Problems from the AIM Workshop on Algorithmic Convex Geometry

Problems compiled by Navin Goyal

Problem 1 (Santosh Vempala) *Let $K \subset \mathbb{R}^n$ be a convex body. How many uniformly randomly chosen points from K are needed to estimate the volume of K within a factor of 2.*

Problem 2 (Santosh Vempala) *Let $S \in \mathbb{R}^n$ be compact, and let $C \in \mathrm{argmin}_{C \in \mathcal{K}} \mathrm{vol}(\Delta(S, C))$ be a convex body* closest *to S ($\mathcal{K}$ stands for the set of compact convex sets with nonempty interior, and the empty set, and $\Delta(\cdot, \cdot)$ stands for the symmetric difference). We say that S is ϵ-convex if $\mathrm{vol}(\Delta(S, C)) \leq \epsilon \mathrm{vol}(S)$. Assume that the center of gravity of S is at the origin. For a pair of points $x, y \neq 0 \in \mathbb{R}^n$, let the subspace spanned by them be $H(x, y)$ and define $P(x, y) := S \cap H(x, y)$.*

Let μ be the distribution on 2-dimensional sections $P(x, y)$ obtained by picking x and y uniformly at random from S. If

$$\Pr_{\mu}(P(x, y) \text{is convex}) \geq 1 - \epsilon,$$

then S is $O(n\epsilon)$-convex.

Problem 3 (Van Vu) *Let $K \subset \mathbb{R}^d$ (d fixed dimension) be a convex body. Let $x_1, \ldots, x_n$ be uniformly random points from K, and let $X := \mathrm{vol}(\mathrm{conv}(x_1, \ldots, x_n))$. Does X satisfy the central limit theorem?*

Remark. The answer is known in the affirmative for the case when K is a smooth body or a polytope. The problem is open for general K even for $d = 2$.

Problem 4 (Ryan O'Donnnell) *Let $K \subset \mathbb{R}^n$ be a convex set, and let X be Gaussian random variable conditioned to lie in K. Does $\mathrm{Var}(X_\theta) = 1$ imply that K is a cylinder in direction θ? If $\mathrm{Var}(X_\theta) = 1 - \epsilon$ then does it mean that K has a small symmetric difference with a cylinder in direction θ?*

Remarks. K is not necessarily symmetric; when it is symmetric, the solution is given by Sidak's lemma. It is known that $\mathrm{Var}(X_\theta) \leq 1$ in every direction θ, where X_θ is the projection of X in direction θ. This can be proved in a number of ways, e.g., using Brascamp–Lieb inequality.

Problem 5 (David Jerison) *For a symmetric convex body K, let S be a least area surface that divides the volume of K into two halves (or any other fixed proportion). Is the surface a graph? (The surface in question may not be a hyperplane in general.)*

Remarks. Simons' cone in $\mathbb{R}^n$ is defined by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2.$$

Simons' cone has the minimum volume among the surfaces that meet the ball in the same boundary, it's not a graph, and the bisecting plane has smaller area. As $n \to \infty$, the area tends to that of the bisecting plane. This is very strange:

Since in the Gaussian space this ties (as $n \to \infty$) with the half-spaces, does it mean that Borell's theorem (for finite n, half-spaces are unique minimizers) does not hold in the limit?

Problem 6 (Mokshay Madiman) *Let X_1, X_2, X_3 be real-valued independent random variables with densities. Is the following inequality true?*

$$e^{2\mathrm{H}(X_1+X_2+X_3)} + e^{2\mathrm{H}(X_2)} \overset{?}{\geq} e^{2\mathrm{H}(X_1+X_2)} + e^{2\mathrm{H}(X_2+X_3)}. \tag{1}$$

Remarks. One gets an equality if X_i are Gaussians even with different variances. If true, (1) implies the following:

$$e^{2\mathrm{H}(X_1+\ldots+X_n)} \geq \sum_{S\subseteq[n]} \beta_S \, e^{2\mathrm{H}(X_S)},$$

where $\{\beta_S\}$ is a fractional covering of $[n] := \{1, \ldots, n\}$; that is, for all $i \in [n]$ we have $\sum_{S:i\in S} \beta_S \geq 1$, and $\beta_S \geq 0$ for all $S \subseteq [n]$.

The above inequality is known to be true when the underlying hypergraph given by sets S consists of all $(n-1)$-subsets of $[n]$ [Artstein, Ball, Naor]. It was generalized to regular hypergraphs (hypergraphs for which $|\{S : i \in S\}|$ is the same for all $i \in [n]$) [Barron, Madiman].

Problem 7 (Grigoris Paouris) *Which bodies are ψ_2-bodies? Are zonoids ψ_2-bodies? Are projections of ψ_2-bodies ψ_2-bodies?*

Remark. It is known that the B_n^p for $p \geq 2$ is ψ_2.

Problem 8 (Grigoris Paouris) *Is it true that every compact set is anti-ψ_2 in some direction?*

Problem 9 (Yuval Peres) *Consider* lazy *random walks on graphs or reversible Markov chains. Define the mixing time $\tau_{\mathrm{TV}}(1/2)$ as minimum t such that the total variation distance between $p^t(x,\cdot)$ and μ (stationary distribution) is at most $1/2$.*

Does there exist $c > 0$ such that

$$\tau_{\mathrm{TV}}(1/2) \leq c \max_{x,A:\mu(A)\geq 1/2} \mathbb{E}\, T_{A,x},$$

where $T_{A,x}$ is the hitting time for hitting A starting from x.

Remark. It was proved by Aldous that there exists a c such that

$$\tau_{\mathrm{TV}}(1/2) \leq c\mu(A) \max_{x,A} \mathbb{E} T_{A,x},$$

Problem 10 (Yuval Peres) *Consider random walks on undirected graphs where each node v is given a positive weight W_v (the probability of going from node u to node v is given by $\frac{W_v}{\sum_{w\in N(u)} W_w}$, where $N(u)$ is the set of neighbors of u in the graph). If we change the weights by a bounded factor, does the mixing time also change by a bounded factor?*

Remark. This is true if the mixing time is replaced by spetral gap.

Problem 11 (Ryan O'Donnell) *Is it true that every ellipsoid has Gaussian surface area bounded by a universal constant?*

Remark. Keith Ball has shown that for a general convex body the answer is $O(n^{1/4})$; this bound is attained by intersection of random half-spaces.