Describing convex semialgebraic sets by linear matrix inequalities

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Introduction

Describing convex semialgebraic sets by LMIs

A semialgebraic set in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ defined by a boolean combination of polynomial inequalities.

A basic closed semialgebraic set in $\mathbb{R}^n$ is the solution set of a finite system of non-strict polynomial inequalities.

In other words, a set $S \subseteq \mathbb{R}^n$ is a basic closed semialgebraic set if S can be written as

$$S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$$

for some $m \in \mathbb{N}$ and some polynomials $g_1, \dots, g_m \in \mathbb{R}[\bar{X}]$.

Here and throughout the talk $\bar{X} := (X_1, \dots, X_n)$ is an n-tuple of variables and $\mathbb{R}[\bar{X}] := \mathbb{R}[X_1, \dots, X_n]$ denotes the algebra of real polynomials in n variables.

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Describing convex semialgebraic sets by LMIs

Finiteness Theorem.
Every closed semialgebraic set is a finite union of basic closed ones.

Theorem (Bröcker & Scheiderer 1989).
Every basic closed semialgebraic set in $\mathbb{R}^n$ can be defined by a system of at most $\frac{n(n+1)}{2}$ non-strict polynomial inequalities.

The proofs are hard and non-constructive. See, e.g.,
Bochnak & Coste & Roy: Real algebraic geometry, Springer (1998)

Very special cases have been done constructively by vom Hofe, Bernig, Grötschel, Henk, Bosse and Averkov, see, e.g.,
Averkov: Representing elementary semi-algebraic sets by a few polynomial inequalities: A constructive approach
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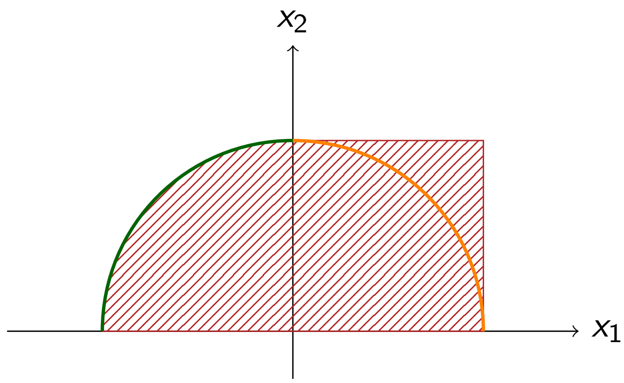
Describing convex semialgebraic sets by LMIs

Example.
$S := (\{(x,y) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\} \cap ([-1,1] \times [0,1])) \cup [0,1]^2$ is closed and semialgebraic but not basic closed. Indeed, by way of contradiction assume $S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$.
Looking at the green points, one of the g_i could be written as $g_i = h \cdot (1 - X_1^2 - X_2^2)^k$ for some odd $k \geq 1$ and $h \in \mathbb{R}[X_1, X_2]$ not divisible by $1 - X_1^2 - X_2^2$.
Looking at the orange points, h would be divisible by $1 - X_1^2 - X_2^2$.



Describing convex semialgebraic sets by LMIs

A convex subset $F \neq \emptyset$ of a convex set S is called a face of S if any line segment $L \subseteq S$ whose relative interior intersects F is actually contained in F.

In particular: If $S \neq \emptyset$, then S is always a face of itself. Any other face of S is contained in the boundary of S. A singleton $F = \{x\}$ is a face of S if and only if x is an extreme point of S.

Proposition. Let $S \subseteq \mathbb{R}^n$ be convex. Then

(a) Any face of a face of S is a face of S.
(b) If F_1, F_2 are faces of S and $F_1 \subsetneq F_2$, then $\dim F_1 < \dim F_2$.
(c) The intersection of any two faces of S is again a face of S.
(d) S is the disjoint union of the relative interiors of its faces.

By a hyperplane, we understand here an affine linear subspace of codimension one in $\mathbb{R}^n$. Any hyperplane divides $\mathbb{R}^n$ into two closed or open half-spaces.

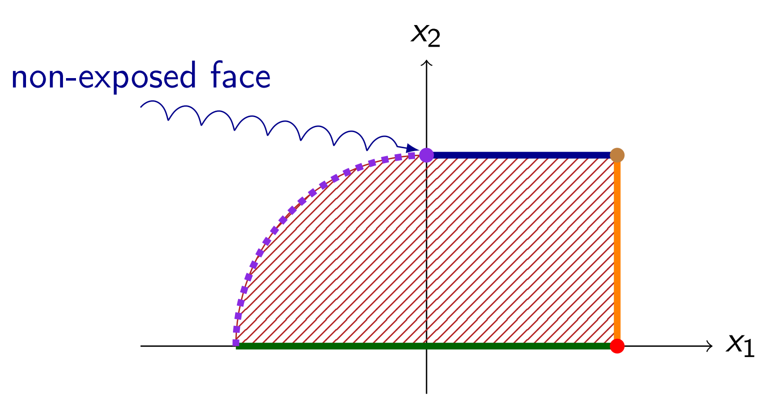
Closed convex sets can be characterized as the intersections of closed half-spaces.

Let $S \subseteq \mathbb{R}^n$ be convex.

A supporting hyperplane of S is a hyperplane H such that $S \cap H \neq \emptyset$ and S is contained entirely in one of the two closed half-spaces determined by H.

If H is a supporting hyperplane of S, then $S \cap H$ is a face of S. These faces as well as S itself are called exposed faces of S.

Example. The faces of $S := (\overline{B_1(0)} \cap ([-1,1] \times [0,1])) \cup [0,1]^2$ are S, $[-1,1] \times \{0\}$, $\{1\} \times [0,1]$, $[0,1] \times \{1\}$, $\{(1,0)\}$, $\{(1,1)\}$ and each point in the second quadrant on the unit circle.
Only one of them is non-exposed, namely $\{(0,1)\}$.



Describing convex semialgebraic sets by LMIs

We will try to describe (in two different ways) convex semialgebraic sets by LMIs.

To define LMIs and for later use, we consider matrix polynomials (also called polynomial matrices), i.e., elements of $\mathbb{R}[\bar{X}]^{s\times t}$.

The degree of a matrix polynomial is the maximal degree of its entries. A linear matrix polynomial is a matrix polynomial of degree at most 1, i.e., of the form $A_0 + X_1A_1 + \cdots + X_nA_n$ for matrices $A_i \in \mathbb{R}^{s\times t}$.

Describing convex semialgebraic sets by LMIs

Let $A \in S\mathbb{R}^{t\times t}$.

$$\begin{aligned}
A \succeq 0 \quad &\iff \quad A \text{ positive semidefinite} \\
&\iff \quad \langle Av, v\rangle \geq 0 \text{ for all } v \in \mathbb{R}^t \\
&\iff \quad \text{all eigenvalues of } A \text{ are } \geq 0 \\
&\iff \quad \text{all coefficients of } \det(A + TI_t) \in \mathbb{R}[T] \text{ are } \geq 0 \\
&\iff \quad \det((A_{ij})_{i,j\in J}) \geq 0 \text{ for all } J \subseteq \{1,\ldots,t\}
\end{aligned}$$

Describing convex semialgebraic sets by LMIs

Let $A \in S\mathbb{R}^{t \times t}$.

$$
\begin{aligned}
A \succ 0 \quad &\iff \quad A \text{ positive } \cancel{\text{semi}}\text{definite} \\
&\iff \quad \langle Av, v \rangle > 0 \text{ for all } v \in \mathbb{R}^t \setminus \{0\} \\
&\iff \quad \text{all eigenvalues of } A \text{ are } > 0 \\
&\iff \quad \text{all coefficients of } \det(A + TI_t) \in \mathbb{R}[T] \text{ are } > 0 \\
&\iff \quad \det((A_{ij})_{i,j \in J}) > 0 \text{ for all } J \subseteq \{1, \dots, t\} \\
&\iff \quad \det((A_{ij})_{i,j \in \{1,\dots,k\}}) > 0 \text{ for all } k \in \{1, \dots, t\}
\end{aligned}
$$

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An inequality of the form

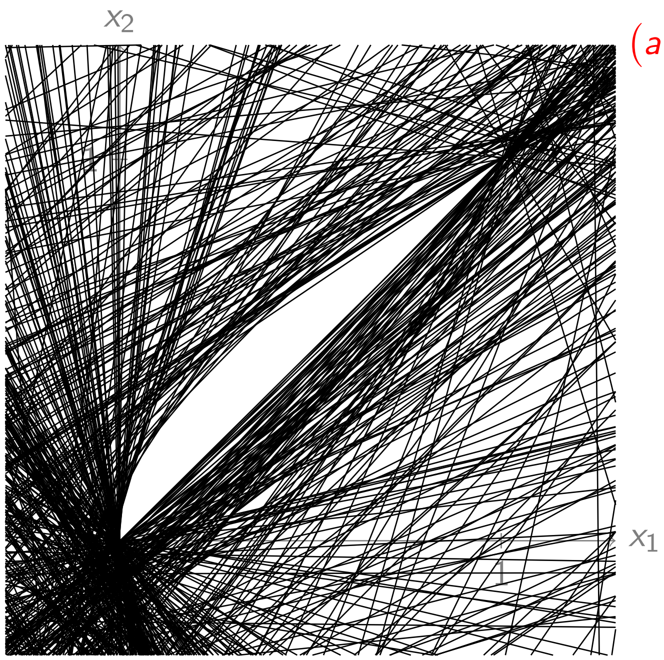
$$A(x) := A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0 \qquad (x \in \mathbb{R}^n)$$

with $A_0, \dots, A_n \in S\mathbb{R}^{t \times t}$ will be called linear matrix inequality.
This corresponds to the family of linear inequalities

$$\langle A(x)v, v \rangle \geq 0 \qquad (x \in \mathbb{R}^n)$$

parametrized by $v \in \mathbb{R}^t$.

Describing convex semialgebraic sets by LMIs



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_1 \\ x_2 & 1 & x_1 \\ x_1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

Describing convex semialgebraic sets by LMIs

Three ways to say what is a polyhedron:

- It is the solution set of a finite system of linear inequalities.
- It is the set of all $x \in \mathbb{R}^n$ such that $A(x) \succeq 0$ for a diagonal linear matrix polynomial.
- It is the intersection of finitely many closed half-spaces.

Two and a half ways to define a spectrahedron:

- It is the solution set of an LMI.
- It is the set of all $x \in \mathbb{R}^n$ such that $A(x) \succeq 0$ for a symmetric linear matrix polynomial.
- It is the intersection of a "nicely parametrized" family of closed half-spaces.

Polyhedra are easy to deal with algorithmically. For example, you can use linear programming to optimize a given linear function on them.

Spectrahedra seem to be easy to deal with algorithmically.
For example, you can use semidefinite programming to optimize a given linear function on them.

Describing convex semialgebraic sets by LMIs

Based on diagonalization of symmetric matrices, spectrahedra share many good properties with polyhedra.

While projections of polyhedra are still polyhedra, projections of spectrahedra are convex and semialgebraic but nothing else is known about them.

In recent years, results of Helton & Vinnikov as well as Helton & Nie showed that surprisingly many convex semialgebraic sets are spectrahedra or projections of spectrahedra.

Describing convex semialgebraic sets by LMIs

Let $S \subseteq \mathbb{R}^n$.

We call a symmetric linear matrix polynomial $A \in S\mathbb{R}[\bar{X}]^{t\times t}$ an LMI representation of S if

$$S = \{x \in \mathbb{R}^n \mid A(x) \succeq 0\}.$$

If $\bar{Y}$ is an m-tuple of additional variables, then we call a symmetric linear matrix polynomial $A \in S\mathbb{R}[\bar{X}, \bar{Y}]^{t\times t}$ a semidefinite representation of S if

$$S = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m : A(x, y) \succeq 0\}.$$

Hence S is a spectrahedron if and only if it is LMI representable, and S is a projection of a spectrahedron if and only if it is semidefinitely representable.

Describing convex semialgebraic sets by LMIs

If V is a finite-dimensional $\mathbb{R}$-vector space, one can identify V with $\mathbb{R}^n$ by fixing a basis.

Then one can speak about the properties of a set $S \subseteq V$ being open, closed, semialgebraic, basic open, basic closed, bounded, convex, a spectrahedron, semidefinitely representable and so on.

All these notions are unambigously defined since they do not depend on the chosen basis as the change of bases is given by an invertible linear map.

Describing convex semialgebraic sets by LMIs

This talk is divided into two parts:

Part I. Spectrahedra
This will lead us to determinantal representations of polynomials.

Part II. Semidefinitely representable sets
This will lead us to sums of squares representations of polynomials.

Part I. Spectrahedra

Spectrahedra and their properties

Let $S \subseteq \mathbb{R}^n$ be a spectrahedron. Then

- S is convex,
- S is a basic closed semialgebraic set, and
- all faces of S are exposed.

Indeed, if $A \in S\mathbb{R}[\bar{X}]^{t \times t}$ is a symmetric linear matrix polynomial such that $S = \{x \in \mathbb{R}^n \mid A(x) \succeq 0\}$, then it is an exercise to show that every face of S is of the form $\{x \in S \mid U \subseteq \ker A(x)\}$ where U is a linear subspace of $\mathbb{R}^n$. But if $U = \mathbb{R}u_1 + \dots + \mathbb{R}u_k$, then

$$\{x \in S \mid U \subseteq \ker A(x)\} = \{x \in S \mid \langle A(x)u_1, u_1\rangle + \dots + \langle A(x)u_k, u_k\rangle = 0\}$$

is empty or an exposed face of S since

$$S \subseteq \{x \in \mathbb{R}^n \mid \langle A(x)u_1, u_1\rangle + \dots + \langle A(x)u_k, u_k\rangle \geq 0\}.$$

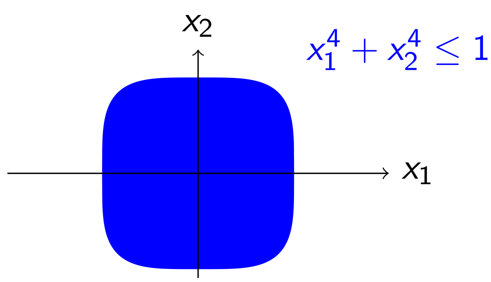
Spectrahedra and their properties

Let $S \subseteq \mathbb{R}^n$ be a spectrahedron. Then

- S is convex,
- S is a basic closed semialgebraic set, and
- all faces of S are exposed.

This three properties do not characterize spectrahedra. We will now learn about another property of polyhedra called rigid convexity which is strictly stronger and which is conjectured to characterize spectrahedra.

The basic closed semialgebraic set $\{x \in \mathbb{R}^2 \mid x_1^4 + x_2^4 \leq 1\}$ is convex and has only exposed faces but we will see that it is not a spectrahedron. The reason for this will be that it is not rigidly convex.



Towards a characterization of spectrahedra

In the following, we will define a condition called rigid convexity for convex sets S with non-empty interior. If such S is rigidly convex, then it will be a basic closed semialgebraic convex set with only exposed faces, and it is conjectured that it is even a spectrahedron.

A convex set has always non-empty interior in its affine hull. By identifying this affine hull with $\mathbb{R}^k$ (for some $k \leq n$), one could define rigid convexity for all convex sets.

Thus the assumption that the interior of S is non-empty is not essential and just made for simplicity.

Towards a characterization of spectrahedra

Let $S \subseteq \mathbb{R}^n$ be a spectrahedron and $x_0 \in S^\circ$. Then one can find $A \in S\mathbb{R}[\bar{X}]^{t\times t}$ with $A(x_0) \succ 0$ such that $S = \{x \in \mathbb{R}^n \mid A(x) \succeq 0\}$. Given such an LMI representation, let

- $p := \det A \in \mathbb{R}[\bar{X}]$ and
- C the connected component of x_0 in $\{x \in \mathbb{R}^n \mid p(x) > 0\}$.

