Growth of lattices in negatively curved manifolds

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Outline

Notations and Problem
Main results
Proofs

Growth functions, entropy and critical exponent
Lattices
The problem

Context

- X complete simply connected Riemannian n-manifold with pinched negative curvature: $-b^2 \leq k(X) \leq -a^2 < 0$
- Γ discrete torsion-free subgroup of isometries of X (a "Kleinian group")

Growth function of X

$v_X(x, R) = \text{vol}(B_X(x, R))$

Growth function of Γ

$v_\Gamma(x, R) = \text{card}(B_X(x, R) \cap \Gamma x)$

Exponential growth rate of X

$\omega(X) = \limsup_{R \to +\infty} \frac{\log v_X(x,R)}{R}$

(also known as the volume-entropy $\text{Ent}(X)$ of the manifold X)
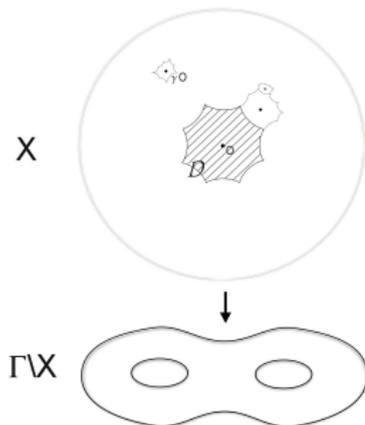
Exponential growth rate of Γ

$\omega(\Gamma) = \limsup_{R \to +\infty} \frac{\log v_\Gamma(x,R)}{R}$

(also known as the critical exponent $\delta(\Gamma)$ of the group Γ)

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The group Γ is a Lattice of X if $\mathrm{vol}(\Gamma\backslash X) < +\infty$

Uniform
if $\Gamma\backslash X$ is compact

Non-uniform
if $\Gamma\backslash X$ is non-compact
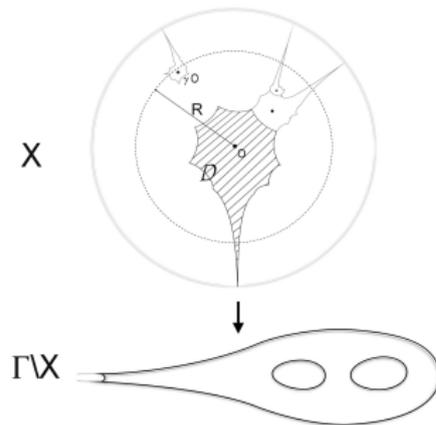
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Folklore (Shvarts, Milnor etc.)

When Γ is uniform:
$v_\Gamma(x, R) \asymp v_X(x, R)$, so $\omega(\Gamma) = \omega(X)$

$f \prec g$ means:
$\exists A, B \geq 0 \mid f(r) \leq Ag(r + B) \; \forall r \gg 0$

$f \asymp g$ means: $f \prec g \prec f$

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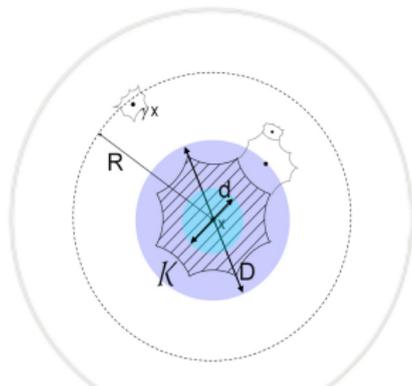$\exists A, B \geq 0 \mid f(r) \leq Ag(r + B) \; \forall r \gg 0$

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Classical argument: the ball $B_X(x, R)$ can be approximated by (about) $v_\Gamma(x, R)$ copies of a compact set.

$$\bigsqcup_{|\gamma| \leq R-d} B_X(\gamma x, d) \subset B_X(x, R) \subset \bigcup_{|\gamma| \leq R+D} B_X(\gamma x, D)$$

$$\Rightarrow \qquad v_\Gamma(x, R) \prec v_X(x, R) \prec v_\Gamma(x, R)$$

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Problem

What happens for a non-uniform lattice?

- *is $v_\Gamma(x, R) \asymp v_X(x, R)$?*
- *is $\omega(\Gamma) = \omega(X)$?*

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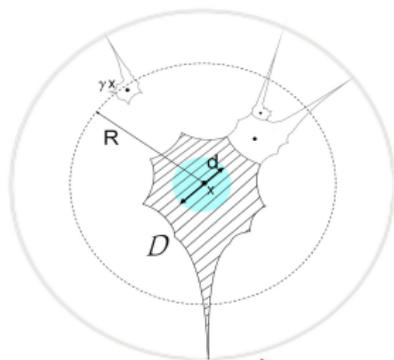
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Problem

What happens for a non-uniform lattice?

- *is $v_\Gamma(x, R) \asymp v_X(x, R)$?*
- *is $\omega(\Gamma) = \omega(X)$?*

Remark. A. Eskin and C. McMullen proved that if X is a symmetric space then $\omega(\Gamma) = \omega(X)$. So, the Problem is for *general* negatively curved manifolds.

