpha} \in \mathfrak{a}$ such that $\alpha(v(s)) = \langle \vec{\alpha}, v(s) \rangle$ $(s \in S)$. We also set $\alpha^\vee := \frac{2\vec{\alpha}}{\|\vec{\alpha}\|^2}$. The affine Weyl group W is a semidirect product $W = W_0 T$ where T is the abelian group generated by $\{\alpha^\vee \mid \alpha \in \Sigma\}$ (see [5], VI, 2.1). With our identifications $\mathfrak{a} \cong \mathscr{A}$ and $0 \equiv x_0$ we have—by some abuse of notation—for all $t \in T$, $t \equiv 0 + t \equiv t \cdot x_0$. Furthermore, the kernel of the map $v : Z \to \mathfrak{a}$ is H and hence $v(Z) \cong Z/H \cong T$.

Fix a basic Weyl chamber $\mathscr{C}_0 \cong \mathfrak{a}^+ \subset \mathfrak{a}$. Let T^+ consist of those $t \in T$ for which $t \cdot x_0$ is in the Weyl chamber $\mathscr{C}_0$. Put $Z^+ := v^{-1}(T^+)$. Note that for $z_1, z_2 \in Z$ with $v(z_1) = v(z_2)$ we have $z_1 z_2^{-1} \in H$ and therefore $Kz_1K = Kz_2K$. Thus, for any $z \in Z$ the double coset KzK depends only on $t = v(z)$ and we denote it by $Kv^{-1}(t)K$. With that notation the following proposition is the p-adic version of a *Cartan decomposition* of G (see [9] and also [21], 2.6.11).

Proposition 2. *For subgroups Z^+ and K of the semisimple group G defined as above, there is a Cartan decomposition $G = KZ^+K$. Moreover there is a one-to-one correspondence from T^+ onto the set of double cosets $K\backslash G/K$ given by $t \mapsto Kv^{-1}(t)K$.*

3.2 Measures on buildings. Let ω be the left-invariant Haar measure on the locally compact group G normalized such that $\omega(K) = 1$. Let dv be the r-dimensional Hausdorff–Lebesgue measure, $r = \operatorname{rank}_{\mathbb{F}} G$, associated to the building metric $d_{\mathscr{B}}$. The dv-measure (or volume) of a dv-measurable subset $A \subset \mathscr{B}$ will be denoted by $V(A)$. We also denote by $B(x,r)$ the ball of radius $r > 0$ and center x in $(\mathscr{B}, d_{\mathscr{B}})$. We have the following elementary estimate for the measures of balls:

Lemma 2. *Let x_0 be the base point of $\mathscr{B}$ (with* $\operatorname{stab}(x_0) = K$*) and $\mathscr{C} \subset A$ a chamber with vertex x_0. Set $D := \operatorname{Diam}(\mathrm{K} \cdot \mathscr{C})$, the diameter of the compact set $K \cdot \mathscr{C} \subset \mathscr{B}$ of the finitely many chambers in $\mathscr{B}$ with common vertex x_0. Then one has for all $r > 0$*

$$V(B(x_0, r)) \leqslant V(K \cdot \mathscr{C})\omega(B(x_0, r) \cap G \cdot x_0) \leqslant V(B(x_0, r + D)).$$

The following lemma asserts that buildings have property (U).

Lemma 3. *Let $\mathscr{C} \subset \mathscr{B}$ be a chamber of the building $\mathscr{B}$. Assume that $\delta < \frac{1}{100} \operatorname{Diam}(\mathscr{C})$. Then there are positive constants $C_i = C_i(\delta)$ $(i = 1, 2)$ such that*

$$0 < C_1 = \inf_{x \in \mathscr{B}} V(B(x, \delta)) \leqslant \sup_{x \in \mathscr{B}} V(B(x, \delta)) = C_2.$$

Proof. Since G acts strongly transitively on $\mathscr{B}$ and preserves the measure dv we can assume that $x \in \overline{\mathscr{C}}$. The size of δ guarantees that there is at least one wall, say E, of $\mathscr{C}$ with $d_{\mathscr{B}}(x, E) > \delta$. Let e be the vertex of the simplex $\mathscr{C}$ opposite to E. Then let $\mathscr{C}'$ be the affine translate of $\mathscr{C}$ (in the vector space $\mathfrak{a}$) from e to x. By construction the intersection of the simplex $\mathscr{C}'$ with the ball $B(x, \delta)$ is completely contained in $\mathscr{C} \cap B(x, \delta)$. Since $\mathscr{C}$ has only finitely many vertices the volume of that intersection is uniformly bounded from below. This argument also yields an upper bound since the building is locally finite (there is $m \in \mathbb{N}$ such that any $x \in \mathscr{B}$ is contained in at most m chambers).