Then $S = \overline{C}$ and p is a real zero polynomial at x_0 in the following sense:

$$\boxed{p(x_0) > 0 \quad \& \quad \forall x \in \mathbb{R}^n\colon \forall \lambda \in \mathbb{C}\colon (p(x_0+\lambda x) = 0 \implies \lambda \in \mathbb{R})}$$

Why? Without loss of generality $x_0 = 0$. Then we have $A_i \in S\mathbb{R}^{t\times t}$ with $A_0 \succ 0$ such that $p = \det A = \det(A_0 + X_1A_1 + \dots + X_nA_n)$. Let $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} 0 &= p(x_0+\lambda x) = p(0+\lambda x) = \det(A(\lambda x)) \\ &= \det(A_0 + \lambda(x_1A_1 + \dots + x_nA_n)) \\ &= \det(P^*(A_0 + \lambda(x_1A_1 + \dots + x_nA_n))P) && (P \in \mathbb{R}^{t\times t}) \\ &= \det(P^*A_0P + \lambda P^*(x_1A_1 + \dots + x_nA_n)P) && (P^*A_0P = I_t) \\ &= \det(I_t + \lambda B) && (B \in S\mathbb{R}^{t\times t}) \end{aligned}$$

and therefore $\det(B + \frac{1}{\lambda}I_t) = 0$ whence $-\frac{1}{\lambda} \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.

Towards a characterization of spectrahedra

Let $S \subseteq \mathbb{R}^n$ be a spectrahedron and $x_0 \in S^\circ$. Then one can find $A \in S\mathbb{R}[\bar{X}]^{t\times t}$ with $A(x_0) \succ 0$ such that $S = \{x \in \mathbb{R}^n \mid A(x) \succeq 0\}$. Given such an LMI representation, let

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S is an algebraic interior in the following sense:

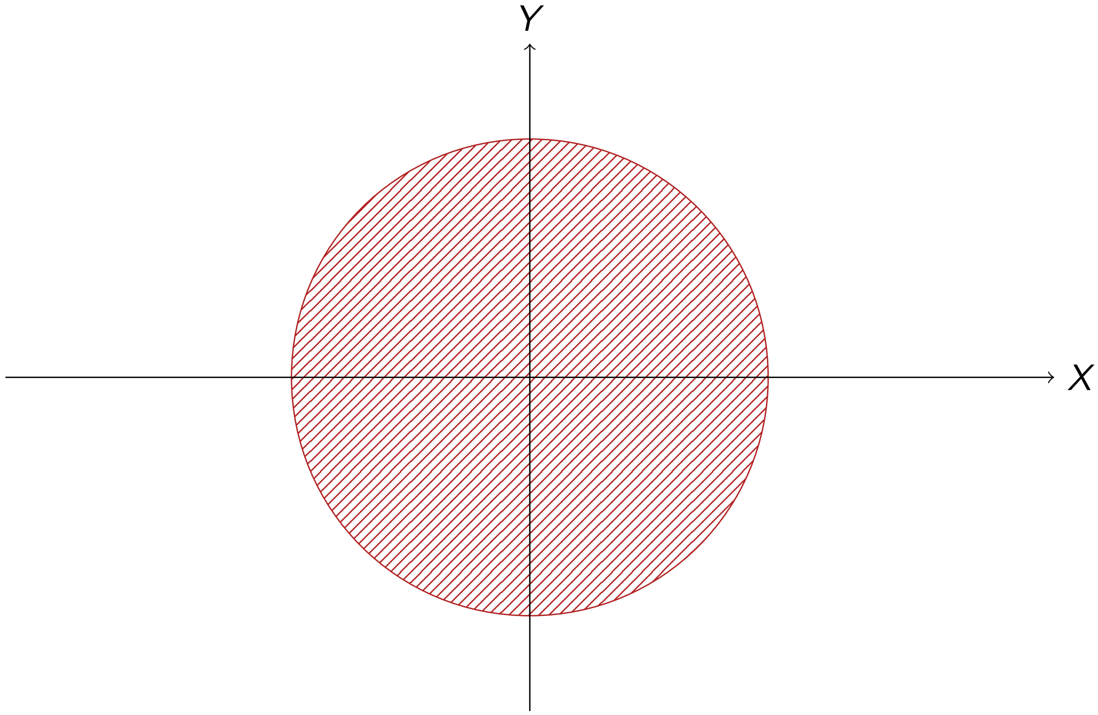
$$\boxed{\exists p \in \mathbb{R}[\bar{X}]\colon \exists \text{ connected component } C \text{ of } \{x \in \mathbb{R}^n \mid p(x) > 0\}\colon S = \overline{C}}$$

If the degree of p is minimal, we call p the minimal polynomial of S (unique up to constant factor $c > 0$). The minimal polynomial of S divides in $\mathbb{R}[\bar{X}]$ every other polynomial p of this kind. In particular, our spectrahedron S is rigidly convex in the following sense:

$$\boxed{S \text{ is an algebraic interior} \quad \& \quad \exists x_0 \in S^\circ\colon \text{min. pol. of } S \text{ is RZ at } x_0}$$

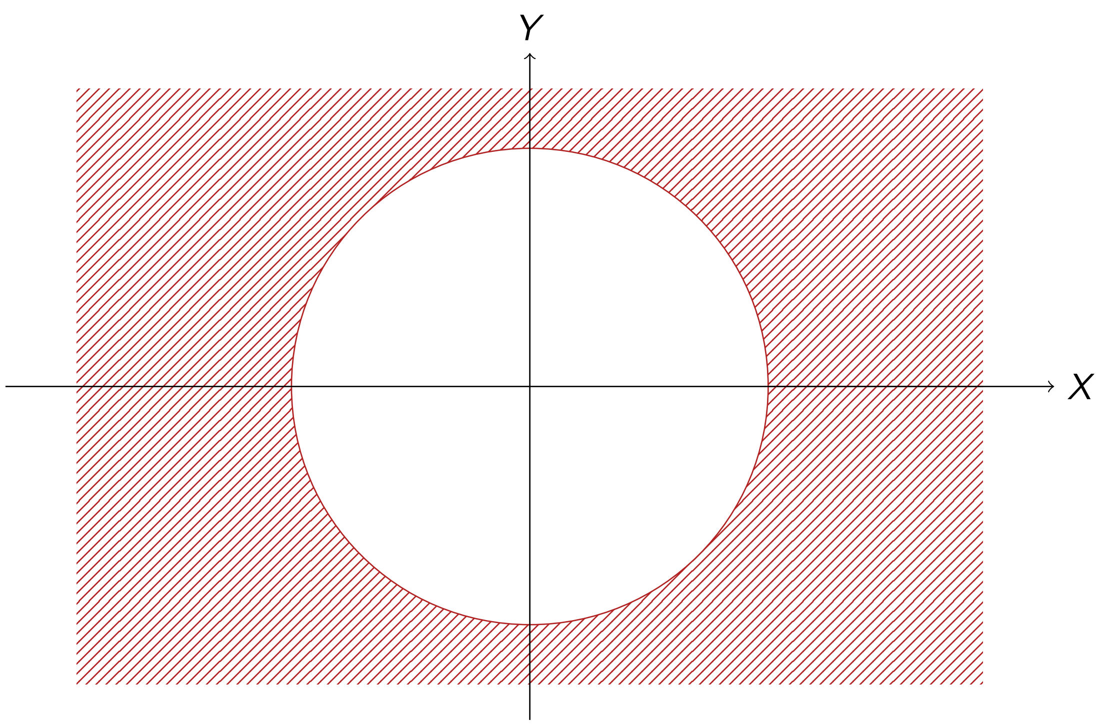
Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $1 - X^2 - Y^2$, rigidly convex



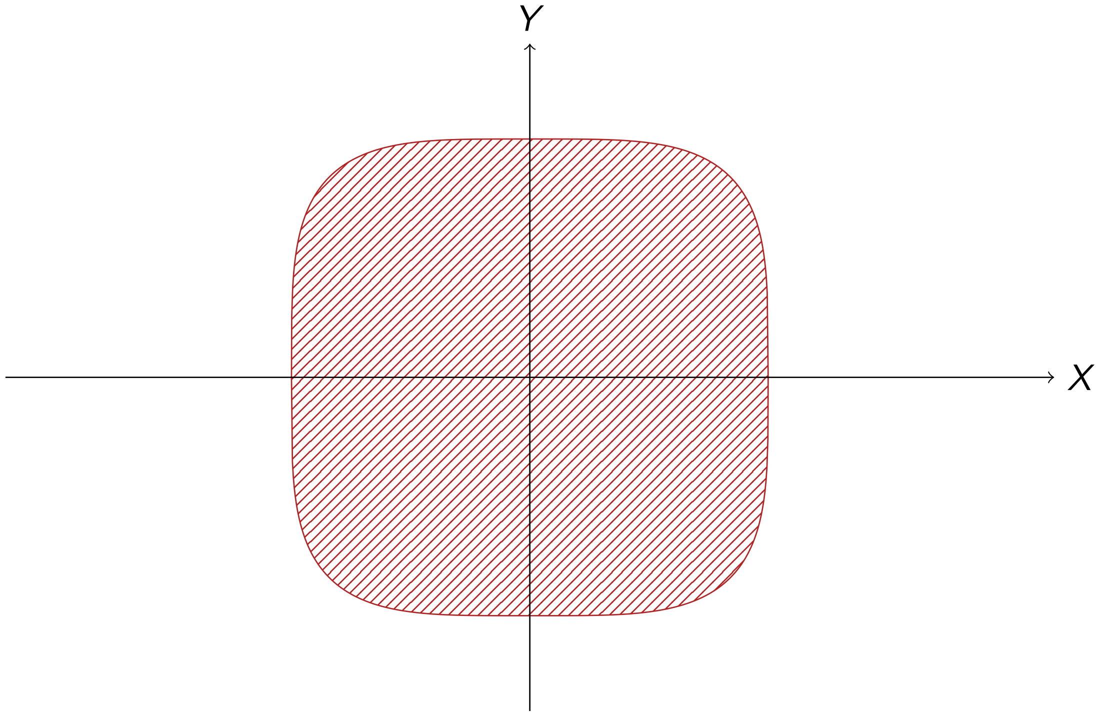
Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $X^2 + Y^2 - 1$, not rigidly convex



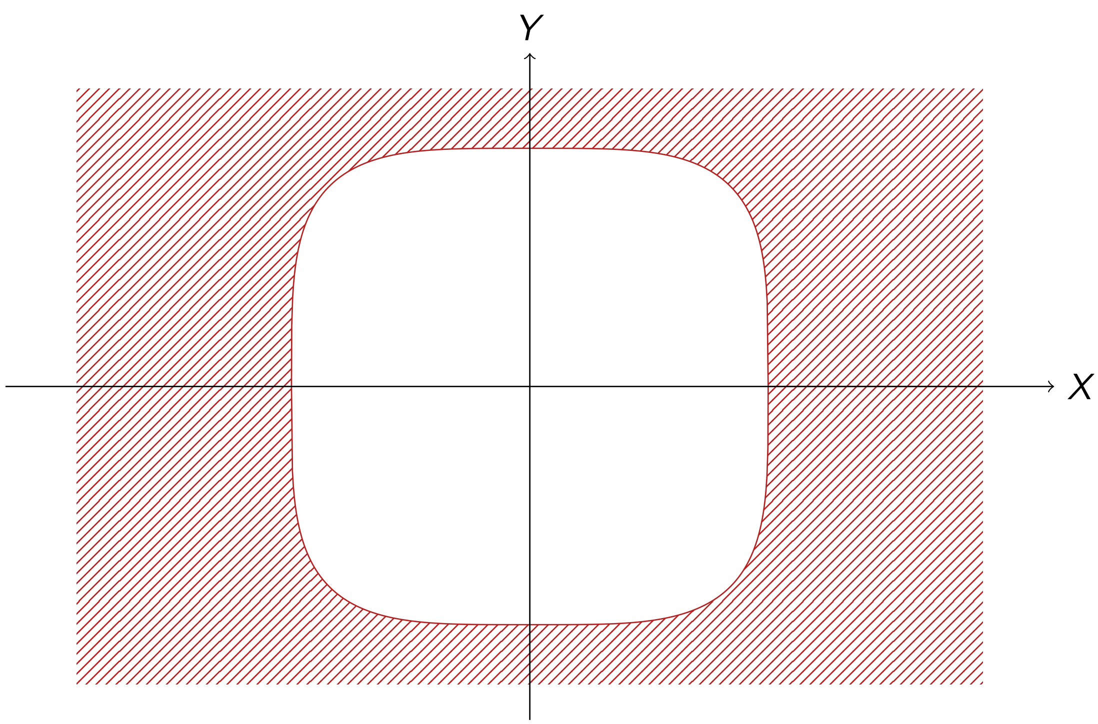
Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $1 - X^4 - Y^4$, not rigidly convex



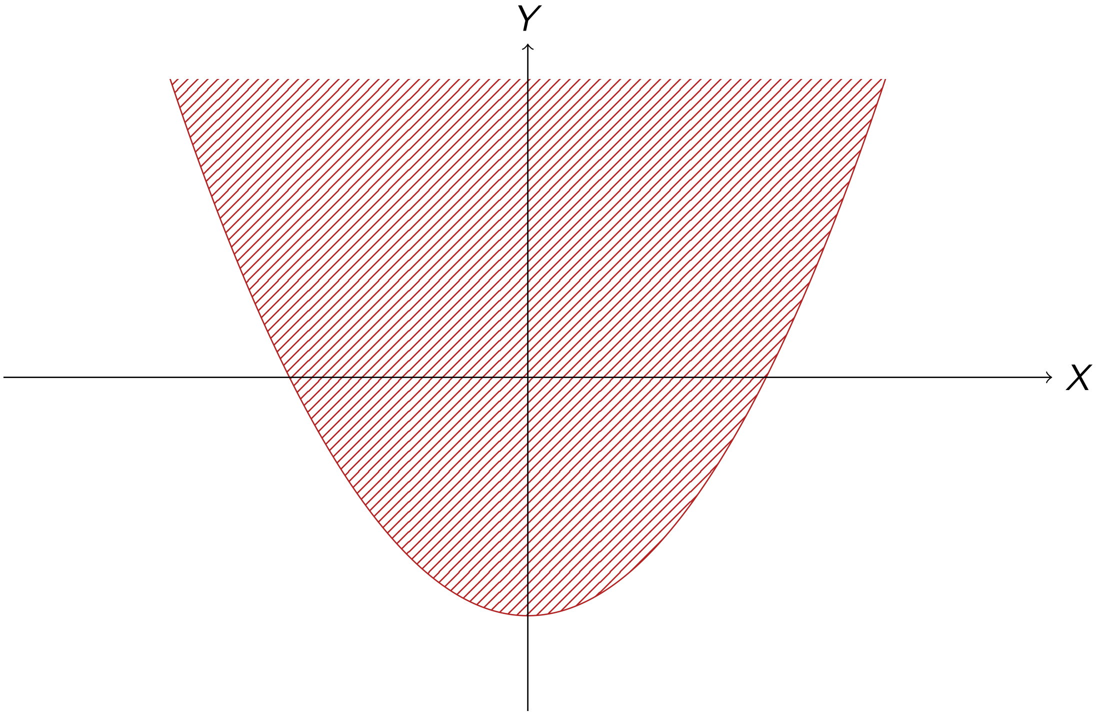
Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $X^4 + Y^4 - 1$, not rigidly convex