Γ non-uniform lattice of X, $\dim(X) = n$, $-b^2 \leq k(X) \leq -a^2$:

- $\omega(\Gamma) \leq \omega(X)$
- $(n-1)a \leq \omega(X) \leq (n-1)b$
- $(n-1)a \leq \omega(\Gamma)$

$$\bigsqcup_{|\gamma| \leq R-d} B_X(\gamma x, d) \subset B_X(x, R) \subset \bigcup_{|\gamma| \leq R+D} B_X(\gamma x, D)$$

$$v_\Gamma(x, R) \prec v_X(x, R) \prec v_\Gamma(x, R)$$

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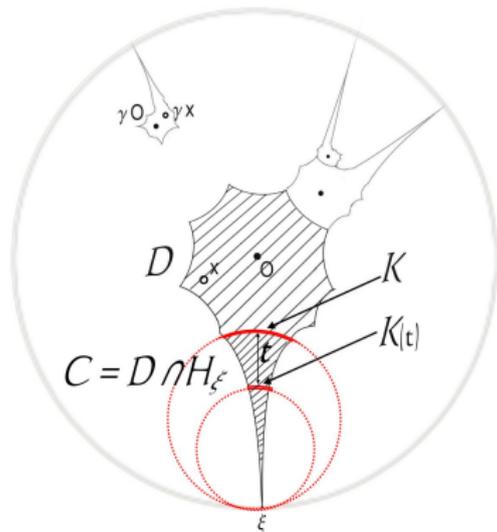
(by Bishop-Gunther's Volume Comparison Theorems)

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Argument:
Break the Poincaré series of X as:
$\int_X e^{-sd(O,x)}dx \ge \sum_\gamma e^{-sd(O,\gamma O)} \cdot \int_D e^{sd(O,x)}dx$

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If $\omega(\Gamma) < s < \omega(X)$ then:

- $\int_X e^{-sd(O,x)} dx = +\infty$,
- $\sum_\gamma e^{-sd(O,\gamma O)} dx < +\infty$

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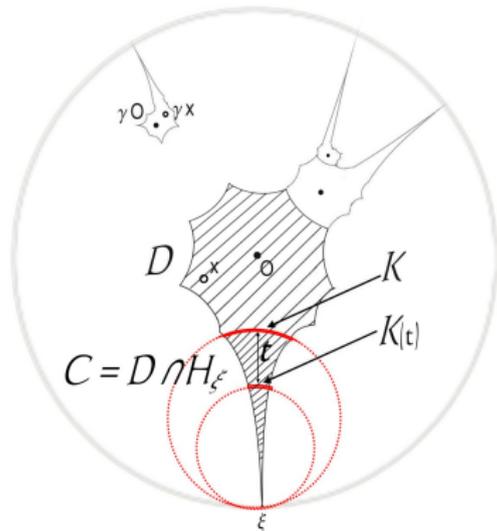- $(n-1)a \leq \omega(\Gamma)$

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Hence: $\int_D e^{sd(O,x)}dx = +\infty$

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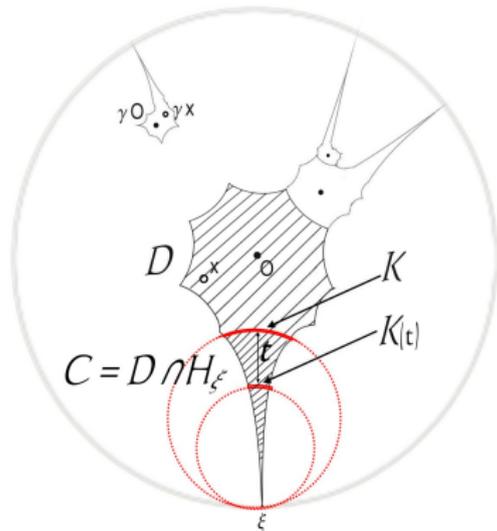
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If $\omega(\Gamma) < s < \omega(X)$ then $\int_C e^{sd(O,x)} dx = +\infty$

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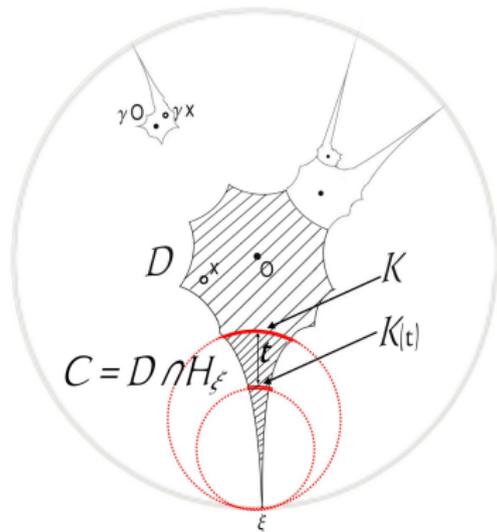
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Compute the integral in horospherical coord.:
$+\infty = \int_0^{+\infty} e^{st} \cdot \text{area}(K(t)) dt$

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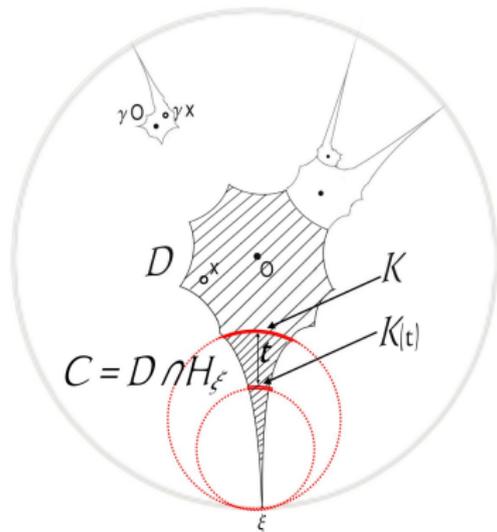- $(n-1)a \leq \omega(\Gamma)$

