Volume entropy rigidity

The American Institute of Mathematics

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Version: Tue Jun 9 19:10:31 2009

Table of Contents

Chapter A: Participant Contributions

A.1 Bucher, Michelle

My mathematical interests belong to the interplay between topology, geometry and group theory which I am studying using bounded cohomology. The framework that bounded cohomology provides, taking its origin in Gromov's seminal 1982 paper "Volume and bounded cohomology", has led me to exact computations of simplicial volumes, combinatorics of triangulations, bounds on the Gromov norm of various characteristic classes, including so-called Milnor-Wood inequalities, and their application to the question of existence of flat structures.

In a recent joint work with Tsachik Gelander, we prove sharp Milnor-Wood inequalities for Riemannian manifolds which are locally isometric to a product of hyperbolic planes. As a corollary, we obtain that such manifolds cannot admit a flat or an affine structure, confirming in this case the old conjecture of Chern that a closed affine manifold has vanishing Euler characteristic.

Beside its important applications to Milnor-Wood type inequalities and its connections to the minimal volume or minimal entropy, the simplicial volume is interesting as it provides us with a new numerical invariant of manifolds. Therefore, I would like to advocate the following questions:

- Find new exact values of nonzero simplicial volumes. (The only known exact computations are for hyperbolic manifolds and for Riemannian manifolds which are locally a product of two hyperbolic planes.) Start with locally symmetric spaces of noncompact type. In particular:

- Give a formula for the simplicial volume of manifolds covered by products of n hyperbolic planes. (First unknown case $n = 3$.) At least, find the asymptotics, letting n tend to infinity.

A.2 Connell, Chris

I am interested in all aspects of the barycenter method. In particular, I am interested in extending the results of Besson, Courtois and Gallot to different settings, including foliations and nonpositive curvature and especially the higher rank rigidity conjecture. Lately I have been applying the method to produce coarse conditions for smooth topological rigidity, e.g. when is a map homotopic to a diffeomorphism.

A.3 Courtois, Gilles

Given a discrete finitely generated group G acting on a non positively curved space X there are relations between the geometry of X and algebraic properties of G. An interesting example is the group $G(x) \subset GL(2,\mathbb{R})$ generated by the two following matrices $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} x & 1 \\ 0 & 1 \end{pmatrix}$, where x is an algebraic integer over $\mathbb{Q}$. It has been observed by E. Breuillard that the entropy of $G(x)$ bounds the Mahler norm of x by below so that the Lehmer problem boils down in uniformly bounding by below the entropy of the $G(x)$'s. We construct a discrete action of $G(x)$ on a product of hyperbolic spaces X and equivariant maps of the Cayley graph of $G(x)$ into X which leads to some bounds of the entropy of $G(x)$. It is a work in progress with G. Besson and S. Gallot.

A.4 Dal'bo, Francoise

About the length spectrum.

Let G be a non elementary discrete group of isometries acting on a Hadamard manifold X.

Is it true that the multiplicative group generated by the numbers $l(g) = \text{Inf } d(x, g(x))$ for each g in G, is dense in $\mathbf{R}_*^+$?

Note that according to [Da], this question can be formulated in the following two different ways:

Is it true that the geodesic flow on $T^1(G/X)$ is topologically mixing in restriction to its non wandering set?

Is it true that the horospherical foliation on $T^1(G/X)$ is topologically transitive in restriction to its non wandering set?

About groups with torsion

Does there exist a geometrical proof of the fact that a finitely generated Fuchsian group admits a subgroup of finite index without torsion?

Is it possible to generalize this statement for geometrically finite groups acting on Hadamard manifolds?

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A.5 Geninska, Slavyana

My areas of research up to now are Fuchsian groups, Kleinian groups, arithmetic groups, symmetric spaces, hyperbolic geometry. My master's thesis was about a geometric characterization of arithmetic Fuchsian groups. Now I consider subgroups of arithmetic groups acting on products of hyperbolic planes and hyperbolic 3-spaces.

Last semester the seminar of our team in Karlsruhe was about volume entropy. This is how I got interested in the topic. Since I am working with an example of higher rank symmetric spaces, this workshop is a very good opportunity for me to start a new research project (related to my current work).

A.6 Kim, Inkang

1. I want to discuss about the local rigidity of lattices in semisimple Lie groups. For example one can show that a lattice in $SU(n, 1), n > 1$ is locally rigid in $Sp(n, 1), U(2n, 2), SO(4n, 4)$, more generally in $Sp(n + k, m)$.