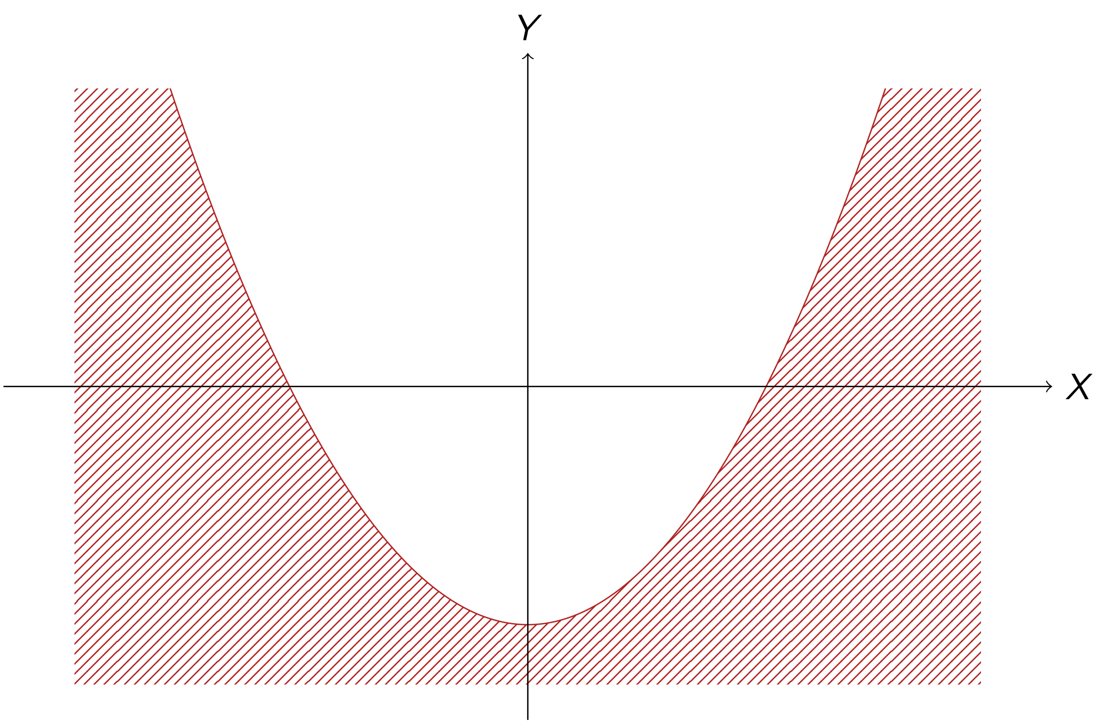
Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $Y - X^2 - 1$, rigidly convex



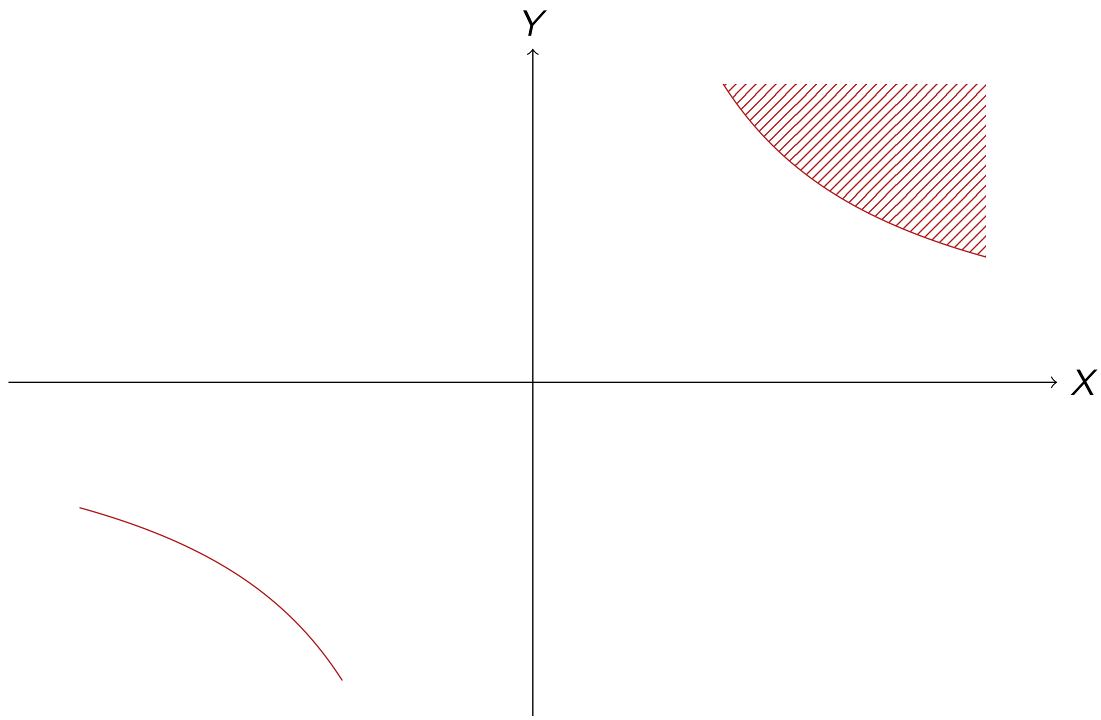
Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $-Y + X^2 + 1$, not rigidly convex



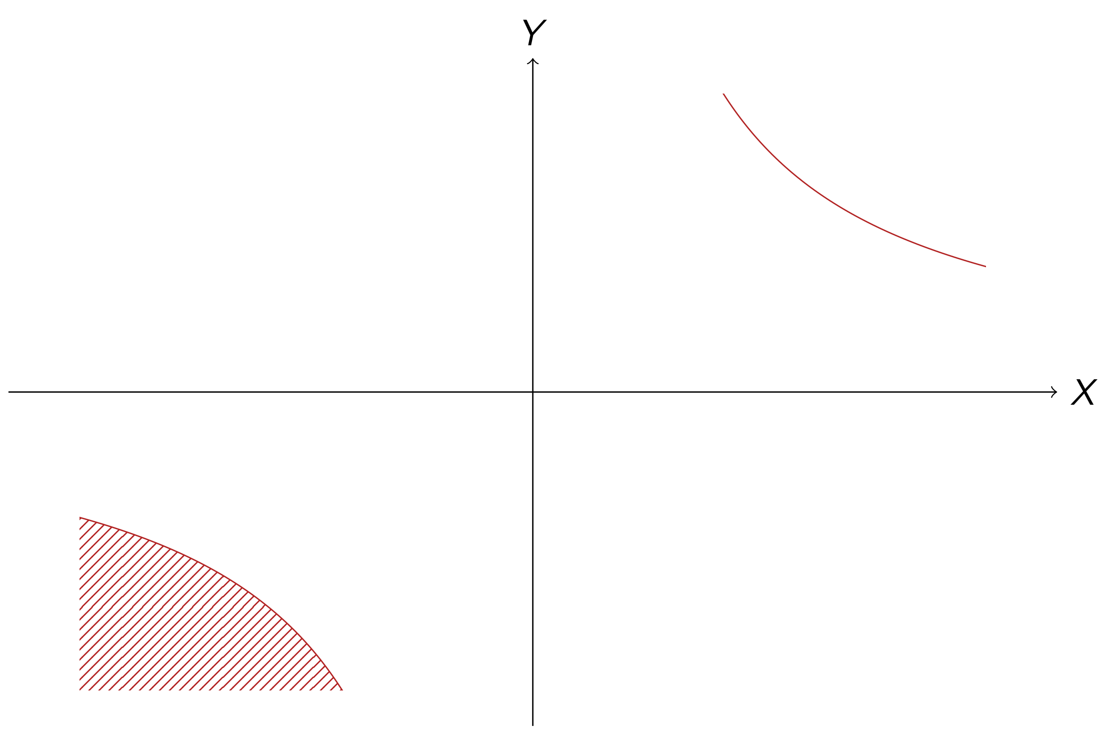
Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $XY - 1$, rigidly convex



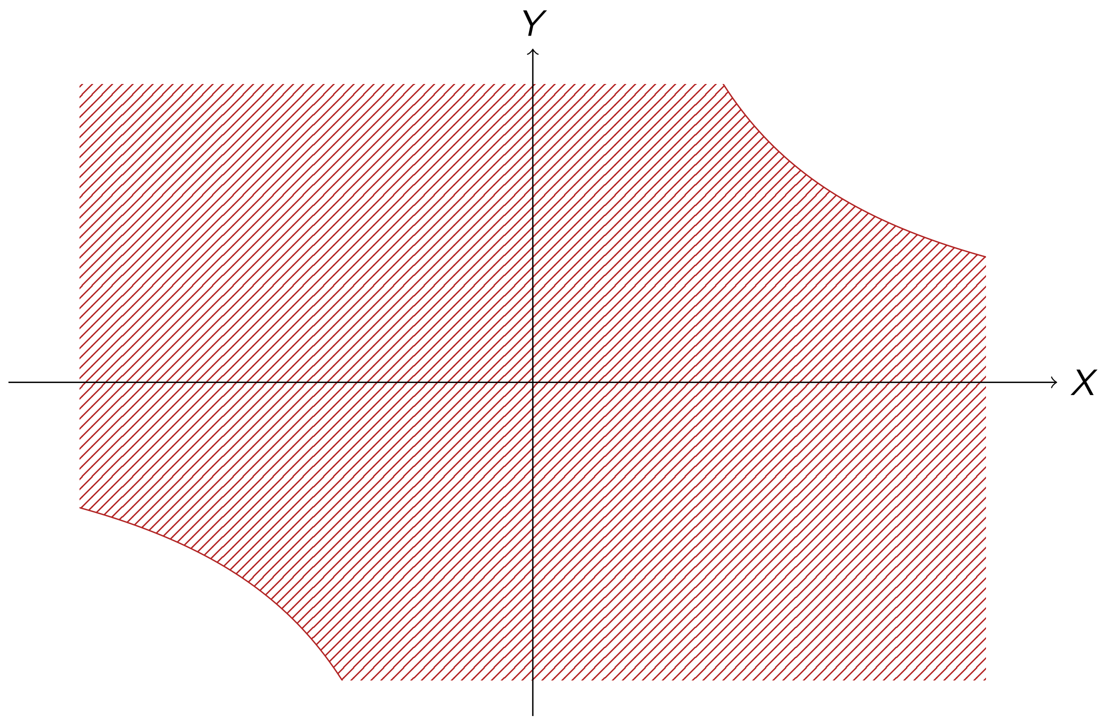
Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $XY - 1$, rigidly convex



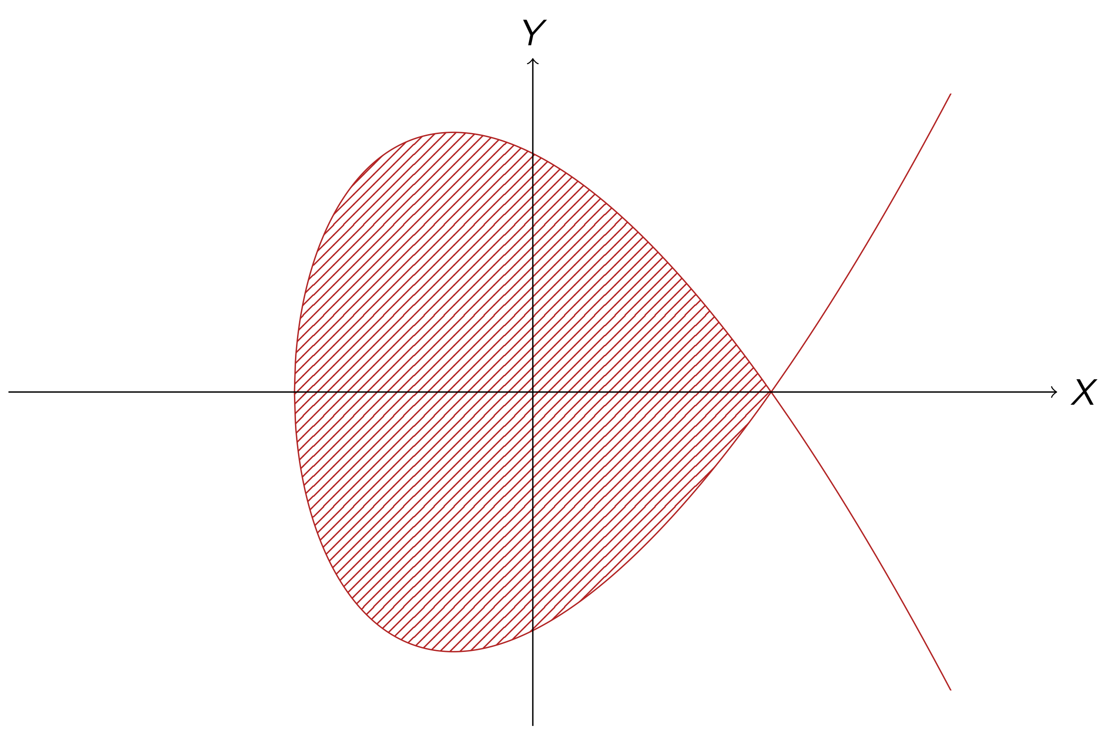
Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $1 - XY$, not rigidly convex



Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $X^3 - X^2 - X - Y^2 + 1$, rigidly convex