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$e^{-(n-1)bt} \leq \dfrac{\text{area}(K(t))}{\text{area}(K)} \leq e^{-(n-1)at}$

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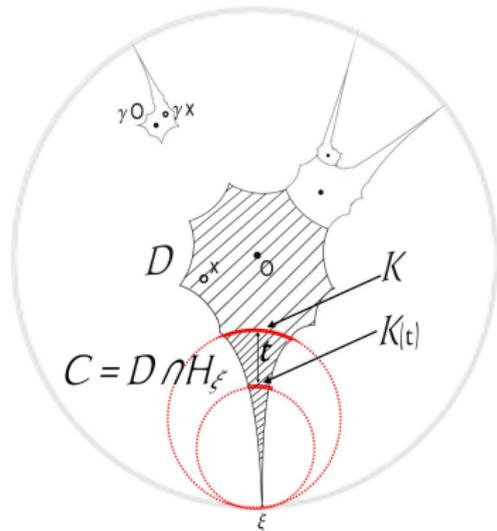$\int_X e^{-sd(O,x)} dx \ge \sum_\gamma e^{-sd(O,\gamma O)} \cdot \int_D e^{sd(O,x)} dx$

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Thus: $\quad s \geq (n-1)a$
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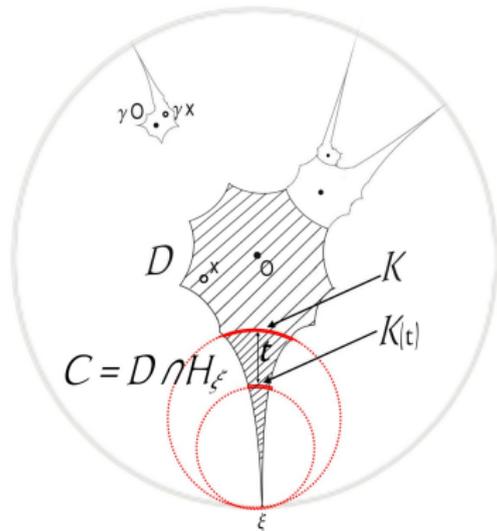If $\omega(\Gamma) < s < \omega(X)$ then $\int_C e^{sd(O,x)} dx = +\infty$

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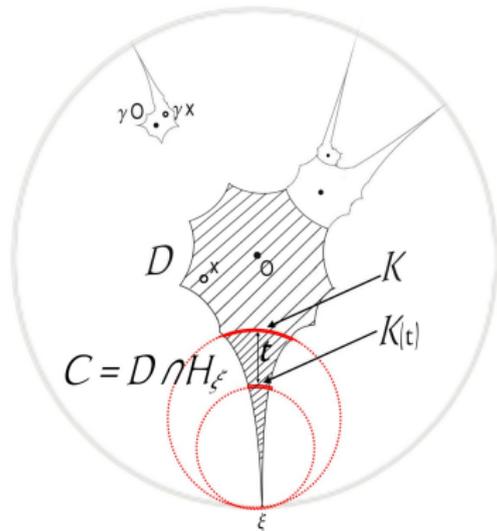$+\infty = \int_0^{+\infty} e^{st} \cdot \text{area}(K(t)) dt$
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Thus: $s \geq (n-1)a$
$\omega(\Gamma) \geq (n-1)a$ □

Remark: this already shows that $\omega(\Gamma) = \omega(X)$ when X has constant cuvature.

Theorem I [DPPS, Crelle 2009]

Let X be any complete, simply connected, Riemanniann n-manifold with pinched negative curvature $-b^2 \leq k(X) \leq -a^2 < 0$.

Assume that X is $\frac{1}{4}$-*pinched* (that is, $b^2/a^2 \leq 4$):

then any lattice Γ of X satisfies $v_\Gamma(x, R) \asymp v_X(x, R)$, so $\omega(\Gamma) = \omega(X)$.

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Theorem II [DPPS, Crelle 2009]

There exists a complete, simply connected Riemannian surface S with pinched negative curvature $-b^2 \leq k(S) \leq -a^2 < 0$ admitting a non-uniform lattice Γ such that $\omega(\Gamma) < \omega(X)$.

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Remark. The same proof works for X homogeneous
(it explains the geometric role of symmetry in the equality $\omega(\Gamma) = \omega(X)$).

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Remark. *Theorem I is (surprisingly) optimal.*
For every $\epsilon > 0$ the surface of Theorem II can be chosen so that
$-4 - \epsilon \le k(S) \le -1 < 0$.

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Remark. Theorem I holds under the weaker assumption that $\Gamma \backslash X$ is
only *asymptotically* $1/4$-pinched.

