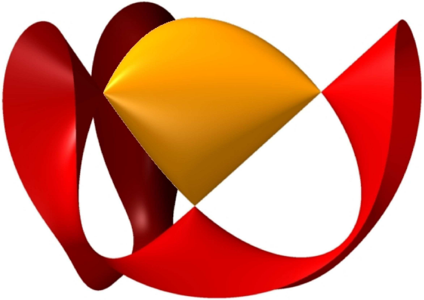
The Geometry of Semidefinite Programming

Bernd Sturmfels
UC Berkeley



Positive Semidefinite Matrices

For a real symmetric $n \times n$-matrix A the following are equivalent:

- All n eigenvalues of A are positive real numbers.
- All 2^n principal minors of A are positive real numbers.
- Every non-zero vector $x \in \mathbb{R}^n$ satisfies $x^T A \cdot x > 0$.

A matrix A is *positive definite* if it satisfies these properties, and it is *positive semidefinite* if the following equivalent properties hold:

- All n eigenvalues of A are non-negative real numbers.
- All 2^n principal minors of A are non-negative real numbers.
- Every vector $x \in \mathbb{R}^n$ satisfies $x^T A \cdot x \geq 0$.

The set of all positive semidefinite $n \times n$-matrices is a convex cone of full dimension $\binom{n+1}{2}$. It is closed and semialgebraic.

The interior of this cone consists of all positive definite matrices.

Semidefinite Programming

A *spectrahedron* is the intersection of the cone of positive semidefinite matrices with an affine-linear space. Its algebraic representation is a linear combination of symmetric matrices

$$A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_m A_m \ \succeq \ 0 \qquad (*)$$

Engineers call this is a *linear matrix inequality*.

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Semidefinite programming is the computational problem of maximizing a linear function over a spectrahedron:

$$\mathrm{Maximize} \ \ c_1 x_1 + c_2 x_2 + \cdots + c_m x_m \ \text{ subject to } \ (*)$$

Example: *The smallest eigenvalue of a symmetric matrix* A *is the solution of the SDP* Maximize x subject to $A - x \cdot \mathrm{Id} \succeq 0$.

Convex Polyhedra

Linear programming is semidefinite programming for diagonal matrices. If $A_0, A_1, \ldots, A_m$ are diagonal $n \times$n-matrices then

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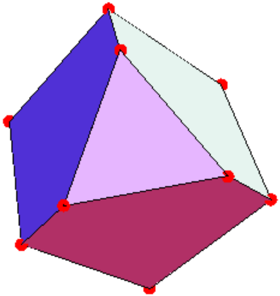
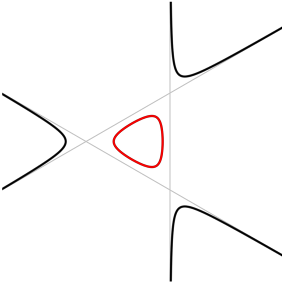
translates into a system of n linear inequalities in the m unknowns. A spectrahedron defined in this manner is a convex polyhedron:



Pictures in Dimension Two

Here is a picture of a spectrahedron for $m = 2$ and $n = 3$:



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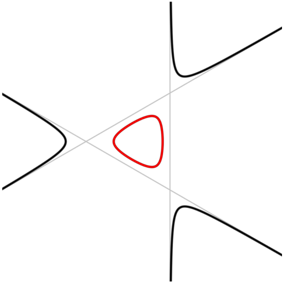
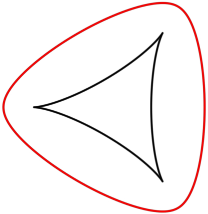


Duality is important in convex optimization:



Example: Multifocal Ellipses

Given m points $(u_1, v_1), \ldots, (u_m, v_m)$ in the plane $\mathbb{R}^2$, and a radius $d > 0$, their *m-ellipse* is the convex algebraic curve

$$\left\{ (x, y) \in \mathbb{R}^2 \,:\, \sum_{k=1}^{m} \sqrt{(x-u_k)^2 + (y-v_k)^2} \,=\, d \right\}.$$

The 1-ellipse and the 2-ellipse are algebraic curves of degree 2.