Algebraic interiors, minimal polynomials and rigid convexity

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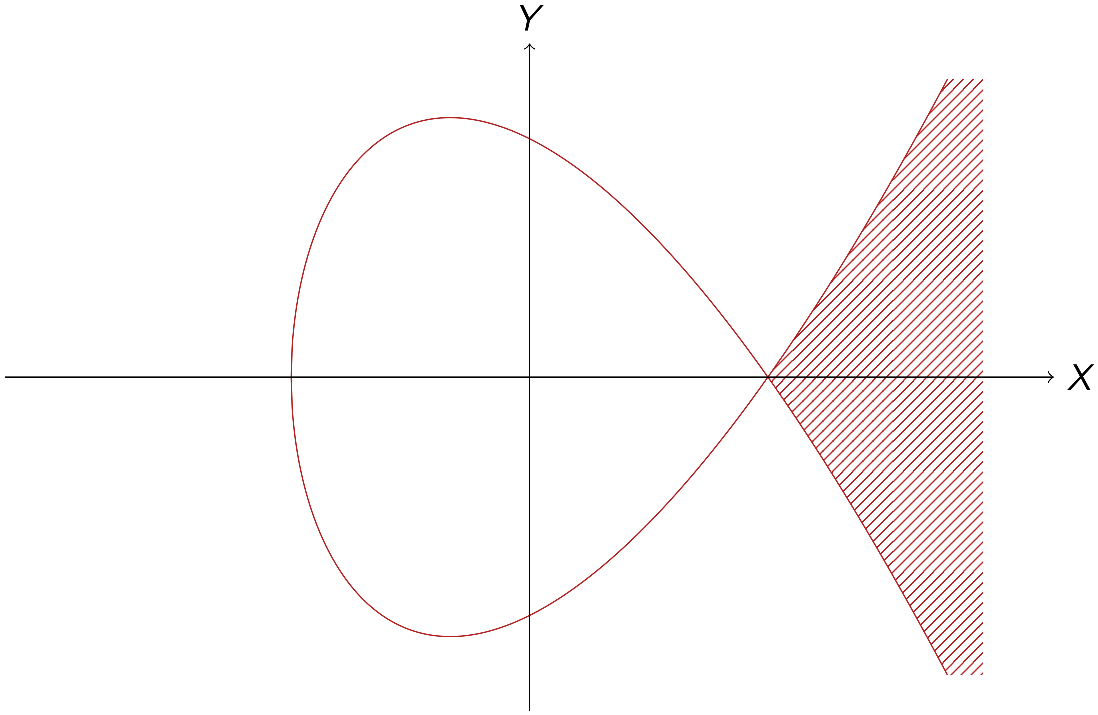
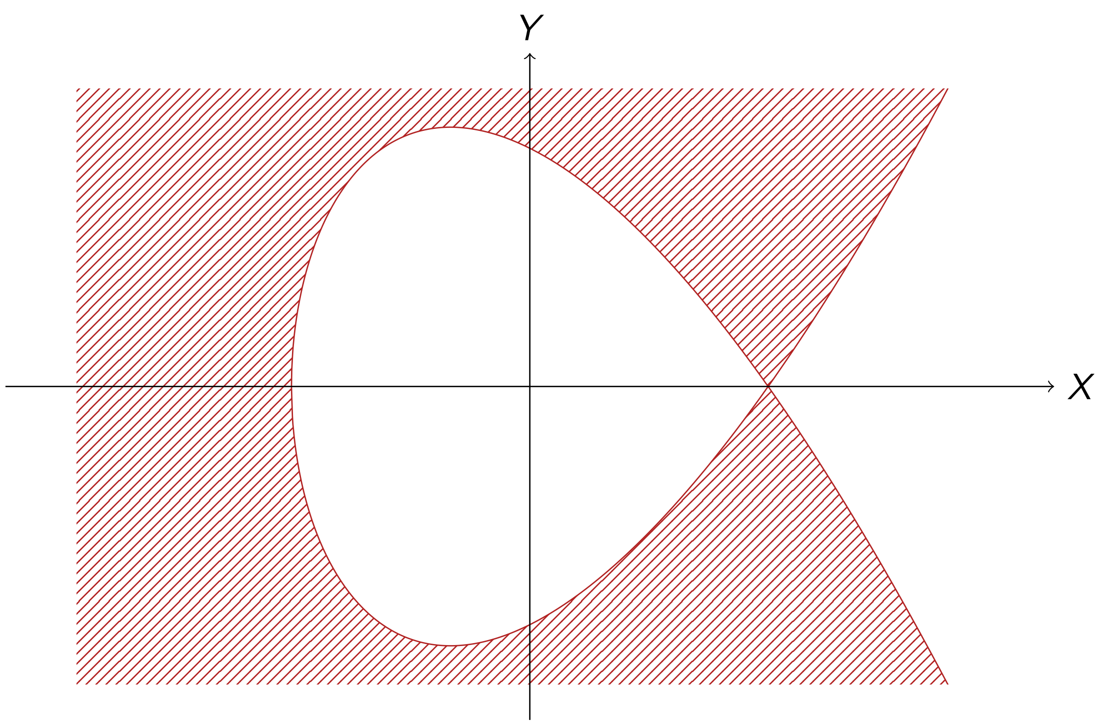


Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $-X^3 + X^2 + X + Y^2 - 1$, not rigidly convex



Towards a characterization of spectrahedra

Proposition (Gårding 1959). If S is rigidly convex, then the minimal polynomial of S has the real zero property at all $x_0 \in S^\circ$ and S is convex.

We have seen that a spectrahedron with non-empty interior is rigidly convex. The first big question of the talk is if the converse is true.

Theorem (Helton & Vinnikov 2007).
Every rigidly convex set $S \subseteq \mathbb{R}^2$ is a spectrahedron.

This is a consequence of the 1958 Lax conjecture:

Lax conjecture (1958)
For all $p \in \mathbb{R}[X_1, X_2]$ RZ at 0 of degree d, there exist $A_i \in S\mathbb{R}^{d \times d}$ such that $A_0 \succ 0$ and $p = \det(A_0 + X_1 A_1 + X_2 A_2)$.

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Conjecture (Helton & Vinnikov 2007).
Every rigidly convex set $S \subseteq \mathbb{R}^n$ is a spectrahedron.

This would be a consequence of the generalized Lax conjecture:

Conjecture (Helton & Vinnikov 2007).
For all $p \in \mathbb{R}[\bar{X}]$ RZ at 0, there exist $t \in \mathbb{N}$ and $A_i \in S\mathbb{R}^{t\times t}$ such that $A_0 \succ 0$ and $p = \det(A_0 + X_1A_1 + \cdots + X_nA_n)$.

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Proposition (Gårding 1959). If S is rigidly convex, then the minimal polynomial of S has the real zero property at all $x_0 \in S^\circ$ and S is convex.

We have seen that a spectrahedron with non-empty interior is rigidly convex. The first big question of the talk is if the converse is true.

Conjecture (Helton & Vinnikov 2007).
Every rigidly convex set $S \subseteq \mathbb{R}^n$ is a spectrahedron.

Theorem (Helton & McCullough & Vinnikov 2006).
For all $p \in \mathbb{R}[\bar{X}]$, there exist $t \in \mathbb{N}$ and $A_i \in S\mathbb{R}^{t\times t}$ such that $p = \det(A_0 + X_1A_1 + \cdots + X_nA_n)$.

New demonstration bypassing polynomials in non-commuting variables and giving an explicit construction: Quarez

Literature on rigid convexity and determinantal representations of real zero polynomials

Helton & Vinnikov: Linear matrix inequality representation of sets
Comm. Pure Appl. Math. 60 (2007), no. 5, 654–674
http://arxiv.org/abs/math.OC/0306180
http://dx.doi.org/10.1002/cpa.20155

Lewis & Parrilo & Ramana: The Lax conjecture is true
Proc. Amer. Math. Soc. 133 (2005), no. 9, 2495–2499
http://arxiv.org/abs/math.OC/0304104
http://dx.doi.org/10.1090/S0002-9939-05-07752-X

Literature on determinantal representations of arbitrary polynomials

Helton & McCullough & Vinnikov: Noncommutative convexity arises from linear matrix inequalities
J. Funct. Anal. 240 (2006), no. 1, 105–191 http://math.ucsd.edu/~helton/osiris/NONCOMMINEQ/convRat.ps
http://dx.doi.org/10.1016/j.jfa.2006.03.018

Quarez: Symmetric determinantal representation of polynomials
http://hal.archives-ouvertes.fr/hal-00275615/fr/

Trivial determinantal representations in one variable

Determinantal representations in several variables go far beyond the scope of this talk. But as an example, we take a closer look at the case of one variable.

By factorization of univariate polynomials over $\mathbb{R}$ into linear and quadratic factors, it is clear that each univariate polynomial has a determinantal representation (useless in practice) since

$$\det \begin{pmatrix} c & X-a & 0 \\ X-a & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (X-a)^2 + c.$$

If $p \in \mathbb{R}[X]$ is a real zero polynomial, i.e., $p(0) > 0$ and $p = \prod_{i=1}^{d} c(X - a_i)$ for some $a_i, c \in \mathbb{R}$, then

$$p = p(0) \prod_{i=1}^{d} (1 - \frac{1}{a_i} X) = p(0) \det \left(I_d - X \operatorname{Diag}\left(\frac{1}{a_1}, \ldots, \frac{1}{a_d} \right) \right).$$

Effective determinantal representations in one variable

Given a polynomial $p \in \mathbb{Q}[X]$ of degree $d = r + 2s$ with at least r real zeros (counted with multiplicity), Quarez constructs $A \in S\mathbb{Q}^{d \times d}$ such that $p = \det(J + XA)$ where $J = \operatorname{Diag}(\underbrace{1, \ldots, 1}_{r \text{ times}}, \underbrace{1, -1, \ldots, 1, -1}_{s \text{ times}})$.

Theorem (Quarez). If $p \in \mathbb{R}[X]$ is of degree $d = r + 2s$ with $p(0) \neq 0$. Then p possesses at least r real zeros if and only if there is $A \in S\mathbb{R}^{d \times d}$ such that $p = \det(J + XA)$ with $J = \operatorname{Diag}(\underbrace{1, \ldots, 1}_{r \text{ times}}, \underbrace{1, -1, \ldots, 1, -1}_{s \text{ times}})$.

Quarez: Sturm and Sylvester algorithms revisited via tridiagonal determinantal representations
`http://hal.archives-ouvertes.fr/hal-00338925/fr/`

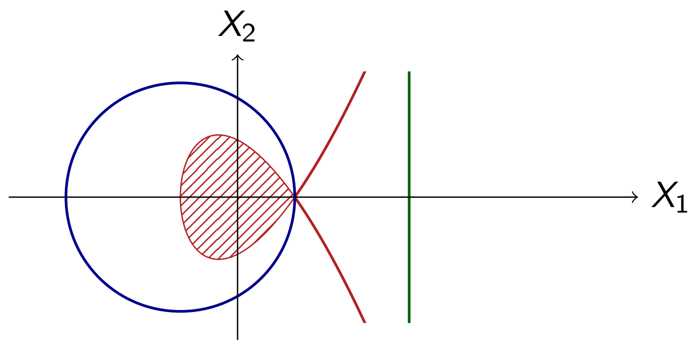
Quarez: Représentations déterminantales effectives des polynômes univariés par les matrices flèches
`http://hal.archives-ouvertes.fr/hal-00318578/fr/`

Definition. Let $p \in \mathbb{R}[\bar{X}]$ be a real zero polynomial of degree d. Then we call

$$R^k p := \frac{\partial^k}{\partial X_0^k} X_0^d \, p\left(\frac{\bar{X}}{X_0}\right)\bigg|_{X_0=1}$$

the k-th Renegar derivative of p. Attention: $R^2 \neq R \circ R$.

Example. Let $p = X_1^3 - X_1^2 - X_1 - X_2^2 + 1 \in \mathbb{R}[X_1, X_2]$. Then p is a real zero polynomial (see picture) and its Renegar derivatives are $Rp = -X_1^2 - 2X_1 - X_2^2 + 3$ and $R^2 p = -2X_1 + 6$.



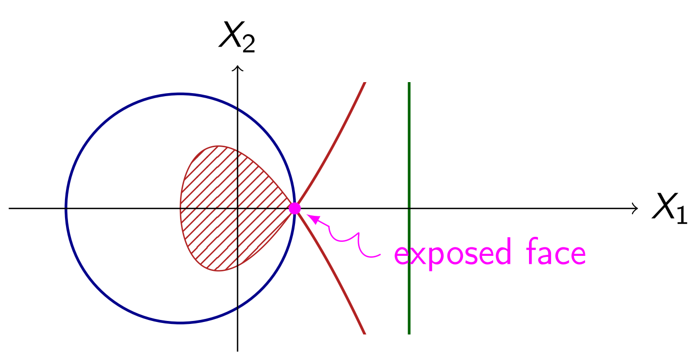
Theorem (Renegar 2006). Let $S \subseteq \mathbb{R}^n$ be rigidly convex with $0 \in S^\circ$ and minimal polynomial p of degree d. Then each $R^k p$ ($k \in \{0, \ldots, d-1\}$) is a real zero polynomial, and the connected components $S^{(k)}$ of 0 in $\{x \in \mathbb{R}^n \mid R^k(x) > 0\}$ form an ascending chain

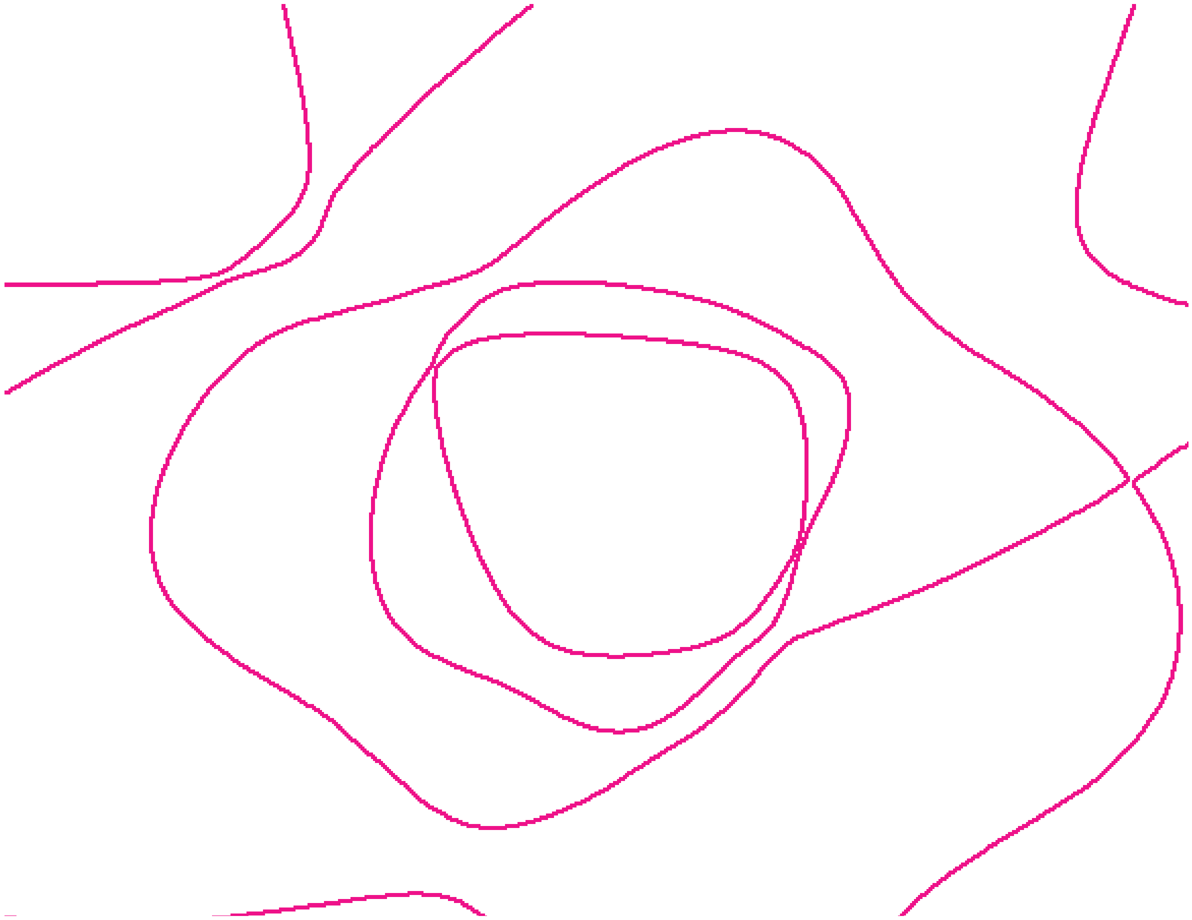
$$S = S^{(0)} \subseteq S^{(1)} \subseteq S^{(2)} \subseteq \cdots \subseteq S^{(d-1)}.$$