3.3 The volume entropy of buildings. The goal of this section is to compute the volume entropy for Bruhat–Tits buildings. It turns out that the value for the volume entropy (see Lemma 4 below) is completely analogous to the case of symmetric spaces mentioned in the introduction (compare also [26]).

Recall from Section 3.1 that the translational part T of the affine Weyl group W is isomorphic to the lattice in $\mathfrak{a}$ spanned by $\{\alpha^\vee \mid \alpha \in \Sigma\}$ and that $T \cong T \cdot x_0 \subset \mathscr{A}$. Let $2\rho = \sum_{\alpha \in \Phi^+(\mathbf{S},\mathbf{G})} m_\alpha \alpha$ be the sum of the positive roots (with multiplicities) of $\mathbf{G}$ with respect to some Weyl chamber in a maximal $\mathbb{F}$-split torus $\mathbf{S}$. Then, again by 3.2, there is $\tau > 0$ such that $x_1 := \tau\rho^\vee \in T$. Note also that the inner product on $\mathfrak{a}$ defines an inner product on $\mathfrak{a}^*$ and we have $\langle \alpha, \beta \rangle = \langle \vec{\alpha}, \vec{\beta} \rangle$ for all roots α, β.

Lemma 4. *Let $\mathbb{F}$ be a non-archimedean local field whose residue class field has finite order q. Let $\mathbf{G}$ be a connected, simply connected, semisimple linear algebraic group defined over $\mathbb{F}$ with associated Bruhat–Tits building $\mathscr{B}$. Let K be the stabilizer of the base point $x_0 \in \mathscr{B}$. The Haar measure on G (normalized such that K has measure one) induces a left-invariant measure ω on G/K. For $k \in \mathbb{N}$, set $r_k := 2k\tau\|\rho\|^{-1}$. Then*

$$\lim_{r_k \to \infty} \frac{1}{r_k} \log_q \omega(B(x_0, r_k) \cap G \cdot x_0) = 2\|\rho\|.$$

Proof. For $t = v(z) \in v(Z^+)$ the point $x := t \cdot x_0$ is in the apartment $\mathscr{A}$. We want to determine the measure of the K-orbit of x, i.e., $\omega(K \cdot x) = |K \cdot x|$ (recall that $\omega(K) = 1$). To that end we will use the Cartan decomposition of G (see Proposition 2). Set $L := t^{-1}Kt \cap K$; then the map $K/L \to Kt^{-1}K/K$ given by $hL \mapsto ht^{-1}K$ is a bijection and hence we have

$$K \cdot x \cong K/(t^{-1}Kt \cap K) \cong Kt^{-1}K/K.$$

Next by [21], 2.3.5, we have

$$KtK = BW_0tW_0B = K(wtw^{-1})K$$

for any $w \in W_0$. Thus, for any $t \in T$ and $w \in W_0$, we get for the index of K in KtK:

$$|KtK : K| = |Kwtw^{-1}K : K|.$$

For $t \in T^+$ that index has been computed in [21], 3.2.15. There is a uniformly bounded rational function $R_t(q)$, i.e., there are constants $0 < C_3 < C_4$ such that $C_3 \leqslant R_t(q) \leqslant C_4$ for all $t \in T$, such that $|KtK : K| = R_t(q)\Delta(t)$. Here $\Delta(t)$ denotes the modular function associated to the minimal parabolic subgroup of G which corresponds to the positive roots $\Phi^+ = \Phi^+(\mathbf{S}, \mathbf{G})$. An explicit formula for $\Delta(t)$ is also determined in [28], Lemma 1.2.1.1 as $\Delta(t) = q^{\langle 2\vec{\rho}, v(z) \rangle}$, with $2\rho := \sum_{\alpha \in \Phi^+} m_\alpha \alpha$. Using these facts, we obtain, for $t \in v(Z^+)$ and w_0 the element of maximal length in W_0, that

$$\begin{aligned}|Kt^{-1}K : K| &= |Kw_0t^{-1}w_0K : K| = R_t(q)\Delta(w_0t^{-1}w_0^{-1})) \\ &= R_t(q)\Delta(w_0tw_0^{-1})^{-1} = R_t(q)\Delta(t) = R_t(q)q^{\langle 2\vec{\rho}, v(z)\rangle}.\end{aligned}$$

Summarizing we get from the above that the cardinality of the K-orbit of $x = t \cdot x_0$ for $t = v(z) \in v(Z^+)$ can be estimated by

$$C_3 q^{\langle 2\vec{\rho}, v(z)\rangle} \leqslant |K \cdot x| = |K/(t^{-1}Kt \cap K)| \leqslant C_4 q^{\langle 2\vec{\rho}, v(z)\rangle}, \tag{6}$$