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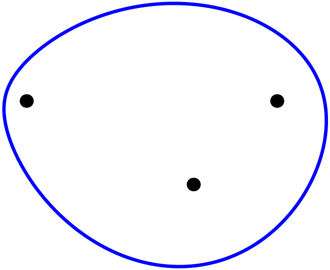
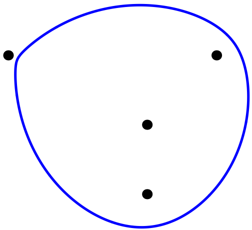
The 3-ellipse is an algebraic curve of degree 8:

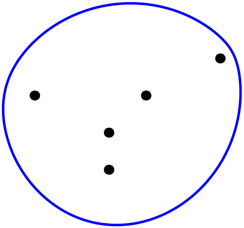


2, 2, 8, 10, 32, ...

The 4-ellipse is an algebraic curve of degree 10:

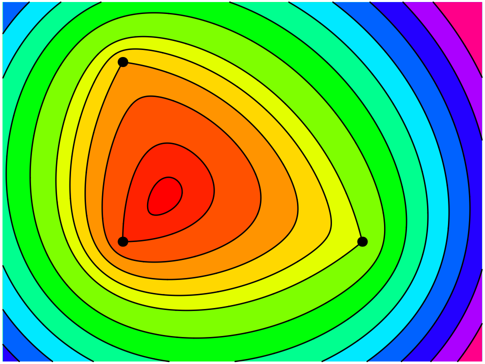


The 5-ellipse is an algebraic curve of degree 32:



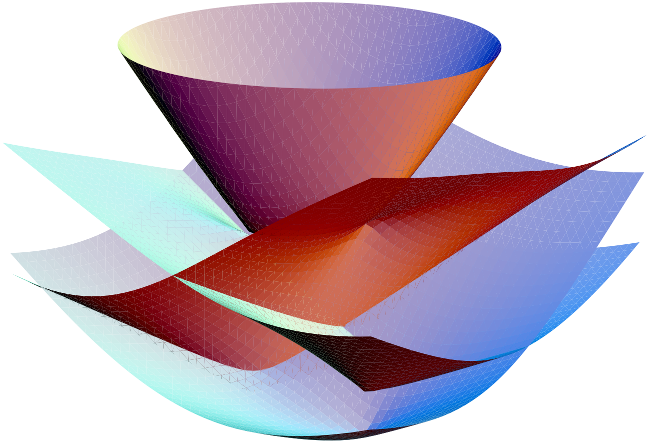
Concentric Ellipses

What is the algebraic degree of the m-ellipse?
How to write its equation?



What is the smallest radius d for which the m-ellipse is non-empty? How to compute the Fermat-Weber point?

3D View



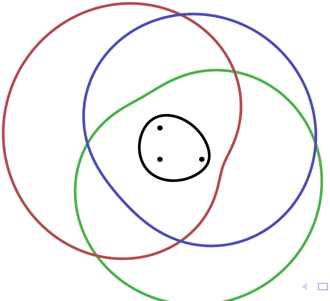
$$\mathcal{C} \;=\; \left\{(x, y, d) \in \mathbb{R}^3 \,:\, \sum_{k=1}^{m} \sqrt{(x-u_k)^2 + (y-v_k)^2} \;\leq\; d \,\right\}.$$

Ellipses are Spectrahedra

The 3-ellipse with foci $(0,0),(1,0),(0,1)$ has the representation

$$\begin{bmatrix} d+3x-1 & y-1 & y & 0 & y & 0 & 0 & 0 \\ y-1 & d+x-1 & 0 & y & 0 & y & 0 & 0 \\ y & 0 & d+x+1 & y-1 & 0 & 0 & y & 0 \\ 0 & y & y-1 & d-x+1 & 0 & 0 & 0 & y \\ y & 0 & 0 & 0 & d+x-1 & y-1 & y & 0 \\ 0 & y & 0 & 0 & y-1 & d-x-1 & 0 & y \\ 0 & 0 & y & 0 & y & 0 & d-x+1 & y-1 \\ 0 & 0 & 0 & y & 0 & y & y-1 & d-3x+1 \end{bmatrix}$$

The ellipse consists of all points (x,y) where this symmetric 8×8-matrix is positive semidefinite. Its boundary is a curve of degree eight:



2, 2, 8, 10, 32, 44, 128, ...