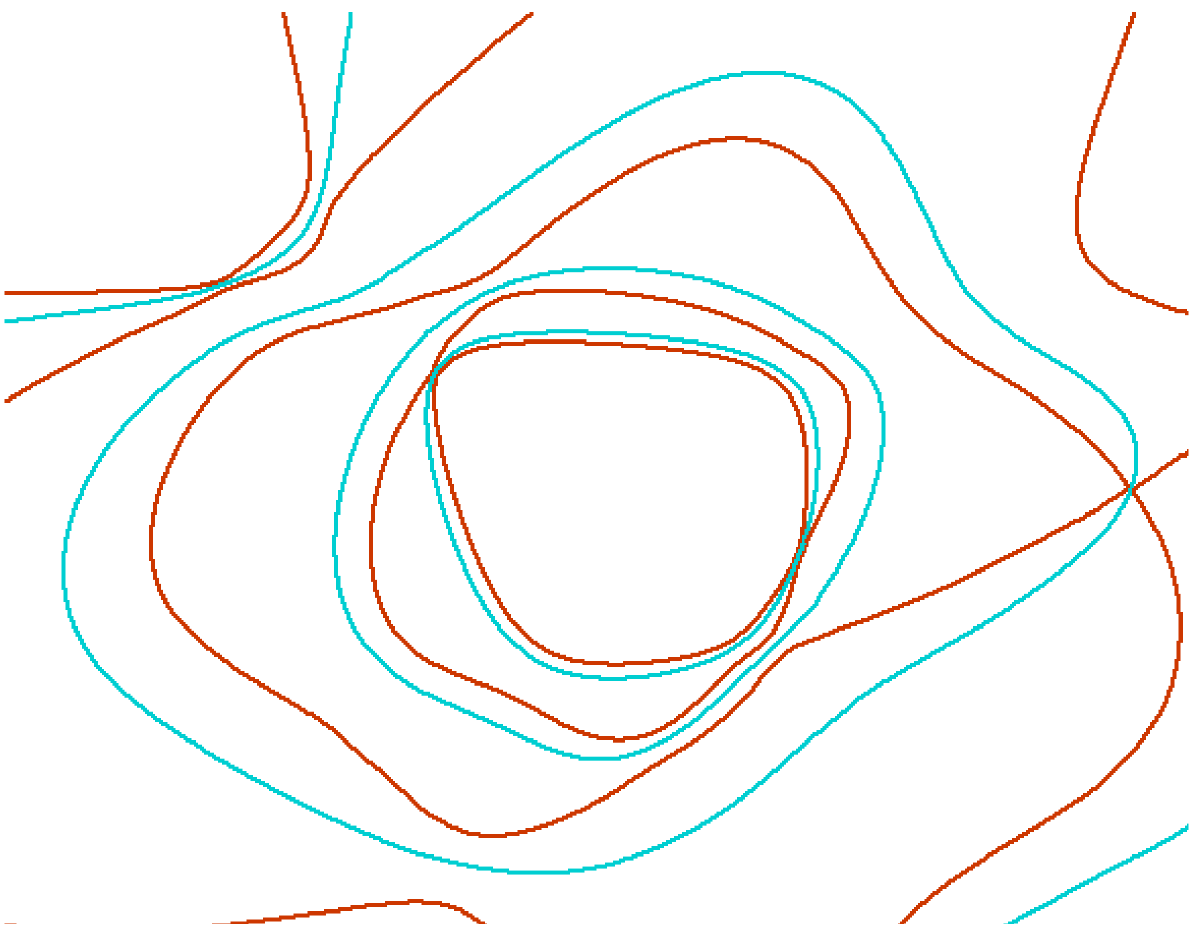
Moreover, S is basic closed and has only exposed faces. More precisely,

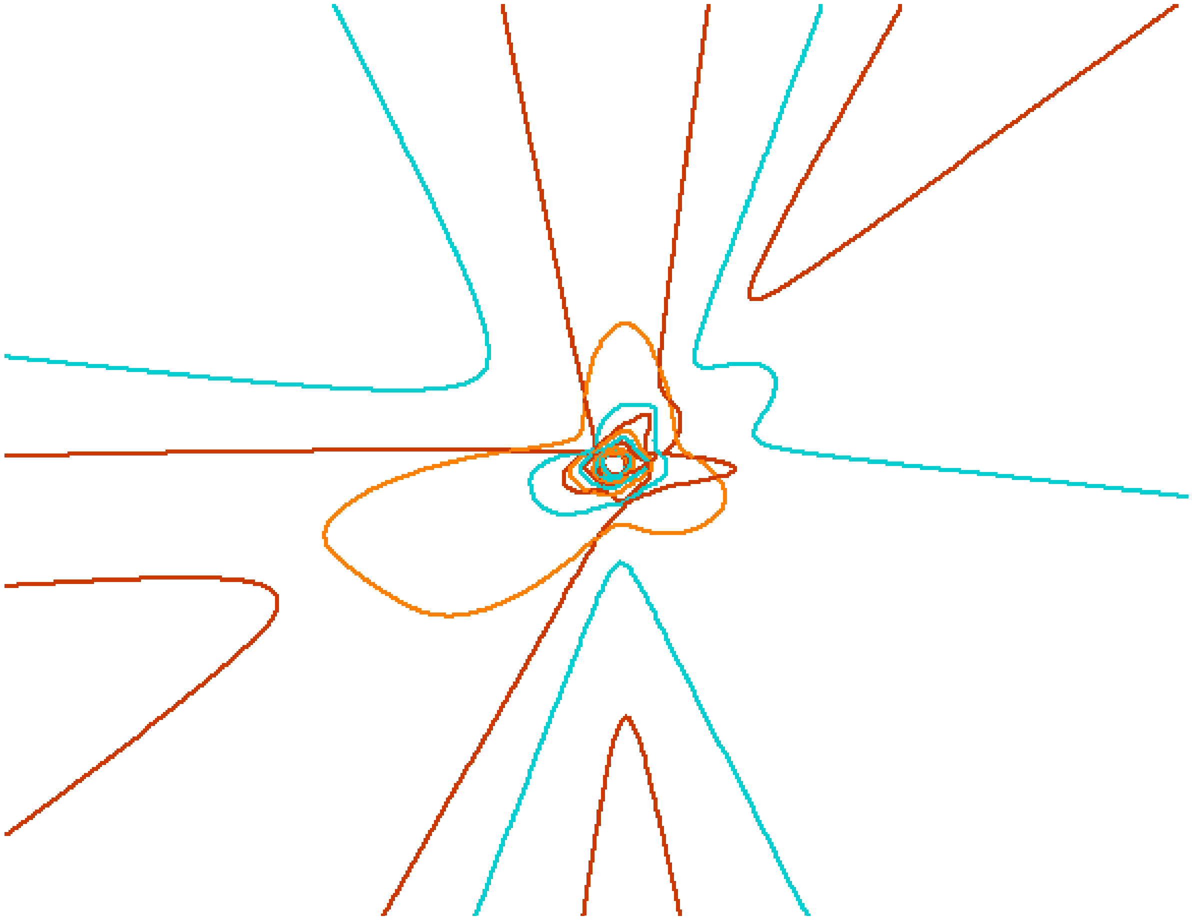
$$S = \{x \in \mathbb{R}^n \mid p(x) \geq 0, Rp(x) \geq 0, \ldots, R^{d-1}p(x) \geq 0\},$$

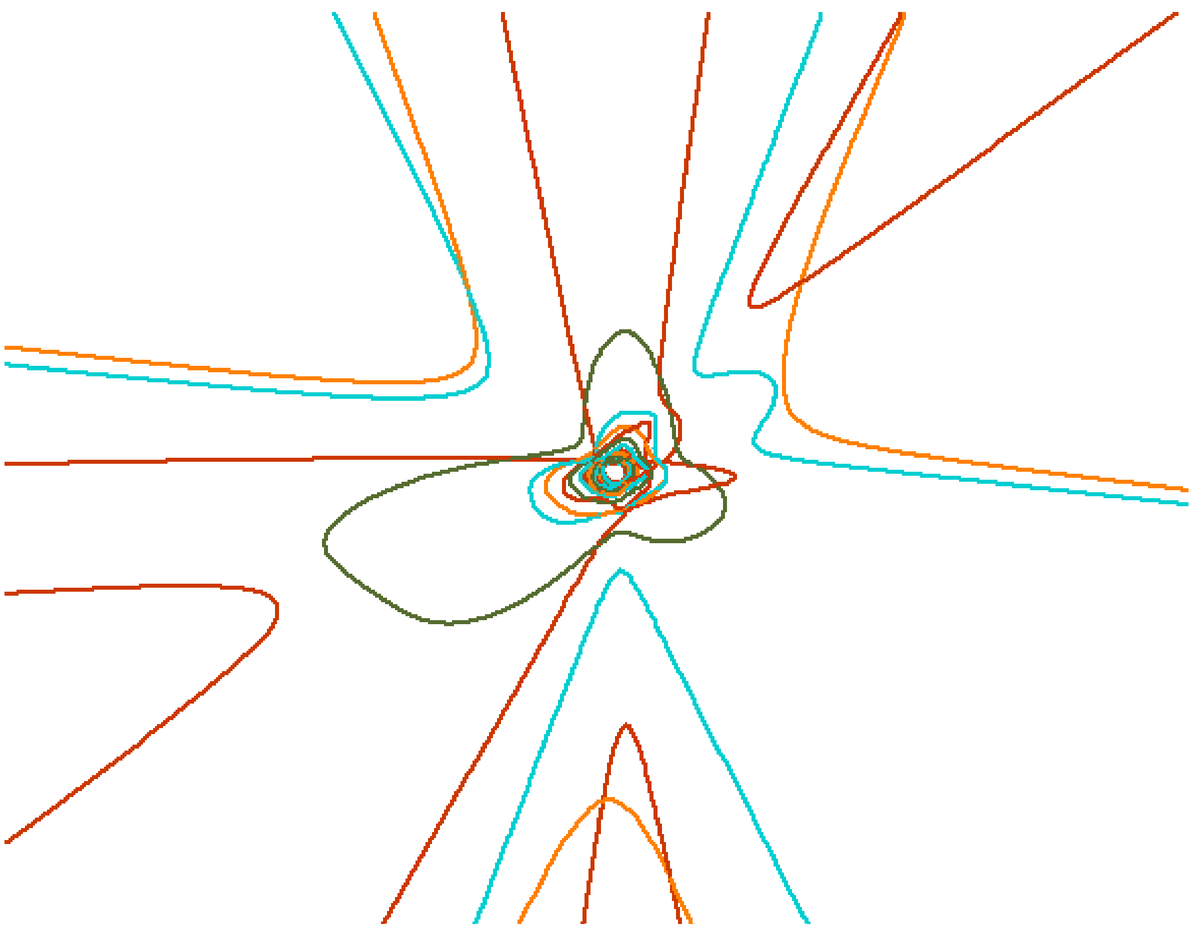
and for $x \in \partial S$ and $k \in \{0, \ldots, d-1\}$ maximal such that $x \in \partial S^{(k)}$, there is a unique supporting hyperplane of $S^{(k)}$ at x, and this hyperplane exposes the face in whose relative interior lies x.