2. Specially I want to address the issue of flexibility and rigidity of surface group representations in semisimple Lie groups. One can give a criterion for deformation to a Zariski dense one, using so-called "balanced condition" of the center of a centraliser of the image. Using this criterion, one can show that a surface group is not flexible only if its image is trapped inside a tube type Hermitian space. Particularly one can conclude that

(a) Any reductive surface group representation in semisimple Lie group is flexible except $\Gamma(1,1) \subset SU(n,1)$. (b) For a real form G of $SL(n,C)$, a surface group representation into G is flexible except $G = SU(p,q)$ and its image is contained in $SU(p,p)$.

3. Also I want to know when the unmarked length spectrum determines a Zariski dense representation in semisimple Lie groups. For example one can show that a given length spectrum of a convex cocompact hyperbolic 3-manifolds, there are finitely many convex cocompact hyperbolic 3-manifolds of the same homotopy type with the same length spectrum. For convex cocompact surface group representations into rank one semisimple Lie groups, the isospectral finiteness holds.

A.7 Lim, Seonhee

I am particularly interested in volume entropy rigidity of buildings or more general polyhedral complexes. Entropy rigidity is not known for hyperbolic buildings, even for Bourdon's buildings (the most well-behaved 2-dimensional right-angled buildings in many senses).

A.8 Paulin, Frederic

Let (M,g) be a compact or finite volume connected complete Riemannian manifold with dimension n at least 2 and sectional curvature at most -1. Let C be a free homotopy class of loops in M, non homotopic to 0, nor into a cusp of M. Let Lk_g be the set of recurrent (locally) geodesic rays (or lines) in (M,g), starting from a given point x_0 (or a given cusp, in which case some normalization is needed). In [PP5], building on the works [HPCMH,HPMZ,HPsurv,HPETDS,HPpre,PP3,PP2,PP4], we study the asymptotic spiraling behaviour of the elements of Lk_g around the closed geodesic C_g in the class C (and around more general convex subsets of more general M's, but we will only consider the above particular case in this announcement).

More precisely, let d' be the natural visual distance on Lk_g. Let $\mathrm{Lk}_{g,C}$ be the (countable, dense) set of elements ρ in Lk_g that spiral indefinitely around C_g. For every r in $\mathrm{Lk}_{C,g}$, let $D(r)$ be the shortest length of a path between x_0 and C which is homotopic (while its endpoints stay in $\{x_0\}$ and C respectively), for any t big enough, to the path obtained by following r from $r(0)$ to $r(t)$, and then a geodesic between $r(t)$ and its closest point on C. This number $D(r)$ naturally measures the wandering of r in M before r seriously starts to spiral indefinitely around C_g.

We define the *spiraling constant* around C of $\xi \in \mathrm{Lk}_g$ by

$$c(\xi) = \liminf_{r \in \mathrm{Lk}_{C,g}\ ,\ D(r) \to +\infty} \quad e^{D(r)}\, d'(\xi, r)\ ,$$

which measures how well ξ is approximated by geodesic lines spiraling indefinitely around C_g, and, when small, says that, asymptotically, ξ has long periods of time during which it spirals around C_g. We define the *spiraling spectrum* around C in (M,g) by

$$\mathrm{Sp}_{C,g} = \left\{ c(\xi)\ :\ \xi \in \mathrm{Lk}_g - \mathrm{Lk}_{C,g} \right\}\ .$$

In [PP5], say in constant curvature with $n \geq 3$ to simplify the statements in this announcement, we prove Dirichlet-type, Cusick-type and Hall-type theorems, that the spiraling spectrum is a closed, bounded subset of $[0,+\infty[$, which contains an interval $[0,c]$ for some $c > 0$.

When C is replaced by a cusp (and spiraling a long time around C is replaced by having a long excursion in a fixed cusp neighborhood), the analogous results are motivated by Diophantine approximation results: see for instance [For,Coh,Pat,Sul,Ser,Haa,CF,Vul,Dal]. In this context, the boundedness of the spectrum was proved in [**?**, Theorem 1.1], the closedness of the spectrum was shown in [Mau], and the existence of a Hall ray was proved in [**?**, Theorem 1.6].

We refer to [PP5,BPP] for various arithmetic applications, in particular for the Diophantine approximation of real or complex numbers by irrational quadratic numbers (see [Bug] for a general reference on the approximation by algebraic numbers).

In connection with the workshop *Volume entropy rigidity* (ETH, Zurich, 15-19 June, 2009), J. Parkkonen and myself would be very interested to understand the variation of these spiraling spectra on the underlying Riemannian metric g. In particular, does the lowest upper bound of the spiraling spectrum varies regularly on g ? Given some class of metric (conformal, Ebin, even in the case $n = 2$ of constant curvature -1, as in [Sch]), do the spiraling spectra provide rigidity statements ? In particular, is a minimum (if it exists, or at least locally) in such a class of the lowest upper bound of the spiraling spectra, for various free homotopy classes, characteristic of a given particularly nice metric g_0 on M ?