Theorem: *The polynomial equation defining the m-ellipse has degree* 2^m *if m is odd* *and degree* $2^m - \binom{m}{m/2}$ *if m is even. We express this polynomial as the determinant of a symmetric matrix of linear polynomials. Our representation extends to weighted m-ellipses and m-ellipsoids in arbitrary dimensions .....*

[J. Nie, P. Parrilo, B.St.: Semidefinite representation of the k-ellipse, in *Algorithms in Algebraic Geometry*, I.M.A. Volumes in Mathematics and its Applications, 146, Springer, New York, 2008, pp. 117-132]

In other words, *m*-ellipses and *m*-ellipsoids are spectrahedra. The problem of finding the Fermat-Weber point is an SDP.

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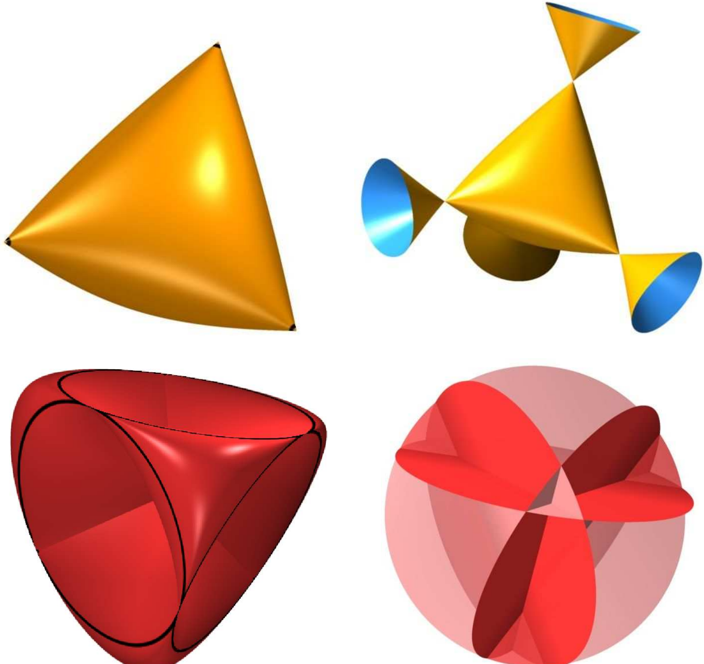
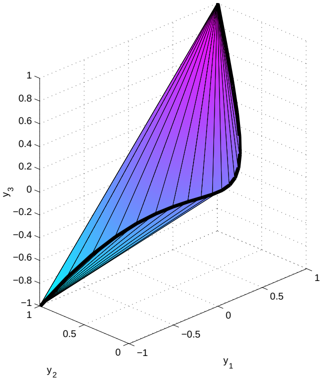
Let's now look at some spectrahedra in dimension three.
Our next picture shows the typical behavior for $m = 3$ and $n = 3$.

A Spectrahedron and its Dual



Non-Linear Convex Hull Computation

Input : $\left\{(t, t^2, t^3) \in \mathbb{R}^3 : -1 \leq t \leq 1\right\}$



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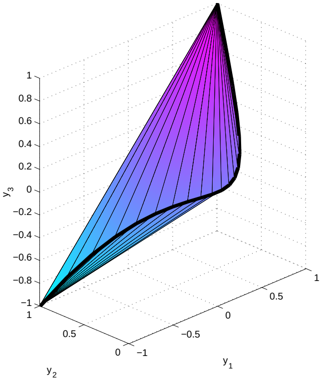


The convex hull of the moment curve is a spectrahedron.

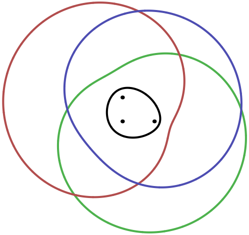
Output : $\begin{pmatrix} 1 & x \\ x & y \end{pmatrix} \pm \begin{pmatrix} x & y \\ y & z \end{pmatrix} \succeq 0$

Characterization of Spectrahedra

A convex hypersurface of degree d in $\mathbb{R}^n$ is *rigid convex* if every line passing through its interior meets (the Zariski closure of) that hypersurface in d real points.