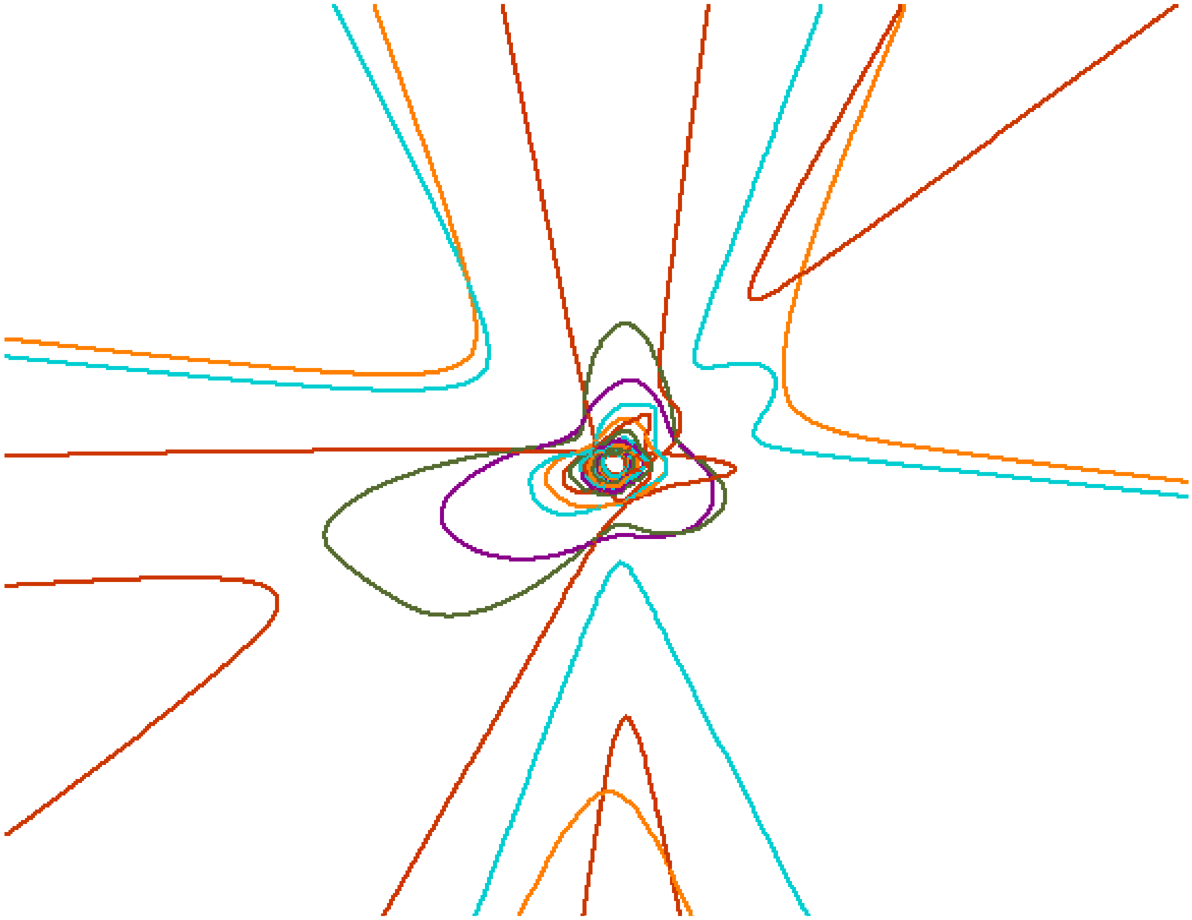


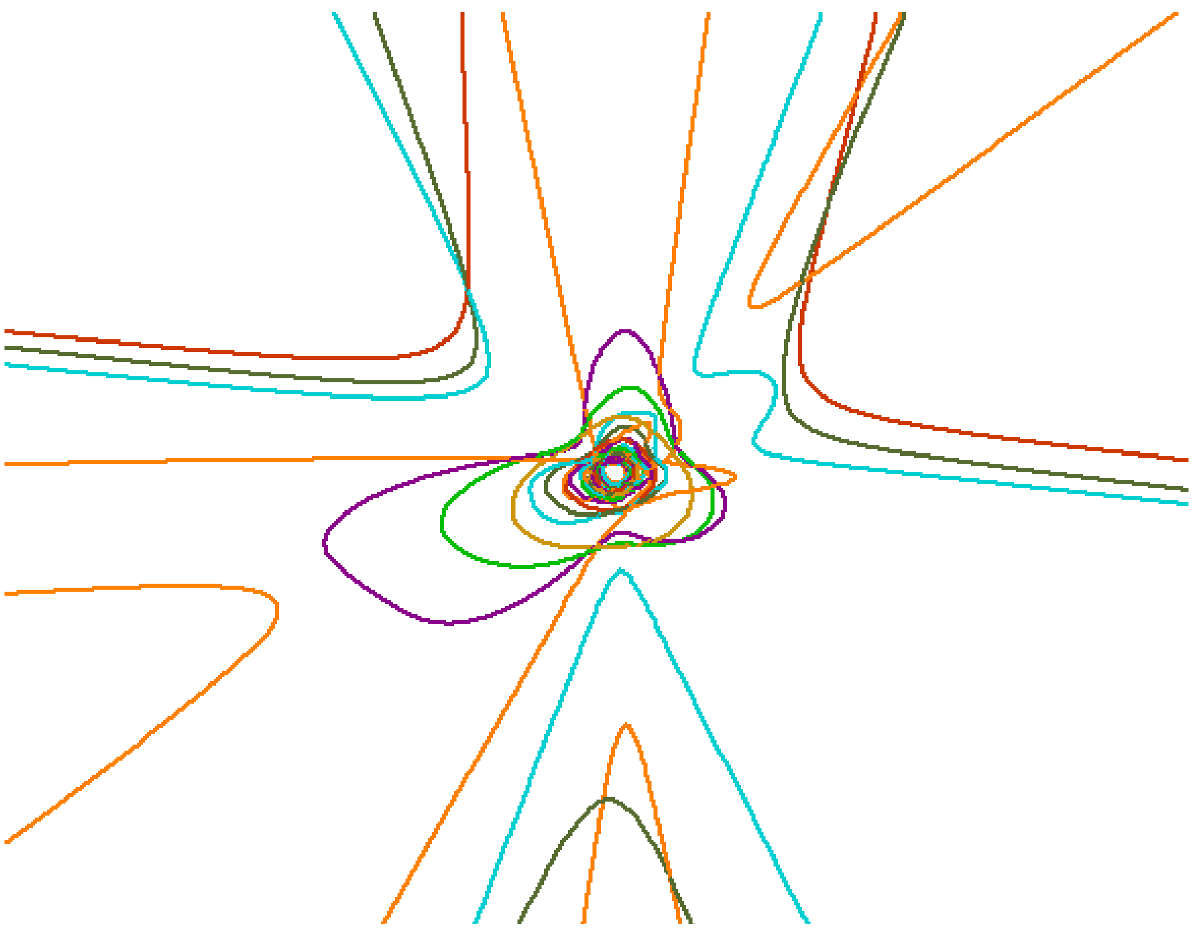


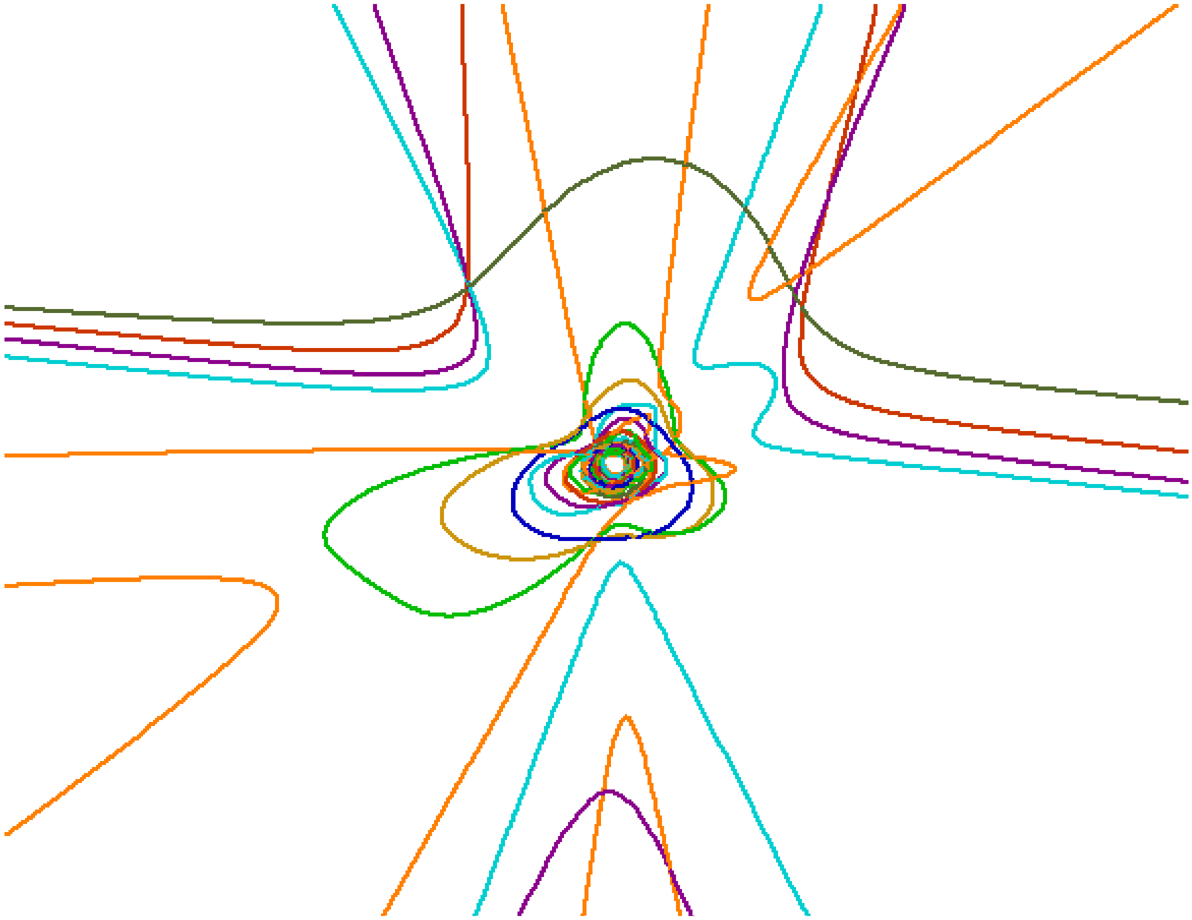


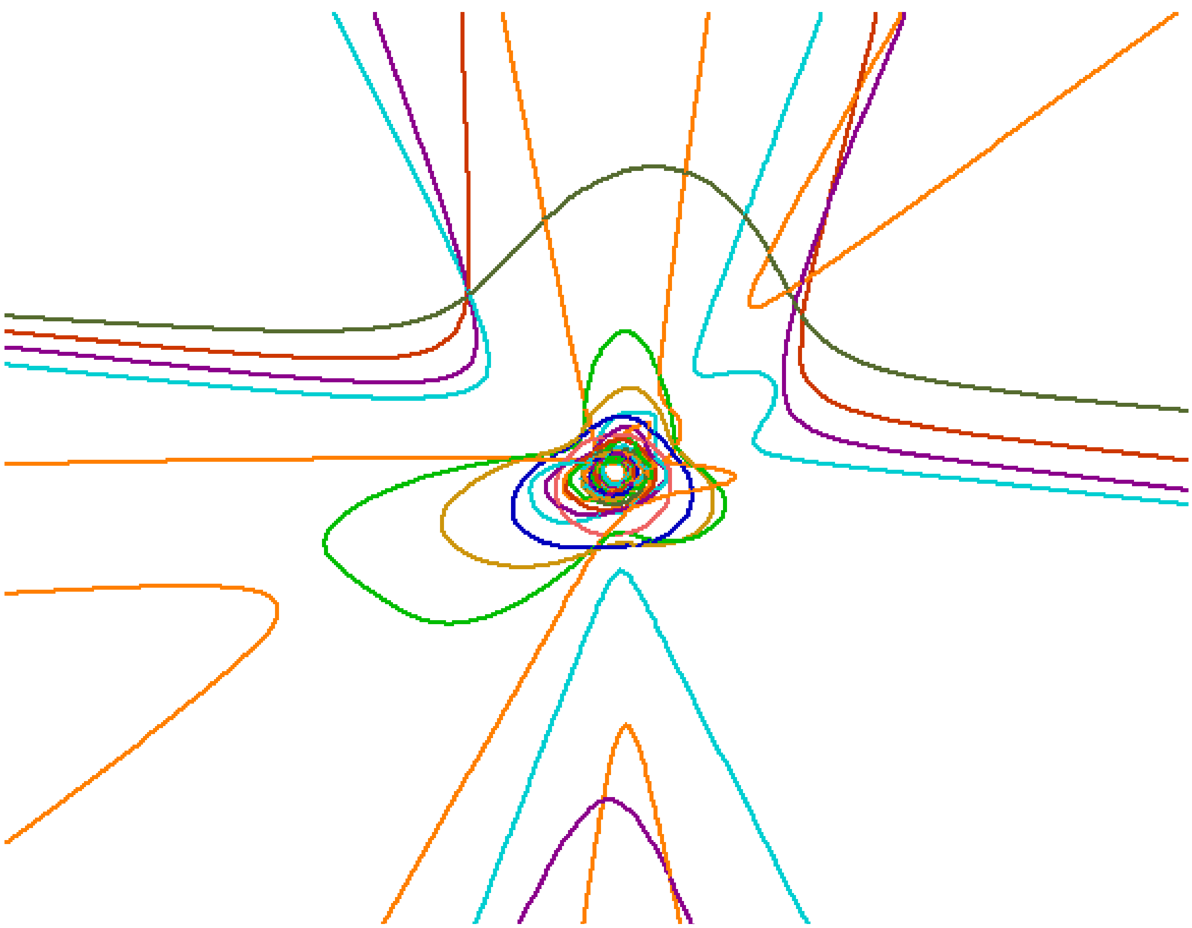


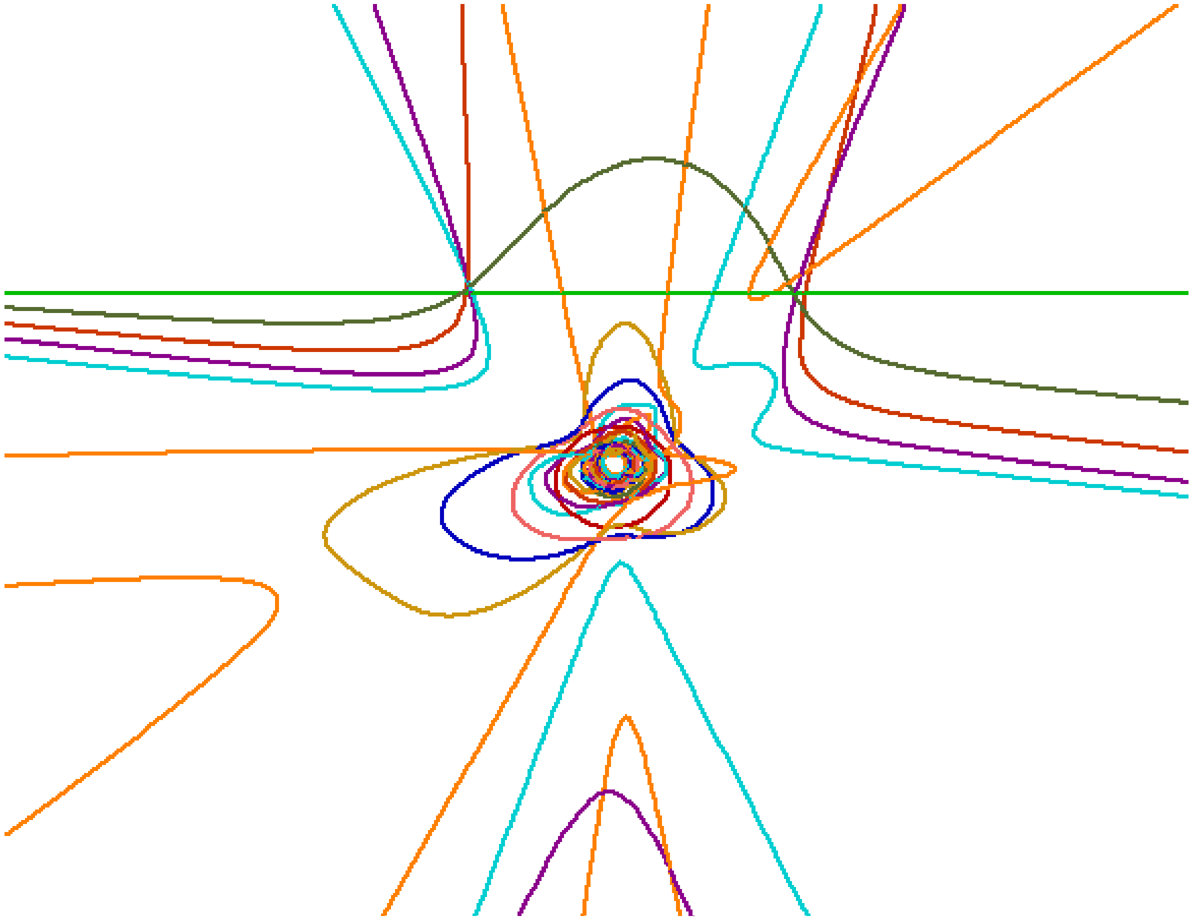












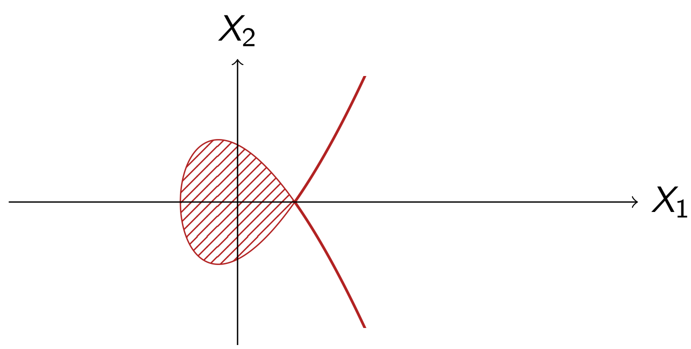
Renegar: Hyperbolic programs, and their derivative relaxations
Found. Comput. Math. 6 (2006), no. 1, 59–79
http://homepage.mac.com/renegar/hyper_progs.pdf
http://dx.doi.org/10.1007/s10208-004-0136-z

Example on the proven Lax conjecture

We have seen geometrically (see below) that

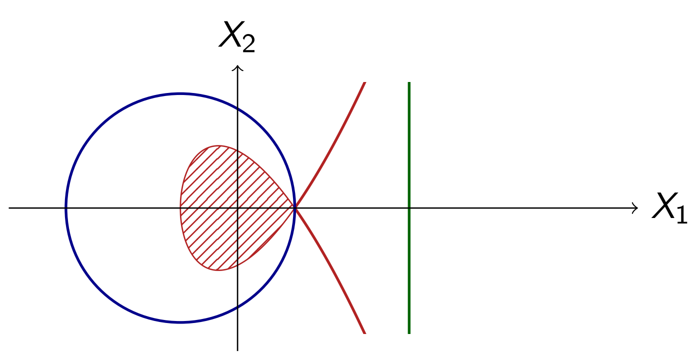
$$p = X_1^3 - X_1^2 - X_1 - X_2^2 + 1 \in \mathbb{R}[X_1, X_2]$$

is a real zero polynomial. Since Helton and Vinnikov have proved the Lax conjecture, there must be $A \in S\mathbb{R}[\bar{X}]^{3\times 3}$ with $A(0) \succ 0$ and $p = \det A$. Indeed, setting $A := \begin{pmatrix} 2-2X_1 & X_2 & 1-X_1 \\ X_2 & 1-X_1 & 0 \\ 1-X_1 & 0 & 1 \end{pmatrix}$, we have $p = \det A$ and $A(0) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \succ 0$. How to compute this in general?



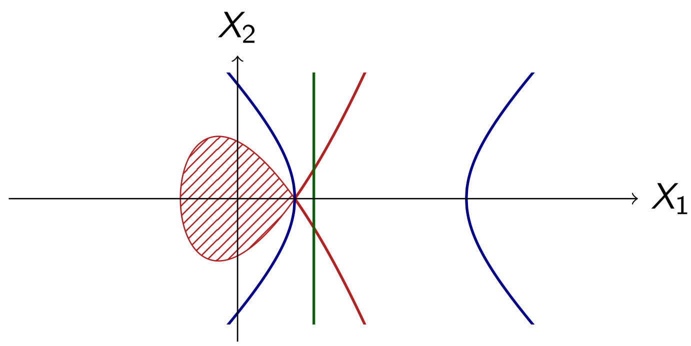
Example on the realization as a basic closed set

Let again $p = \det A$ with $A = \begin{pmatrix} 2-2X_1 & X_2 & 1-X_1 \\ X_2 & 1-X_1 & 0 \\ 1-X_1 & 0 & 1 \end{pmatrix}$. We have already seen how to realize the connected of component S of 0 in $\{x \in \mathbb{R}^2 \mid p(x) \geq 0\}$ as a basic closed set by writing $S = \{x \in \mathbb{R}^2 \mid p(x) \geq 0, Rp(x) \geq 0, R^2p(x) \geq 0\}$.