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Proof of Theorem I
Improving the formula
Proof of Theorem II

Dissection of a Ball under the action of a nonuniform lattice Γ

- Γ\$\backslash$X has a *finite* number of cusps C
- each cusp $C = P \backslash \mathcal{H}_\xi$:
  $P < \Gamma$ maximal parabolic sbgr. fixing $\xi \in \partial X$
  $\mathcal{H}_\xi$ horoball centered in ξ
- P acts cocompactly on the horosphere $\partial\mathcal{H}_\xi$

$\mathcal{D}$ = Dirichlet domain centered at O for $\Gamma \curvearrowright X$

$\mathcal{C} = \mathcal{D} \cap \mathcal{H}_\xi$ fundam. domain for $P \curvearrowright \mathcal{H}_\xi$

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$$\bigcup_{|\gamma| \leq R} \gamma \mathcal{K} \; \widetilde{\subset} \; B_X(O, R) \; \widetilde{\subset} \; \bigcup_{|\gamma| \leq R} \gamma \mathcal{K} \bigcup_{\text{cusps } C_i} \bigcup_{|\gamma| \leq ?} [B_X(O, R) \cap \gamma \mathcal{C}_i]$$

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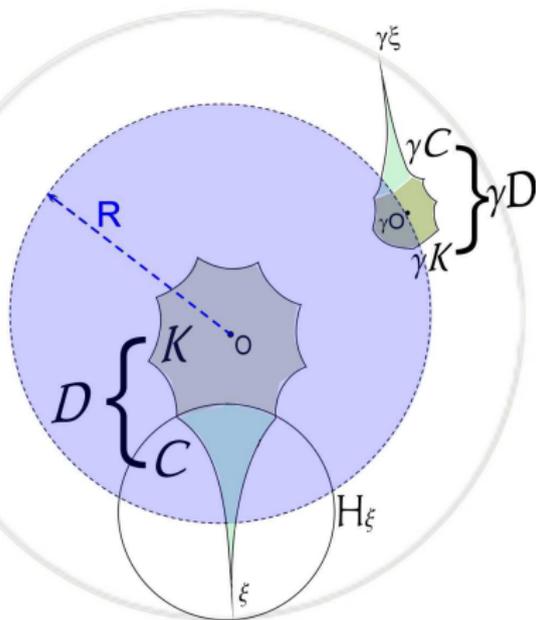# Dissection of a Ball under the action of a nonuniform lattice Γ

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$\mathcal{K} = \mathcal{D}\backslash\mathcal{C}$ (the compact part of $\mathcal{D}$) $y \in \partial H_{xi}$

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$$\cong \bigcup_{|\overline{\gamma}|\leq R} \overline{\gamma}\,[B_X(y, R - |\overline{\gamma}|) \cap \mathcal{H}_{\xi_i}]$$

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$$y_i \in \partial \mathcal{H}_{\xi_i}$$

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$$\bigcup_{|\gamma| \leq R} \gamma \mathcal{K} \overset{\sim}{\subset} B_X(O, R) \overset{\sim}{\subset} \bigcup_{|\gamma| \leq R} \gamma \mathcal{K} \bigcup_{cusps \ c_i} \bigcup_{|\gamma| \leq R} \gamma \left[B_X(y_i, R - |\gamma|) \cap \mathcal{H}_{\xi_i}(y_i) \right]$$

1st Key Fact: a ball of radius r in X intersects a horoball passing through its center in a ball of radius (about) $r/2$

$$B\left(y', \frac{r}{2}\right) \overset{\sim}{\subset} B(y, r) \cap H_\xi(y) \overset{\sim}{\subset} B(y', \frac{r}{2})$$

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Dissection of a Ball under the action of a nonuniform lattice Γ

$$\bigcup_{|\gamma| \leq R} \gamma \mathcal{K} \quad \widetilde{\subset} \quad B_X(O, R) \quad \widetilde{\subset} \quad \bigcup_{|\gamma| \leq R} \gamma \mathcal{K} \bigcup_{\text{cusps } c_i} \bigcup_{|\gamma| \leq R} \gamma \left[B_X(y_i, R - |\gamma|) \cap \mathcal{H}_{\xi_i}(y_i) \right]$$

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$$\Downarrow$$

Formula: $\omega(X) = \max\left\{ \omega(\Gamma), \frac{1}{2}\omega\left(\sup_y v_X(y, R) \right) \right\}$

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In fact: $\quad v_\Gamma(O, R) \prec v_X(O, R) \prec v_\Gamma(O, R) + \displaystyle\sum_{cusps} \sum_{k=1}^{[R]} v_\Gamma(O, k) \cdot \sup_y v_X \left(y, \frac{R-k}{2} \right)$

$$v_\Gamma(O, R) \prec v_X(O, R) \prec v_\Gamma(O, R) + \left(v_\Gamma(O, R) * \sup_y v_X(y, R/2) \right)$$

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$$\omega(\Gamma) \leq \omega(X) \leq \max \left\{ \omega(\Gamma), \frac{1}{2}\omega \left(\sup_y v_X(y, R) \right) \right\}$$

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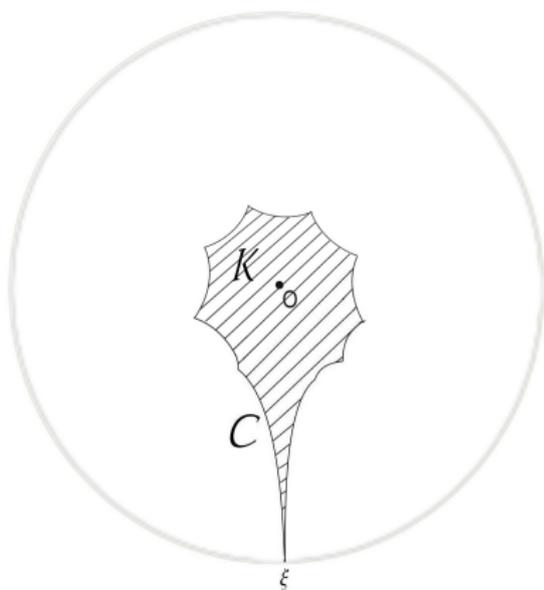
em If $-b^2 \leq k(X) \leq -a^2$ with $0 < b \leq 2a$:

$$\Rightarrow \omega\left(\sup_y v_X(y, R)\right) \leq b(n-1), \quad \omega(\Gamma) \geq a(n-1)$$

$$\Rightarrow \omega(X) = \max\left\{ \omega(\Gamma), \frac{2a(n-1)}{2} \right\} = \omega(\Gamma) \quad \square$$