Theorem (Helton–Vinnikov (2006))

Every spectrahedron is rigid convex. The converse is true for $n = 2$.



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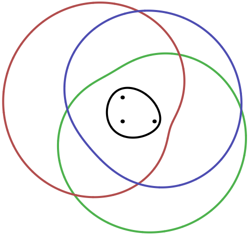

Open problem: Is every compact convex basic semialgebraic set $\mathcal{S}$ the projection of a spectrahedron in higher dimensions?

Theorem (Helton–Nie (2008))

The answer is yes if the boundary of $\mathcal{S}$ is "sufficiently smooth".

Questions about 3-Dimensional Spectrahedra

What are the edge graphs of spectrahedra in $\mathbb{R}^3$?

How can one define their *combinatorial types*?

Is there an analogue to Steinitz' Theorem for polytopes in $\mathbb{R}^3$?

Consider 3-dimensional spectrahedra whose boundary is an irreducible surface of degree n. Can such a spectrahedron have $\binom{n+1}{3}$ isolated singularities in its boundary? How about $n = 4$?

A Pinch of Statistics

Every positive definite $n \times n$-matrix $\Sigma = (\sigma_{ij})$ is the covariance matrix of a multivariate normal distribution. A Gaussian graphical model is specified by requiring that some entries of Σ^{-1} are zero.

Maximum likelihood estimation is a *matrix completion problem*.

For example, under which condition on the visible entries σ_{ij} can we find x and y which make the following matrix positive definite?

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & x & \sigma_{14} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} & y \\ x & \sigma_{23} & \sigma_{33} & \sigma_{34} \\ \sigma_{14} & y & \sigma_{34} & \sigma_{44} \end{pmatrix}$$

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The MLE is the point $(\hat{x}, \hat{y})$ which maximizes the determinant. In optimization, this is the *analytic center* of the spectrahedron.

[C. Uhler, B.St.: Multivariate Gaussians, Semidefinite Matrix Completion and Convex Algebraic Geometry, 2009]

Minimizing Polynomial Functions

Let $f(x_1, \ldots, x_m)$ be a polynomial of even degree $2d$.
We wish to compute the global minimum x^* of $f(x)$ on $\mathbb{R}^m$.

This optimization problem is equivalent to

Maximize λ such that $f(x) - \lambda$ is non-negative on $\mathbb{R}^m$.

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Empirically, the optimal value of the SDP almost always agrees with the global minimum. In that case, the optimal matrix of the dual SDP has rank one, and the optimal point x^* can be recovered from this. **How to reconcile this with Blekherman's results?**

SOS Programming: A Univariate Example

Let $m = 1$, $d = 2$ and $f(x) = 3x^4 + 4x^3 - 12x^2$. Then

$$f(x) - \lambda \quad = \quad \begin{pmatrix} x^2 & x & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & \mu - 6 \\ 2 & -2\mu & 0 \\ \mu - 6 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$$

Our problem is to find (λ, μ) such that the 3×3-matrix is positive semidefinite and λ is maximal. The optimal solution of this SDP is

$$(\lambda^*, \mu^*) \;=\; (-32, -2).$$

Cholesky factorization reveals the SOS representation

$$f(x) - \lambda^* \quad = \quad \left((\sqrt{3}\,x - \frac{4}{\sqrt{3}}) \cdot (x+2)\right)^2 + \frac{8}{3}(x+2)^2.$$

We see that the global minimum is $x^* = -2$.
This approach works for many polynomial optimization problems.

Hankel Matrices

Consider the intersection of the cone of 6×6 PSD matrices with the 15-dimensional linear space consisting of all Hankel matrices

$$H \quad = \quad \begin{pmatrix} \lambda_{400} & \lambda_{220} & \lambda_{202} & \lambda_{310} & \lambda_{301} & \lambda_{211} \\ \lambda_{220} & \lambda_{040} & \lambda_{022} & \lambda_{130} & \lambda_{121} & \lambda_{031} \\ \lambda_{202} & \lambda_{022} & \lambda_{004} & \lambda_{112} & \lambda_{103} & \lambda_{013} \\ \lambda_{310} & \lambda_{130} & \lambda_{112} & \lambda_{220} & \lambda_{211} & \lambda_{121} \\ \lambda_{301} & \lambda_{121} & \lambda_{103} & \lambda_{211} & \lambda_{202} & \lambda_{112} \\ \lambda_{211} & \lambda_{031} & \lambda_{013} & \lambda_{121} & \lambda_{112} & \lambda_{022} \end{pmatrix}.$$

Dual to this intersection is the projection

$$\mathrm{Sym}_2(\mathrm{Sym}_2(\mathbb{R}^3)) \;\rightarrow\; \mathrm{Sym}_4(\mathbb{R}^3)$$

taking a 6×6-matrix to the ternary quartic it represents. Its image is a cone whose algebraic boundary is a *discriminant* of degree 27.