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A.9 Picaud, Jean-Claude

Main interests : 1) estimations of critical exponents of Kleinian groups and their quotients in the variable curvature setting, 2) volume entropy in Hilbert spaces, 3) metrics with non-positive sectionnal curvatures on manifolds and ergodic properties of the geodesic flow in that setting.

A.10 Remy, Bertrand

Strictly speaking, my work on buildings is not connected to entropy questions so far. Still, I am interested in measure-theoretic techniques on buildings and their automorphism groups because a joint project with Y. Guivarc'h is to study Martin compactifications of Bruhat-Tits buildings. Presumably, this would provide new boundaries for these buildings and related algebraic groups. So far, we have obtained compactifications of buildings by means of the (compact) Chabauty topology on the space of closed subgroups (with Y. Guivarc'h) and by means of analytic geometry in the sense of Berkovich spaces (with A. Thuillier and A. Werner). The first compactification provides a geometric parametrization of the maximal amenable subgroups of a non-archimedean Lie group, the second one provides more compactifications, closer in spirit to the Satake compactifications for real Lie groups.

A.11 Sauer, Roman

I got interested in the barycenter method and the work of Besson, Courtois, and Gallot through a more recent joint project with Uri Bader and Alex Furman where we prove a cocycle rigidity theorem for hyperbolic lattices (generalizing Mostow rigidity); strictly speaking we do not use the barycenter method so far but we see some potential use in this context and also in the context of orbit equivalence theorems for such lattices. The workshop gives me the opportunity to discuss this with the experts.

My other mathematical interests are L^2-invariants and bounded cohomology. I proved an inequality between minimal volume and L^2-Betti numbers of aspherical manifolds outlined by Gromov. Moreover, together with Clara Löh, I worked on the simplicial volume of noncompact nonpositively curved manifolds. I find the general problem (initiated by Gromov) of relating L^2-invariants, simplicial volume to geometric quantities like minimal volume, entropy etc. quite fascinating.

A.12 Smirnova-Nagnibeda, Tatiana

Among the topics to be discussed at the workshop I am particularly interested in the problem of volume entropy rigidity for singular spaces, such as buildings or CW-complexes. In a joint work with I. Kapovich, we have proved a "minimal volume entropy" type theorem in the context of finite graphs of a given rank. This is related to the study of the Outer space – the model space for the action of the group of outer automorphisms of the free group. It would be very interesting to see what kind of techniques have to be developed in order to approach the higher-dimensional case.

A.13 Storm, Peter

I would like to learn about the recent work of Besson-Courtois-Gallot studying volume growth entropy of groups acting on nonpositively curved spaces, and potential (if tenuous) connections to number theory. Also, I think the recent collaboration of M. Bucher and T. Gelander would interest many of the participants.

A.14 Vernicos, Constantin

I am interested in the volume entropy of Hilbert Geometries. These geometries are a generalisation of the projective model of the Hyperbolic geometry, and are defined using the cross-ratio inside a convex set. The conjecture is that the volume entropy is always smaller than $n-1$. This conjecture has been proved in dimension 2 in full generality by Berck, Bernig and myself, and in all dimension for strictly convex divisible Hilbert Geometries by M. Crampon which also showd that in that family a rigidity occured, mainly the inequality is strict if the geometry is not the Hyperbolic one. Our work made us introduced an invariant called the centro-projective area of a convex set (related to a known invariant to the specialist of the theory of Valuation called centro-affine area) and which is a generalisation of Margulis's function in the setting of Hilbert Geometries. The rigidity conjecture which generalises M. Crampon's result is that the entropy is equal to $n-1$ if and only if the centro-projective area of the convex in which the Hilbert Geometry is defined is not zero, and that the centro-affine area does not depend on the base point (centre of a familly of balls) if and only if the convex is an ellispoid (in which case the geometry is the usual Hyperbolic one).

Two other questions are interesting in that setting.

A. First it is not clear that the entropy is a limit. This is something I started investigating with G. Link and hopefully this workgroup might allow us to conclude by combining our respective knowledge.

B. Second: What are the Hilbert Geometries with null Volume Entropy ? We know that convex polytopes give rise to geometries which are bi-lipshitz to a normed vector space (Independently Bernig/Vernicos) hence have zero Volume Entropy. Are they the only ones (some kind of Tits alternative for Hilbert Geometries) ?

A.15 Xie, Xiangdong

My interests are in nonpositive curvature, geometric group theory and their connections with quasiconformal analysis. I am particularly interested in various rigidity questions. Many years ago, as a visiting student at the University of Michigan , I sat in Ralf Spatzier's class on "BCG methods". I tried to follow the work by Connell and Farb.

I would like to see more discussions of the difficulty of the conjecture in higher rank. I also would like to know more about the analogous question for singular spaces.