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Formula: $\omega(X) = \max\left\{\omega(\Gamma), \frac{1}{2}\omega\left(\sup_y v_X(y, R)\right)\right\}$

The same argument works if X is homogeneous:

$$\Rightarrow \omega\left(\sup_y v_X(y, R)\right) = \omega(X),$$

$$\Rightarrow \omega(X) = \max\left\{\omega(\Gamma), \frac{1}{2}\omega(X)\right\} = \omega(\Gamma)$$

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Horospherical Area

P = maximal parabolic subgroup of $\Gamma \curvearrowright X$, Fix$(P) = \xi$

$P \curvearrowright \partial H_\xi$ cocompactly

Fix $x \in X$:

- $\partial H_\xi(x)$ = horosphere centered in ξ through x

- $\mathcal{F}_x$ = compact fundamental domain for $P \curvearrowright \partial H_\xi(x)$

- $\Phi_{\xi,t}$ radial flow in the direction of ξ
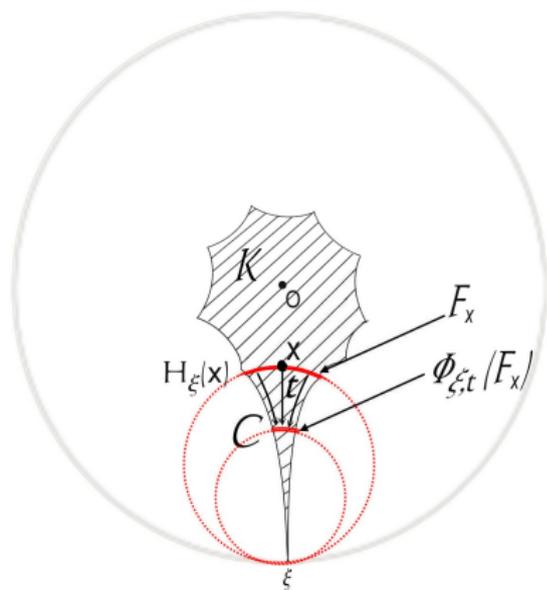
Horospherical Area Function of P

$$\mathcal{A}_P(x, t) = \text{Area}(\Phi_{\xi,t}(\mathcal{F}_x))$$

(only depends on x and on the subgroup P)

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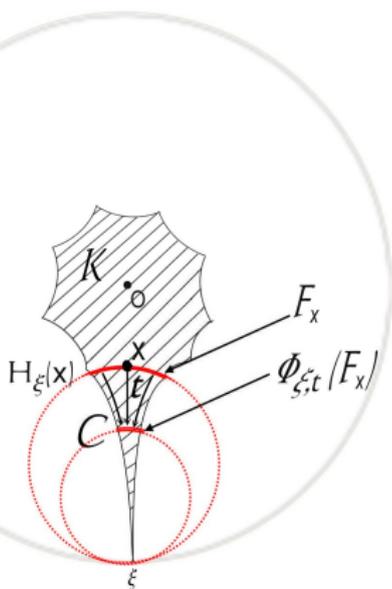
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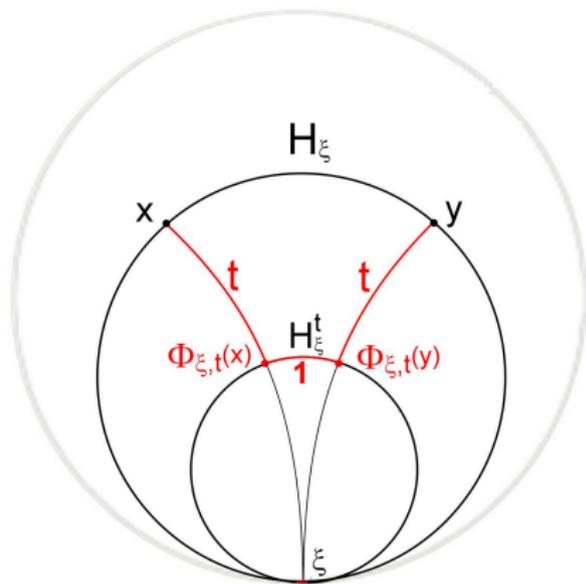
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Horospherical area

The function $\mathcal{A}_P(x, R)$ satisfies: $\mathcal{A}_P(x, R) \asymp 1/v_P(x, 2R)$