Recall that the map $T = v(Z) \to \mathfrak{a}$, $t \mapsto t \cdot x_0$ is an isomorphism from T onto the lattice spanned by $\alpha^\vee = \frac{2\vec{\alpha}}{\|\alpha\|^2}$, $\alpha \in \Sigma^+$. For $x \in v(Z^+) \cdot x_0$, we have

$$\langle \vec{\rho}, v(z)\rangle \leqslant \|\vec{\rho}\| \, \|v(z)\| = \|\rho\| d_{\mathscr{B}}(x_0, v(z) \cdot x_0). \tag{7}$$

For $k \in \mathbb{N}$ we now set $x_k := k\tau\rho^\vee \cdot x_0 \in T \cdot x_0$ and set $r_k := \|x_k\| = 2k\tau\|\rho\|^{-1}$. The intersection of the balls $B(x_0, r_k)$ in $\mathscr{B}$ with the apartment $\mathscr{A} \cong \mathfrak{a}$ contains a finite number of vertices in $G \cdot x_0$. Since $G \cdot x_0 \cap \mathscr{A}$ is quasi-isometric to $\mathscr{A}$ the cardinality of that intersection is bounded by a polynomial P in r_k:

$$|B(x_0, r_k) \cap G \cdot x_0 \cap \mathscr{A}| \leqslant P(r_k).$$

By the Cartan decomposition, Proposition 2, $B(x_0, r_k) \cap G \cdot x_0$ is the union of the K-orbits of all points in the previous intersection. Combining this with (6), (7) and since $2\langle \vec{\rho}, k\tau\rho^\vee\rangle = 4k\tau$, we eventually obtain the upper bound

$$|B(x_0, r_k) \cap G \cdot x_0| \leqslant P(r_k) C_4 q^{4k\tau}.$$

On the other hand, by definition of r_k, $B(x_0, r_k) \cap G \cdot x_0$ contains the K-orbit of $x_k = k\tau\rho^\vee \cdot x_0$. Therefore using (6) we have also the lower bound

$$C_3 q^{4k\tau} \leqslant |B(x_0, r_k) \cap G \cdot x_0|.$$

Hence

$$\|\rho\| \left(\frac{1}{2k\tau} \log_q C_3 + 2\right) \leqslant \frac{1}{r_k} \log_q |B(x_0, r_k) \cap G \cdot x_0| \leqslant \|\rho\| \left(\frac{1}{2k\tau} \log_q (P(r_k) C_4) + 2\right).$$

Taking the limit, $k \to \infty$, yields the claim.

3.4 Proof of Part (c) of the Main Theorem. The building $\mathscr{B}$ is a geodesically complete Hadamard space (see Section 3.1) and has property (U) by Lemma 3. Hence the assumptions of the Main Theorem Part (a) and (b) are satisfied and we obtain $h_{\text{top}}(\varphi) = h_{\text{vol}}(dv)$. Next Lemma 2 implies that $h_{\text{vol}}(dv) = h_{\text{vol}}(\omega)$. And finally Lemma 4 yields $h_{\text{vol}}(\omega) = \|2\rho\|$.

Remark. If the local field has $\mathbb{F}$ has characteristic zero, then a theorem of Borel and Harder asserts that G contains cocompact lattices (see [4]). Moreover, in characteristic zero, a lattice is necessarily compact (see [23], IX 3.7). If the characteristic of $\mathbb{F}$ is positive, cocompact lattices do not always exist (see [23], IX 1.6, IX 4.5). If $\operatorname{rank}_{\mathbb{F}} G = 1$, then the building $\mathscr{B}$ is a tree. In that case cocompact lattices always exist, also in positive characteristic (see [20]).

4 An application: The growth of fundamental groups

4.1 Critical exponents of lattices. Consider a geodesic space (X, d_X) and a group Γ which acts isometrically and properly discontinuously on X. For $x, y \in X$ denote by $N(x, y; R)$ the number of orbit points of $y \in X$ under Γ contained in the ball $B(x, R)$. The *exponent of growth* (*or critical exponent*) *of* Γ is defined as

$$\delta(\Gamma) := \limsup_{R \to \infty} \frac{1}{R} \log_a N(x, y; R).$$

This number is independent of the chosen points $x, y \in X$.