Example on the realization as a basic closed set

Let again $p = \det A$ with $A = \begin{pmatrix} 2-2X_1 & X_2 & 1-X_1 \\ X_2 & 1-X_1 & 0 \\ 1-X_1 & 0 & 1 \end{pmatrix}$. We have already seen how to realize the connected of component S of 0 in $\{x \in \mathbb{R}^2 \mid p(x) \geq 0\}$ as a basic closed set by writing $S = \{x \in \mathbb{R}^2 \mid p(x) \geq 0, Rp(x) \geq 0, R^2p(x) \geq 0\}$.
Another way of doing this is to calculate $\det(A + TI_3) = T^3 + (4 - 3X_1)T^2 + (X_1^2 - 5X_1 - X_2^2 + 4)T + p$ and write $S = \{x \in \mathbb{R}^2 \mid p(x) \geq 0, x_1^2 - 5x_1 - x_2^2 + 4 \geq 0, 4 - 3x_1 \geq 0\}$.



Part II. Semidefinitely representable sets

Projections of spectrahedrons

Recall: If $S = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^k \colon A(x,y) \succeq 0\}$ for some symmetric linear matrix polynomial $A \in \mathbb{R}[\bar{X}, \bar{Y}]^{t \times t}$, we call A a semidefinite representation of S and we say that S is semidefinitely representable.

Example $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} \colon \left(\begin{smallmatrix} x & 1 \\ 1 & y \end{smallmatrix}\right) \succeq 0\}$

Let S be semidefinitely representable. Then

- S is convex and
- S is semialgebraic.

Indeed, recall that by Tarski's real quantifier elimination every projection of a semialgebraic set is semialgebraic.

Second big question of the talk:

Question (Nemirovski, International Congress of Mathematicians, Madrid 2006)
Is every convex semialgebraic set semidefinitely representable?

Nemirovski's question

Question (Nemirovski, International Congress of Mathematicians, Madrid 2006)
Is every convex semialgebraic set semidefinitely representable?

Nemirovski: Advances in convex optimization: conic programming
International Congress of Mathematicians. Vol. I, 413–444, Eur. Math. Soc., Zürich, 2007 http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.94.1539&rep=rep1&type=pdf

Example. We have seen that $S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 + x_2^4 \leq 1\}$ is not a spectrahedron. However, it is semidefinitely representable since

$$\begin{aligned} S &= \{(x_1,x_2) \in \mathbb{R}^2 \mid \exists y_1, y_2 \in \mathbb{R} : \\ &\qquad\qquad 1 - y_1^2 - y_2^2 \geq 0 \quad \& \quad y_1 \geq x_1^2 \quad \& \quad y_2 \geq x_2^2\} \\ &= \Big\{(x_1,x_2) \in \mathbb{R}^2 \mid \exists y_1, y_2 \in \mathbb{R} : \\ &\qquad \begin{pmatrix} 1+y_1 & y_2 \\ y_2 & 1-y_1 \end{pmatrix} \succeq 0 \quad \& \quad \left(\begin{smallmatrix} y_1 & x_1 \\ x_1 & 1 \end{smallmatrix}\right) \succeq 0 \quad \& \quad \left(\begin{smallmatrix} y_2 & x_2 \\ x_2 & 1 \end{smallmatrix}\right) \succeq 0\Big\}. \end{aligned}$$

How to find semidefinite representations

Let U be a subset of a convex set S. Recall that S is the disjoint union of the relative interiors of its faces. Netzer defines $U \looparrowleft S$ as the union of the relative interiors of all faces intersecting U.

Theorem (Netzer). If $U \subseteq S \subseteq \mathbb{R}^n$ are semidefinitely representable sets. Then $U \looparrowleft S$ is again semidefinitely representable.

The proof of Netzer is constructive and gives rise to simple explicit constructions which preserve for example rational coefficients in the semidefinite representation.

Netzer: On semidefinite representations of sets
`http://arxiv.org/abs/0907.2764`

How to find semidefinite representations

Helton and Nie conjectured that every convex semialgebraic set is semidefinitely representable.

This is based on their seminal work in which they prove that surprisingly many compact basic closed convex semialgebraic sets are semidefinitely representable.

They have two methods:

- The simple and explicit Lasserre moment constructions. The proof that these relaxations are exact is very deep but works under fairly general hypotheses.
- A local version of these constructions which is glued together by a non-constructive compactness argument. The proofs are simpler though still deep, and the hypotheses are very general.

Each of the methods is scattered over both of the following papers.

How to find semidefinite representations

First paper
Helton & Nie: Semidefinite representation of convex sets
to appear in Math. Prog.
http://arxiv.org/abs/0705.4068
http://dx.doi.org/10.1007/s10107-008-0240-y

Second paper
Helton & Nie: Sufficient and necessary conditions for semidefinite representability of convex hulls and sets
http://arxiv.org/abs/0709.4017

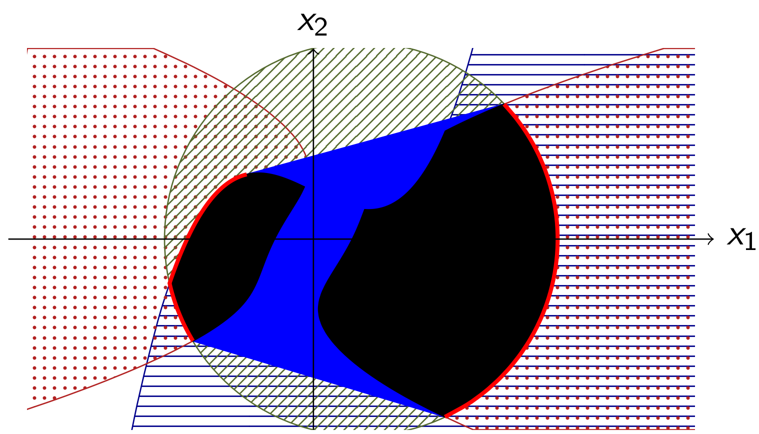
How to find semidefinite representations

The basic idea is to use the Lasserre moment relaxation of a basic closed semialgebraic set, or more precisely of a finite system of non-strict polynomial inequalities. We will explain this now.

Lasserre: Convex sets with semidefinite representation
Math. Prog. 120, no. 2 (2009), 457–477
http://hal.archives-ouvertes.fr/docs/00/33/16/65/PDF/SDR-final.pdf http://dx.doi.org/10.1007/s10107-008-0222-0

System of polynomial inequalities

$$
\begin{array}{lcccccccccccc}
{\color{blue}A} & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
{\color{red}B} & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
{\color{green}C} & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



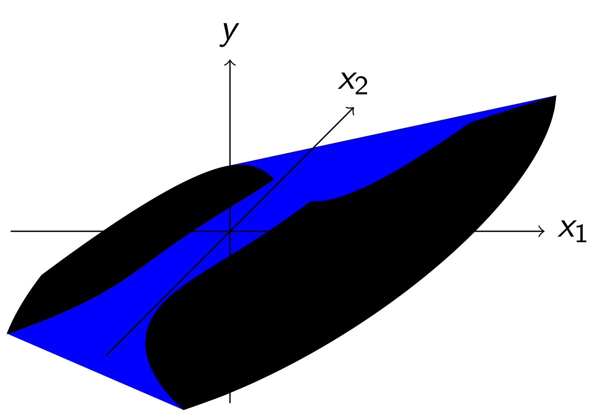
S

conv S

$S \cap \partial \operatorname{conv} S$

System of polynomial inequalities

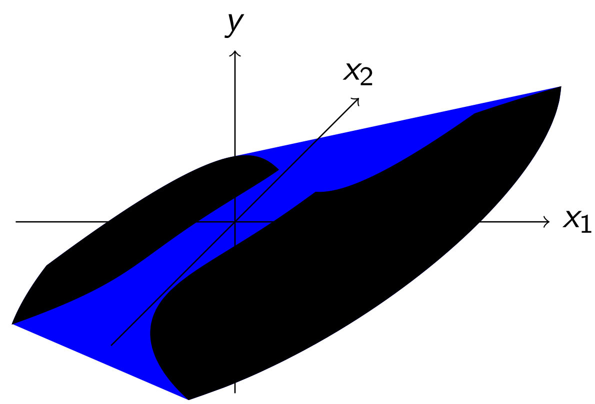
$$
\begin{array}{lcccccccccccc}
{\color{blue}A} & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
{\color{red}B} & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
{\color{green}C} & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



conv S

System of linear inequalities

$$
\begin{array}{lrcl}
A & -y_1 + x_1 + 2x_2 - 1 & \geq & 0 \\
B & -y_2 + 2y_3 - 2y_4 + y_5 - \frac{1}{3} & \geq & 0 \\
C & -y_3 - y_5 + x_1 + 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrcl}
A & -x_1^3 + x_1 + 2x_2 - 1 & \geq & 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$

redundant:

$$
\begin{array}{lrcl}
AB & x_1^3x_2^4 - \dots - x_2^2 - \frac{2}{3}x_2 + \frac{1}{3} & \geq & 0 \\
AC & x_1^5 + \dots - x_1 + 8x_2 - 4 & \geq & 0 \\
ABC & -x_1^5x_2^4 + \dots - \frac{13}{3}x_2^2 - \frac{8}{3}x_2 + \frac{4}{3} & \geq & 0 \\
D^2 & x_1^2 - 2x_1x_2 + x_2^2 & \geq & 0 \\
D^2C & -x_1^4 + \dots + 4x_1^2 + 4x_1x_2 + 4x_2^2 & \geq & 0
\end{array}
$$

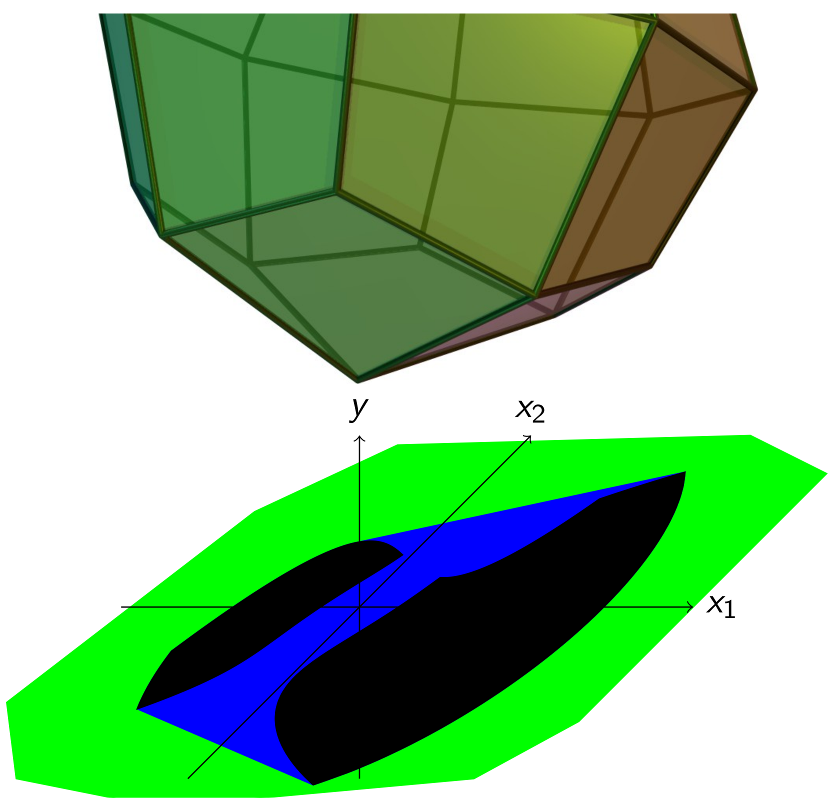
System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrl}
A & & -y_1 & +x_1 & +2x_2 & -1 & \geq 0 \\
B & -y_2 & +2y_3 & -2y_4 & +y_5 & -\frac{1}{3} & \geq 0 \\
C & & -y_3 & -y_5 & +x_1 & +4 & \geq 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrl}
AB & y_6 & -\dots & -y_5 & -\frac{2}{3}x_2 & +\frac{1}{3} & \geq 0 \\
AC & y_{10} & +\dots & -x_1 & +8x_2 & -4 & \geq 0 \\
ABC & -y_{13} & +\dots & -\frac{13}{3}y_5 & -\frac{8}{3}x_2 & +\frac{4}{3} & \geq 0 \\
D^2 & & & y_3 & -2y_4 & +y_5 & \geq 0 \\
D^2C & -y_{18} & +\dots & +4y_3 & +4y_4 & +4y_5 & \geq 0
\end{array}
$$



conv S

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \Longleftrightarrow$$

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix}
\begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \Longleftrightarrow$$

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\ y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\ y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2 \end{pmatrix} \succeq 0$$

System of polynomial inequalities

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$$
\begin{array}{lcccccccccccc}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c \end{pmatrix} (-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

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C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - x_2^2 + x_1 + 4 & \dots & \dots \\ -y_1 - x_1x_2^2 + y_3 + 4x_1 & \dots & \dots \\ -x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - y_5 + x_1 + 4 & \dots & \dots \\ -y_1 - y_6 + y_3 + 4x_1 & \dots & \dots \\ -y_7 - x_2^3 + y_4 + 4x_2 & \dots & \dots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
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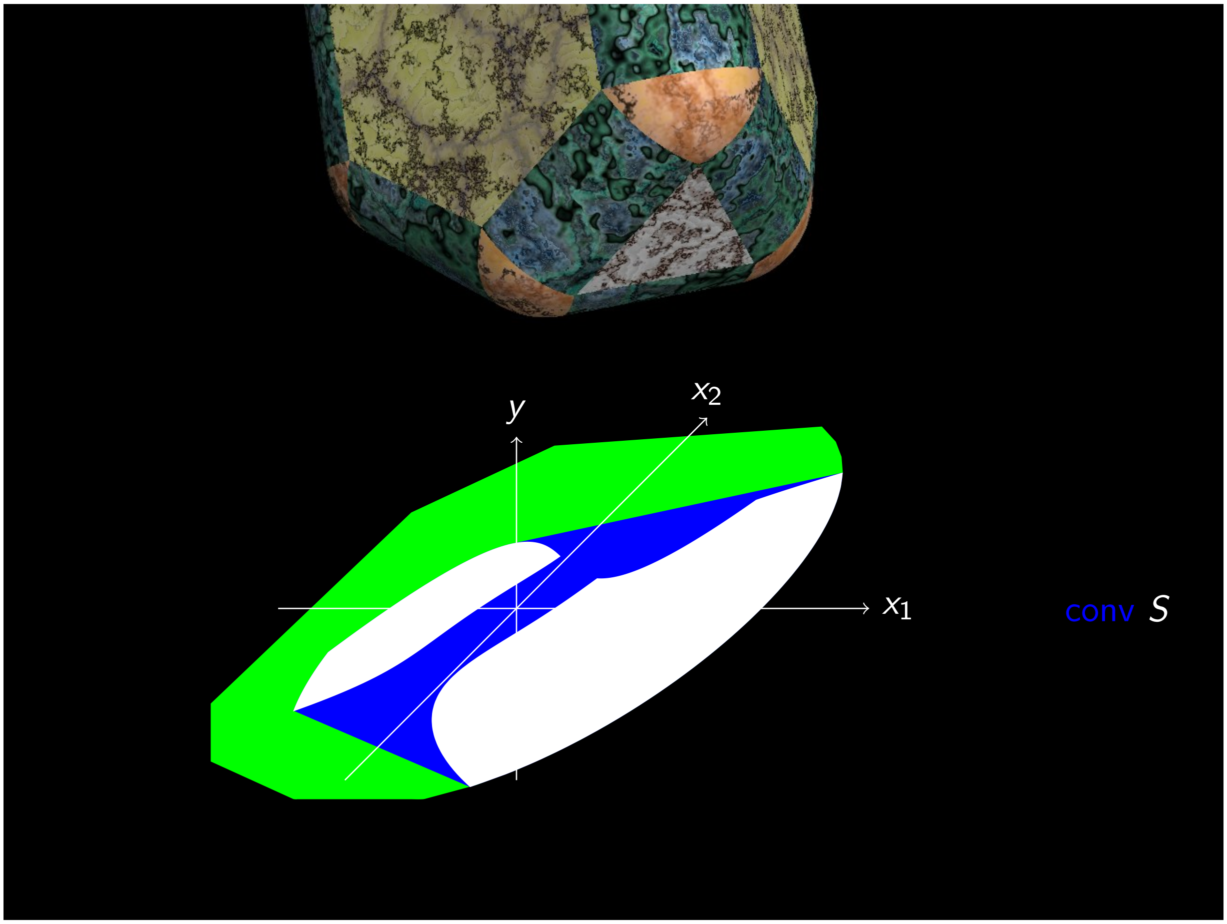
irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} -y_3 - y_5 + x_1 + 4 & \dots & \dots \\ -y_1 - y_6 + y_3 + 4x_1 & \dots & \dots \\ -y_7 - y_8 + y_4 + 4x_2 & \dots & \dots \end{pmatrix} \succeq 0$$