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by hyperbolic geometry $\rightsquigarrow$ $d_{H_\xi^t}(\Phi_{\xi,t}(x), \Phi_{\xi,t}(y)) = 1 \Leftrightarrow d(x, y) \simeq 2t$
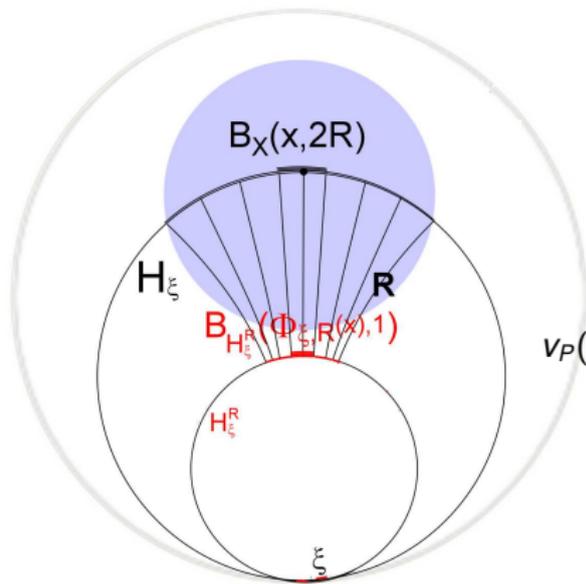
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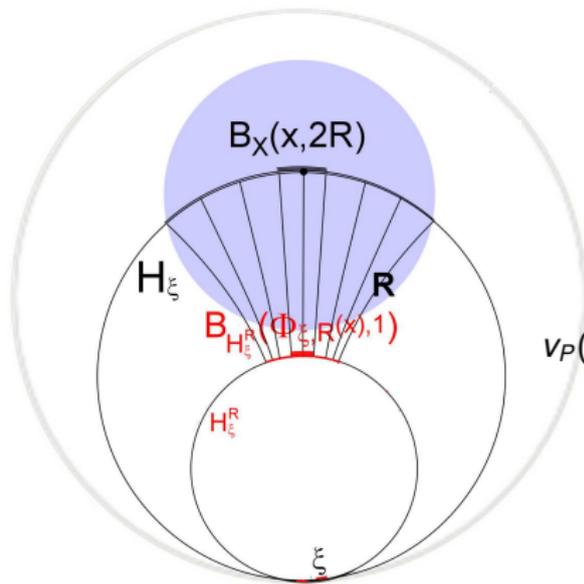
$$\Phi_{\xi,R}[B_x(x, 2R) \cap H_\xi] \cong B_{H_\xi^R}(\Phi_{\xi,R}(x), 1)$$

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$\cong v_P(x, 2R)$
disjoint copies
of $\mathcal{F}_x$

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disjoint copies   disjoint copies
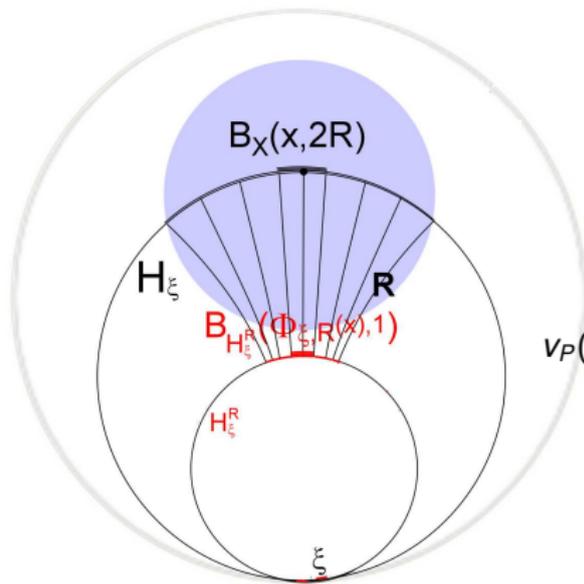of $\mathcal{F}_x$ of $\Phi_{\xi,R}(\mathcal{F}_x)$

$\Downarrow$

$$v_P(x, 2R) \cdot \text{Area}[\Phi_{\xi,R}(\mathcal{F}_x)] \simeq \text{Area}[B_{H_\xi^R}(\Phi_{\xi,R}(x), 1)]$$

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$\simeq v_P(x, 2R)$ — disjoint copies of $\mathcal{F}_x$

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bounded between positive constants v^-, v^+

(since $-b^2 \le k(X) \le -a^2$)

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$$\Rightarrow \quad \mathcal{A}_P(x, R) \asymp 1/v_P(x, 2R)$$

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Dissection of a Ball under the action of a nonuniform lattice Γ

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1st **Key Fact (sharper estimate):** the volume of the intersection of a ball $B_X(x, r)$ with a horoball $H_\xi(x)$ passing through its center x is given by

$$\mathrm{vol} \left[B_X(x, R) \cap H_\xi(x) \right] \asymp \int_0^R \frac{\mathcal{A}_P(x, t)}{\mathcal{A}_P \left(x, \frac{t+R}{2} \right)} dt$$

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$\mathcal{F}(x, R)$**: the _Cuspidal Function_ of the cusp**

$\Downarrow$

Sharper Formula: $\omega(X) = \max\{\omega(\Gamma), \omega(\mathcal{F}_1), ..., \omega(\mathcal{F}_N)\}$

where $\mathcal{F}_1, ..., \mathcal{F}_N$ are the cuspidal functions of the cusps of $\Gamma \backslash X$

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Construction of a lattice Γ of a space X with $\omega(\Gamma) < \omega(X)$
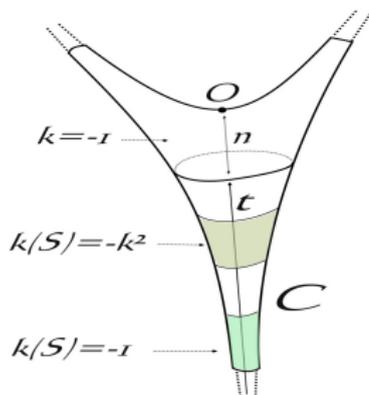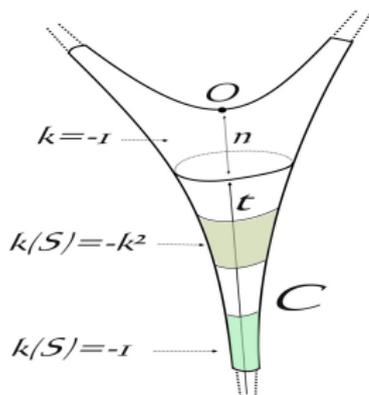
Deform the metric of a finite volume, hyperbolic surface S
on an infinite number of bands $[r_i, R_i]$ along a cusp:

$S_n = $ surface obtained perturbing **only outside the ball** $B_S(O, R_n)$
$C_n = $ the perturbed cusp
$\mathcal{A}_n = $ the horospherical area
$\mathcal{F}_n = $ the cuspidal function

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Construction of a lattice Γ of a space X with $\omega(\Gamma) < \omega(X)$

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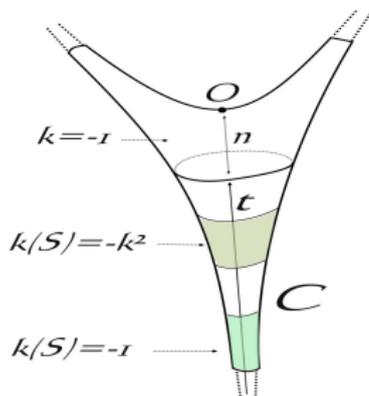$\mathcal{A}_n$ = the horospherical area

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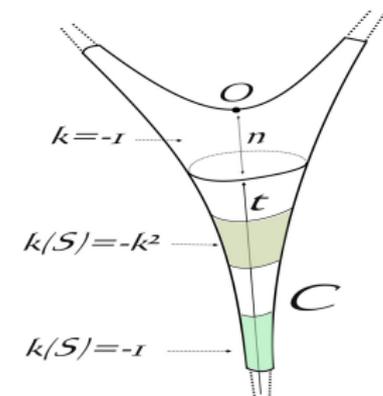
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$X_n = \tilde{S}_n$

$g_n = \mathcal{A}_n(t)^2 d\vartheta^2 + dt^2$

$\mathcal{A}_n(t) = e^{-t}$ when $k(X_n) = -1$

$\mathcal{A}_n(t) = e^{-kt}$ when $k(X_n) = -k^2$

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Construction of a lattice Γ of a space X with $\omega(\Gamma) < \omega(X)$
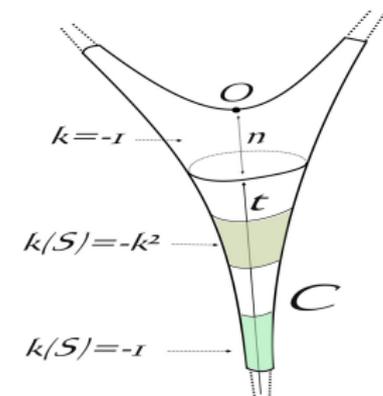
Deform the metric of a finite volume, hyperbolic surface S
on an infinite number of bands $[r_i, R_i]$ along a cusp:

S_n = surface obtained perturbing **only outside the ball** $B_S(O, R_n)$

C_n = the perturbed cusp

$\mathcal{A}_n$ = the horospherical area

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$$X_n = \tilde{S}_n$$
$$g_n = \mathcal{A}_n(t)^2 d\vartheta^2 + dt^2$$
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- $\omega(P_n) = \limsup_{R \to \infty} \frac{1}{R} \ln v_P(R)$

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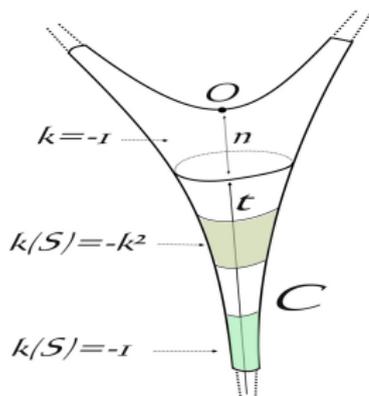
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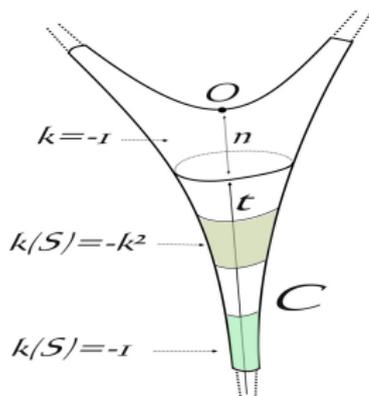
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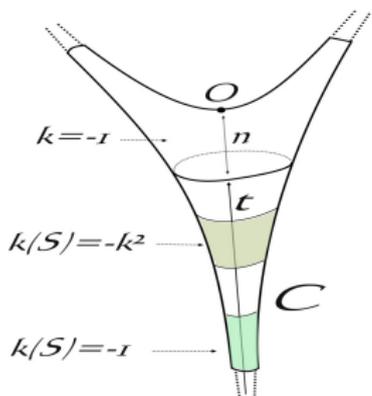
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C

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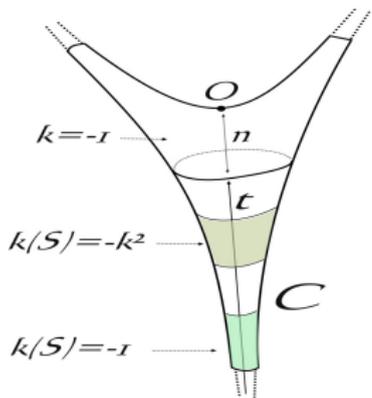
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Remark: if $\omega(\mathcal{F}) \leq \frac{k}{2}$