Problem: Determine the variety of all Bézout matrices H^{-1}.

Orbitopes

An *orbitope* is the convex hull of an orbit under a real algebraic representation of a compact Lie group. Primary examples are the groups $\mathrm{SO}(n)$ and their products. Orbitopes for their adjoint representations are continuous analogues of *permutohedra*.

Many of these special orbitopes are projections of spectrahedra.

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Quiz: Is this orbitope a spectrahedron?

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Quiz: Is this orbitope a spectrahedron?

Answer: Yes, it is the set of psd Hankel matrices H that satisfy

$$\lambda_{400} + \lambda_{040} + \lambda_{004} + 2\lambda_{220} + 2\lambda_{202} + 2\lambda_{022} \;=\; 9.$$

Problem. Classify all $\mathrm{SO}(n)$-orbitopes that are spectrahedra.

Tautological Orbitopes

... are obtained by taking the convex hull of a matrix group.

Example (P. Parrilo): $\mathrm{conv}(SO(3))$ is the set of 3×3-matrices

$$\begin{pmatrix} u_{11}+u_{22}-u_{33}-u_{44} & 2u_{23}-2u_{14} & 2u_{13}+2u_{24} \\ 2u_{23}+2u_{14} & u_{11}-u_{22}+u_{33}-u_{44} & 2u_{34}-2u_{12} \\ 2u_{24}-2u_{13} & 2u_{12}+2u_{34} & u_{11}-u_{22}-u_{33}+u_{44} \end{pmatrix}$$

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Proof: Psd matrices having both trace 1 and rank 1 are of the form

$$U \;=\; \frac{1}{a^2+b^2+c^2+d^2} \begin{pmatrix} a^2 & ab & ac & ad \\ ab & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{pmatrix}$$

Their images under the linear map parametrize the group $SO(3)$.

Barvinok-Novik Orbitopes

Consider the $\mathrm{SO}(2)\times\mathrm{SO}(2)$-orbitope BN_4 determined by the curve

$$\theta \;\mapsto\; \big(\cos(\theta), \sin(\theta), \cos(3\theta), \sin(3\theta)\big) \quad \in \; \mathbf{R}^4.$$

This is the projection of a 6-dimensional Hermitian spectrahedron:

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under the map $(c_1, c_2, c_3, s_1, s_2, s_3) \mapsto (c_1, c_3, s_1, s_3)$. Here the unknown c_j represents $\cos(j\theta)$, the unknown s_j represents $\sin(j\theta)$.

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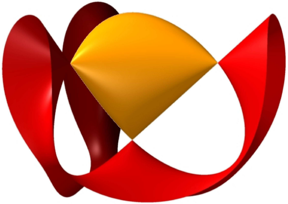
The faces of BN_4 are certain edges and triangles. Its algebraic boundary is the threefold defined by the degree 8 polynomial

$$x_3^2y_1^6 - 2x_1^3x_3y_1^3y_3 + x_1^6y_3^2 + 4x_1^3y_1^3 - 6x_1x_3y_1^4 - 6x_1^4y_1y_3 + 12x_1^2x_3y_1^2y_3 \\ - 2x_3^2y_1^3y_3 - 2x_1^3x_3y_3^2 - 3x_1^2y_1^2 + 4x_3y_1^3 + 4x_1^3y_3 - 6x_1x_3y_1y_3 + x_3^2y_3^2.$$

Conclusion

Spectrahedra and orbitopes deserve to be studied in their own right, independently of their important uses in applications.



A true understanding of these convex bodies will require the integration of three different areas of mathematics:

- Classical Convexity
- Algebraic Geometry
- Optimization Theory