Proposition 3. *Let (X, d_X, μ) be a complete, geodesic metric measure space. Assume that X has property (U) and that there exists a group Γ of isometries of X which acts totally discontinuously and cocompactly on X and preserves the measure μ. Then the critical exponent of Γ satisfies*

$$\delta(\Gamma) = h_{\text{vol}}(\mu).$$

Proof. Pick a point $x \in X$ and let $\mathscr{F}$ be a fundamental domain for Γ in X which contains x. For $R > 0$ set $\Gamma_R := \{\gamma \in \Gamma \mid \gamma \cdot x \in B(x, R)\}$. Then $N(x, x; R) = |\Gamma_R|$, the cardinality of Γ_R. If D is the diameter of $\mathscr{F}$ we have by the triangle inequality that $\Gamma_R \cdot \mathscr{F} \subseteq B(x, R + D)$. On the other hand, using that $\mathscr{F}$ is a fundamental domain, we get again by the triangle inequality and for $R \geqslant D$, $B(x, R - D) \subseteq \Gamma_R \cdot \mathscr{F}$. In conclusion we have for large R

$$\mu(B(x, R - D)) \leqslant |\Gamma_R| \mu(\mathscr{F}) \leqslant \mu(B(x, R + D)),$$

and the claim follows from the definitions and Proposition 1.

Remark. Critical exponents of arbitrary discrete subgroups of real and p-adic Lie groups have been investigated in [19] and [26].

The Main Theorem Part (c) and Proposition 3 immediately yield the

Corollary 1. *Let* $(\mathscr{B}, d_{\mathscr{B}})$ *be the euclidean Bruhat–Tits building associated to an algebraic group* $\mathbf{G}$ *as in the Main Theorem Part* (c). *Let* Γ *be a cocompact lattice in* $G = \mathbf{G}(\mathbb{F})$. *Then the critical exponent of* Γ *satisfies* $\delta(\Gamma) = h_{\mathrm{vol}}(dv) = 2\|\rho\|$.

4.2 An extension of a theorem of Dinaburg. Let (X, d_X) be a geodesic metric space and let (Y, d_Y) be a compact metric space whose universal covering space is X. Let Γ be the fundamental group of Y and let $\mathscr{F}$ be a fundamental domain of Γ in X. The set $S := \{g \in \Gamma \mid g\mathscr{F} \cap \mathscr{F} \neq \emptyset\}$ is a finite set of generators for Γ. For each positive integer m let $\beta(\Gamma, S; m)$ be the number of distinct group elements which can be expressed as word of length $\leqslant m$ in the elements of S. Then the limit $\lim_{m\to\infty} \frac{1}{m} \log \beta(\Gamma, S; m) =: h(\Gamma, S)$ exists and the property $h(\Gamma, S) > 0$ is independent of the specific set S (see [24] or [25], Lemma 5.16). The number $h(\Gamma, S)$ is called *exponential growth rate of* Γ or the *entropy of* Γ with respect to S (see [16], 5.11). The following proposition is proved in the same way as in the case of manifolds (compare [22], 5.17).

Proposition 4. *Let* (X, d_X, μ) *and* Γ *be as in Proposition* 1. *Let* D *be the diameter of a fundamental domain* $\mathscr{F} \subset X$ *of* Γ. *Then the exponential growth rate of* Γ *with respect to a finite generating set* $S = S(\mathscr{F})$ *satisfies*

$$h_{\mathrm{vol}}(\mu) \geqslant \frac{1}{2D} h(\Gamma, S).$$

We have the following extension of a theorem of Dinaburg (see [12], [25], 5.18).

Corollary 2. *Let* (X, d_X, μ), Γ, S *and* D *be as in Proposition* 4. *Then the topological entropy of the geodesic flow* φ *on the space of geodesics* (*parametrized by arc-length*) *of the compact quotient* $\Gamma\backslash X$ *satisfies*

$$h_{\mathrm{top}}(\varphi) \geqslant \frac{1}{2D} h(\Gamma, S).$$

Proof. The claim directly follows from Part (a) of the Main Theorem and Proposition 4.

Corollary 3. *Let* $(\mathscr{B}, d_{\mathscr{B}})$ *be the euclidean Bruhat–Tits building associated to an algebraic group* $\mathbf{G}$ *as in the Main Theorem* (c). *Let* Γ *be a cocompact lattice in* $G = \mathbf{G}(\mathbb{F})$ *and let* $S = S(\mathscr{F})$ *be a* (*finite*) *generating set for* Γ *with respect to some fundamental domain* $\mathscr{F}$ *of diameter* D. *Then the entropy of* Γ *satisfies*

$$h(\Gamma, S) \leqslant 4D\|\rho\| = 2D\delta(\Gamma).$$

Proof. This is a direct consequence of Part (c) of the Main Theorem, Corollary 1 and Corollary 2.

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