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Remark: if $\omega(\mathcal{F}) \leq \frac{k}{2} = \omega(P_n) \leq \omega(\Gamma_n)$
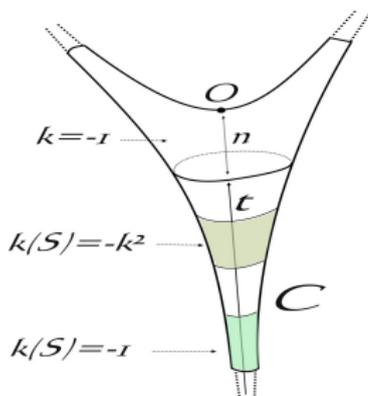$\rightsquigarrow$ no hope to get $\omega(X_n) > \omega(\Gamma_n)$

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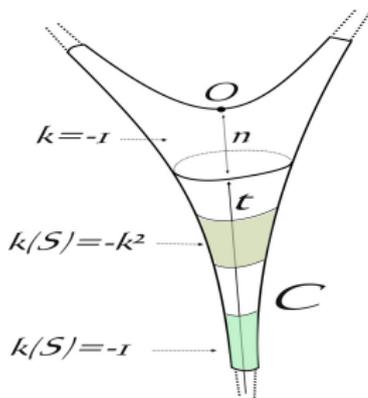$\omega(X_n) = \max\{\omega(\Gamma_n), \omega(\mathcal{F})\}$

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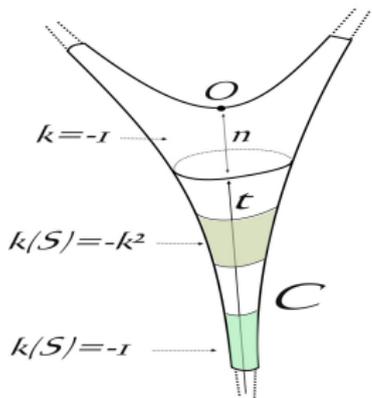
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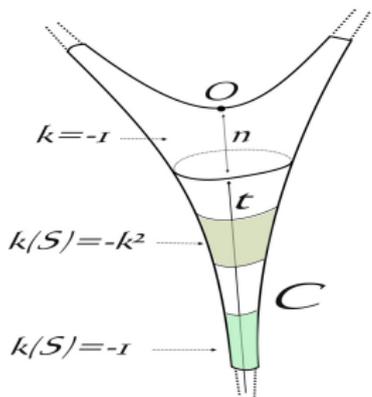
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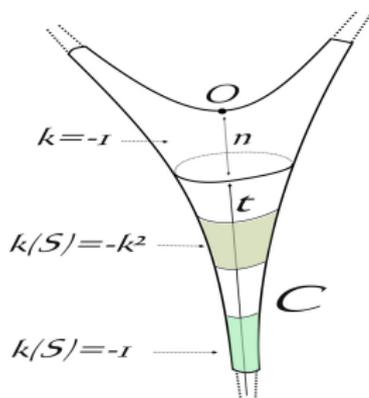
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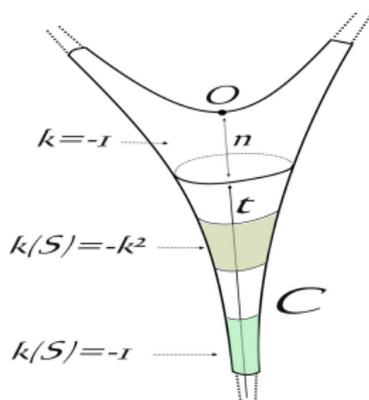Notations and Problem
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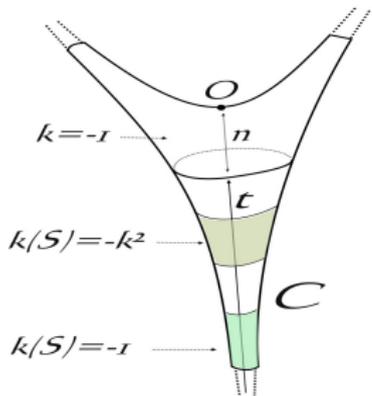
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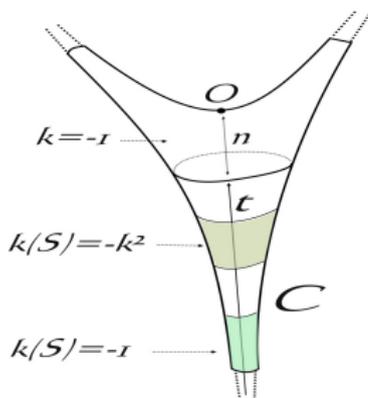- $\omega(\mathcal{F}) = k/2 + \epsilon$ \qquad (if $k^2 > 4$)

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2nd Key Fact: perturbing outside $B_S(O, R_n)$ for $n \gg 0 \Rightarrow \omega(P_n) \to \omega(\Gamma_n)$.

$$\Rightarrow \omega(X_n) = \max\{ \underbrace{\omega(\Gamma_n)}_{\sim \omega(P_n)}, \underbrace{\omega(\mathcal{F})}_{> \frac{k}{2} + \epsilon} \} > \omega(\Gamma_n) \ \square$$