- $\bar{X} = (X_1, \dots, X_n)$ variables
- $\mathbb{R}[\bar{X}]$ polynomials
- $g_1, \dots, g_m \in \mathbb{R}[\bar{X}]$ polynomials defining ...
- ... the set $S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$
- $T := \{ \sum_{\delta \in \{0,1\}^m} s_\delta g_1^{\delta_1} \cdots g_m^{\delta_m} \mid s_\delta \in \sum \mathbb{R}[\bar{X}]^2 \}$
  convex cone in $\mathbb{R}[\bar{X}]$
- $\mathcal{L} := \left\{ L \mid L \colon \mathbb{R}[\bar{X}] \to \mathbb{R} \text{ linear}, L(1) = 1, L(T) \subseteq \mathbb{R}_{\geq 0} \right\}$
  solution set of the "linearized" system
- $S' := \{(L(X_1), \dots, L(X_n)) \mid L \in \mathcal{L}\}$
  Schmüdgen relaxation

- $\bar{X} = (X_1, \dots, X_n)$ variables
- $\mathbb{R}[\bar{X}]_k$ polynomials of degree at most k
- $g_1, \dots, g_m \in \mathbb{R}[\bar{X}]$ polynomials defining ...
- ... the set $S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$
- $T_k := \{\sum_{\delta \in \{0,1\}^m} s_\delta g_1^{\delta_1} \cdots g_m^{\delta_m} \mid s_\delta \in \sum \mathbb{R}[\bar{X}]^2, \deg(s_\delta g^\delta) \leq k\}$
  convex cone in $\mathbb{R}[\bar{X}]_k$
- $\mathcal{L}_k := \{L \mid L \colon \mathbb{R}[\bar{X}]_k \to \mathbb{R} \text{ linear}, L(1) = 1, L(T_k) \subseteq \mathbb{R}_{\geq 0}\}$
  solution set of the "linearized" system (spectrahedron in $\mathbb{R}[\bar{X}]_k^*$)
- $S_k{}' := \{(L(X_1), \dots, L(X_n)) \mid L \in \mathcal{L}_k\}$
  k-th Lasserre relaxation (semidefinitely representable)

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We have $S \subseteq \operatorname{conv} S \subseteq S' \subseteq \dots \subseteq S_4' \subseteq S_3' \subseteq S_2' \subseteq S_1'$.
The question is whether $\operatorname{conv} S = S_k'$ for some $k \in \mathbb{N}$.

$\boxed{\text{Suppose } S \neq \emptyset \text{ and fix } k \in \mathbb{N} := \{1, 2, 3, \dots\}.}$

Proposition (Powers & Scheiderer 2005).
If S has non-empty interior, then T_k is closed in $\mathbb{R}[\bar{X}]_k$.

Proposition. If S is compact, then $\operatorname{conv} S$ is closed in $\mathbb{R}^n$.

Proposition. $\overline{T_k} = \{f \in \mathbb{R}[\bar{X}] \mid \forall L \in \mathcal{L}_k : L(f) \geq 0\}$.

Remark. $\overline{\operatorname{conv} S} = \bigcap\{f^{-1}(\mathbb{R}_{\geq 0}) \mid f \in \mathbb{R}[\bar{X}]_1, f \geq 0 \text{ on } S\}$

Proposition. $\overline{S'_k} = \bigcap\{f^{-1}(\mathbb{R}_{\geq 0}) \mid f \in \mathbb{R}[\bar{X}]_1 \cap \overline{T_k}\}$.

Proposition. If $\operatorname{conv} S$ is closed, then
$\operatorname{conv} S = S'_k \iff \forall f \in \mathbb{R}[\bar{X}]_1 : (f \geq 0 \text{ on } S \implies f \in \overline{T_k})$.

$\boxed{\text{Suppose } S \text{ is compact.}}$

Theorem (Schmüdgen 1991).
(a) $\forall L \in \mathcal{L} \colon \exists$ probability measure μ on $S \colon \forall p \in \mathbb{R}[\bar{X}] \colon L(p) = \int p \, d\mu$
(b) $\forall f \in \mathbb{R}[\bar{X}] \colon (f > 0 \text{ on } S \implies f \in T)$

Corollary. $\operatorname{conv} S = S'$

Theorem (2004). For $f \in \mathbb{R}[\bar{X}]$, $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \binom{\alpha_1 + \dots + \alpha_n}{\alpha_1 \ \dots \ \alpha_n} \bar{X}^\alpha$, $a_\alpha \in \mathbb{R}$, we define $\|f\| := \max\{|a_\alpha| \mid \alpha \in \mathbb{N}^n\}$. Suppose $\emptyset \neq S \subseteq (-1, 1)^n$. Then there is a constant $c \in \mathbb{N}$ (depending only on n, m and $g_1, \dots, g_m$) such that, for each $f \in \mathbb{R}[\bar{X}]_d$ with $f^* := \min\{f(x) \mid x \in S\} > 0$, we have $f \in T_k$ for some

$$k \leq cd^2 \left(1 + \left(d^2 n^d \frac{\|f\|}{f^*}\right)^c\right).$$

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$$k \le c\left(1 + \left(\frac{\|f\|}{f^*}\right)^c\right).$$

Corollary. $\exists c \in \mathbb{N} : \forall k \in \mathbb{N}_{\ge c} : \forall x \in S'_k : \operatorname{dist}(x, \operatorname{conv} S) \le \frac{c}{\sqrt[c]{k}}$

$\boxed{\text{Suppose } S \text{ is compact.}}$

Theorem (Schmüdgen 1991). For all $f \in \mathbb{R}[\bar{X}]$:
$f > 0$ on $S \implies \exists p_\delta \in \mathbb{R}[\bar{X}]^{1\times *}$: $f = \sum_{\delta \in \{0,1\}} p_\delta p_\delta^T g^\delta$

Corollary (Hol & Scherer 2008). For all $F \in S\mathbb{R}[\bar{X}]^{t\times t}$:
$F \succ 0$ on $S \implies \exists P_\delta \in \mathbb{R}[\bar{X}]^{t\times *}$: $F = \sum_{\delta \in \{0,1\}} P_\delta P_\delta^T g^\delta$

Proof (S.). Given $F \in \mathbb{R}[\bar{X}]^{t\times t}$ with $F \succ 0$ on S, we consider $f := Y \in \mathbb{R}[\bar{X}, Y]$ and observe that $f > 0$ on

$$\begin{aligned} S_F &:= \{(x,y) \in \mathbb{R}^{n+1} \mid x \in S,\ \text{y eigenvalue of F(x)}\} \\ &= \{(x,y) \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0, P_F(x,y) = 0\} \end{aligned}$$

where $P_F \in \mathbb{R}[\bar{X}][Y] = \mathbb{R}[\bar{X}, Y]$ is the characteristic polynomial of F. Apply Schmüdgen to $f = Y$. Use $\mathbb{R}[\bar{X}, Y] \to \mathbb{R}[\bar{X}, F] \subseteq \mathbb{R}[\bar{X}]^{t\times t}$ ($\mathbb{R}[\bar{X}, F]$ is commutative). Since $P_F(\bar{X}, F) = 0$ by Cayley-Hamilton, p_F disappears in this representation. Now use that matrix calculations can be done in blocks!
Problem: We do not get degree bounds like for Schmüdgen in this way.

Theorem (Helton & Nie). For $F = \sum_{\alpha \in \mathbb{N}^n} A_\alpha \binom{\alpha_1 + \dots + \alpha_n}{\alpha_1\ \dots\ \alpha_n} \bar{X}^\alpha$, $A_\alpha \in S\mathbb{R}^{t\times t}$, we define $\|F\| := \max\{\|A_\alpha\| \mid \alpha \in \mathbb{N}^n\}$.
Suppose $\emptyset \neq S \subseteq (-1,1)^n$.Then there is a constant $c \in \mathbb{N}$ (depending only on n, m and $g_1, \dots, g_m$) such that, for each $F \in S\mathbb{R}[\bar{X}]_d^{t\times t}$ with $F^* := \min\{\lambda_{\min}(F(x)) \mid x \in S\} > 0$, we have $F = \sum_{\delta \in \{0,1\}} P_\delta P_\delta^T g^\delta$ for certain $P_\delta \in S\mathbb{R}[\bar{X}]_k^{t\times *}$ with

$$k \leq cd^2 \left(1 + \left(d^2 n^d \frac{\|F\|}{F^*}\right)^c\right).$$

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Concavity

The following terminology is not standard but suitable to us. It is a kind of local concavity of a function which can be detected by looking at its second derivative.

Definition. Let $p \in \mathbb{R}[\bar{X}]$ and $U \subseteq \mathbb{R}^n$.

$$p \text{ strictly concave on } U \;:\Longleftrightarrow\; D^2p \prec 0 \text{ on } U \;\Longleftrightarrow\; \forall x \in U\colon \forall v \in \mathbb{R}^n \setminus \{0\}\colon D^2p(x)[v,v] < 0$$

$$p \text{ strictly quasiconcave on } U \;:\Longleftrightarrow\; \forall x \in U\colon \forall v \in \mathbb{R}^n \setminus \{0\}\colon (Dp(x)[v] = 0 \implies D^2p(x)[v,v] < 0)$$

Suppose S is compact, convex and has non-empty interior.

Lemma (Helton & Nie). If each g_i is strictly concave on S, then $S = S'_k$ for some $k \in \mathbb{N}$.

Idea of proof. Let $u \in \partial S$ and $f \in \mathbb{R}[\bar{X}]_1 \setminus \{0\}$ with $f \geq 0$ on S and $f(u) = 0$. To show: $f \in T_k$ for some $k \in \mathbb{N}$ which is independent of f. Since the Slater condition is satisfied, we get Lagrange multipliers $\lambda_i \geq 0$, $i \in I := \{i \mid g_i(u) = 0\}$, such that $D(f - \sum_{i\in I} \lambda_i g_i)(u) = 0$. Now we have for $x \in \mathbb{R}^n$

$$f(x) - \sum_{i\in I} \lambda_i g_i(x) = \int_0^1 \int_0^t D^2(f - \sum_{i\in I} \lambda_i g_i)(u + s(x-u))[x-u, x-u]\,ds\,dt$$

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$F_{i,u} \in S\mathbb{R}[X]^{n \times n}$, $F_{i,u} \succ 0$ on S, use Hol & Scherer with bounds!

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Now we have

$$f - \sum_{i \in I} \lambda_i g_i = -\sum_{i \in I} \lambda_i (\bar{X} - u)^T F_{i,u} (\bar{X} - u)$$

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$$f - \sum_{i \in I} \lambda_i g_i = \sum_{i \in I} \lambda_i (\bar{X} - u)^T \Big(\sum_{\delta \in \{0,1\}^m} P_{i,u,\delta} P_{i,u,\delta}^T g^\delta \Big) (\bar{X} - u)$$

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Lemma (Helton & Nie). If each g_i is strictly concave on S, then $S = S'_k$ for some $k \in \mathbb{N}$.

Idea of proof. Let $u \in \partial S$ and $f \in \mathbb{R}[\bar{X}]_1 \setminus \{0\}$ with $f \geq 0$ on S and $f(u) = 0$. To show: $f \in T_k$ for some $k \in \mathbb{N}$ which is independent of f. Since the Slater condition is satisfied, we get Lagrange multipliers $\lambda_i \geq 0$, $i \in I := \{i \mid g_i(u) = 0\}$, such that $D(f - \sum_{i \in I} \lambda_i g_i)(u) = 0$. Now we have

$$f - \sum_{i \in I} \lambda_i g_i = \sum_{i \in I} \lambda_i \sum_{\delta \in \{0,1\}^m} (P_{i,u,\delta}^T (\bar{X} - u))^T (P_{i,u,\delta}^T (\bar{X} - u)) g^\delta$$

$F_{i,u} \in S\mathbb{R}[X]^{n \times n}$, $F_{i,u} \succ 0$ on S, use Hol & Scherer with bounds!

$\boxed{\text{Suppose } S \text{ is compact, convex and has non-empty interior.}}$

Theorem (Helton & Nie). If each g_i is strictly quasiconcave on S, then $S = S'_k$ for some $k \in \mathbb{N}$.

Theorem (Helton & Nie). Suppose each g_i is strictly quasiconcave on $S \cap \{g_i = 0\}$ and a very ugly additional hypothesis is fulfilled that might follow from this. Then $S = S'_k$ for some $k \in \mathbb{N}$.

Lemma (Netzer & Sinn, bounded case: Helton & Nie).
The convex hull of finitely many semidefinitely representable sets is again semidefinitely representable.

Netzer & Sinn: A note on the convex hull of finitely many projections of spectrahedra `http://arxiv.org/abs/0908.3386`

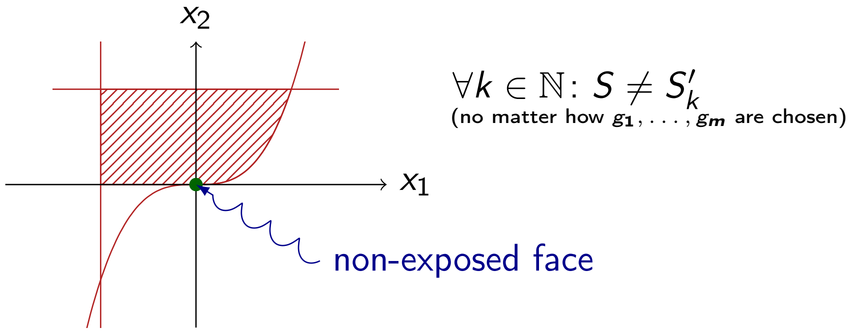
This enables Helton and Nie to show non-constructively the following theorem, glueing together local moment constructions.

Theorem (Helton & Nie). Suppose S is compact, each g_i is strictly quasiconcave on $S \cap (\partial \operatorname{conv} S) \cap \{g_i = 0\}$ and the boundary of S is contained in the closure of the interior of S. Then $\operatorname{conv} S$ is semidefinitely representable.

$\boxed{\text{Suppose } S \text{ is convex and } S^\circ \neq \emptyset.}$

Theorem (Netzer & Plaumann & S.) If $S = S'_k$ for some $k \in \mathbb{N}$, then all faces of S are exposed.

Example. $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 \leq x_1,\ x_1^3 \leq x_2,\ 0 \leq x_2 \leq 1\}$



Netzer & Plaumann & S.: Exposed faces of semidefinite representable sets http://arxiv.org/abs/0902.3345